Longest paths through an arc in strongly connected in-tournaments

Lutz Volkmann

Lehrstuhl II für Mathematik, RWTH Aachen,
52056 Aachen, Germany
e-mail: volkm@math2.rwth-aachen.de

Abstract

An in-tournament is an oriented graph such that the in-neighborhood of every vertex induces a tournament. Recently, we have shown that every arc of a strongly connected tournament of order n is contained in a directed path of order $\lceil (n+3)/2 \rceil$. This is no longer valid for strongly connected in-tournaments, because there exist examples containing an arc with the property that the longest directed path through this arc consists of three vertices. But in this paper we shall see that every strongly connected in-tournament has at most one such arc. More general, we shall prove that if a strongly connected in-tournament D of order n contains $m-2 \leq n-3$ arcs $a_3, a_4, \ldots, a_m$ such that the longest directed path through a_k consists of k vertices for $3 \leq k \leq m$, then all other arcs of D belong to directed paths of order at least $m+1$. Furthermore, we shall show that every arc of a strongly connected in-tournament is contained in a directed path of order $k+2$, when $\max\{\delta^+, \delta^-\} \geq k$, where δ^+ and δ^- is the minimum outdegree and the minimum indegree, respectively.

1. Terminology and introduction

The vertex set and the arc set of a digraph D are denoted by $V(D)$ and $E(D)$, respectively. The number $|V(D)|$ is the order of the digraph D. Throughout this paper we will consider digraphs without multiple arcs, loops, or directed cycles of length two. Such digraphs are called oriented graphs. If there is an arc from x to y in D, then y is a positive neighbor of x and x is a negative neighbor of y, and we also say that x dominates y, denoted by $x \rightarrow y$. More generally, let A and B be two disjoint subdigraphs of D or subsets of $V(D)$. If $x \rightarrow y$ for every vertex x in A and every vertex y in B, then we write $A \rightarrow B$ and say that A dominates B. Two vertices x and y of a digraph are adjacent when $x \rightarrow y$ or $y \rightarrow x$. The outset $N^+(x)$ of a vertex x is the set of vertices dominated by x, and the inset $N^-(x)$ is the set of vertices dominating x. The numbers $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ are called outdegree and indegree, respectively. The minimum outdegree δ^+ and the minimum indegree δ^- of D are given by $\min\{d^+(x) \mid x \in V(D)\}$ and $\min\{d^-(x) \mid x \in V(D)\}$, respectively. For $A \subseteq V(D)$, we define $D[A]$ as the subdigraph induced by A. By a cycle (path) we mean a directed cycle (directed path). A cycle or a path of order m is called an m-cycle or an m-path, respectively. A cycle (path) of a digraph D is Hamiltonian if it includes all the vertices of D. We speak of a connected digraph if the underlying graph is connected. A digraph D is said to be strongly connected or just strong, if for every pair x, y of vertices of D, there is a path from x to y. A strong component of D is a maximal induced strong subdigraph of D. A digraph D is k-connected if for any set S of at most $k-1$ vertices, the subdigraph $D - S$ is strong. A minimal separating set of a strong digraph D is a subset $S \subset V(D)$ such that $D - S$ is not strong, but $D - S'$ is strong for any $S' \subset S$. An in-tournament is an oriented graph with the property that the inset of every vertex induces a tournament, i.e., every pair of distinct vertices that have a common positive neighbor are adjacent. A local tournament is an oriented graph such that the inset as well as the outset of every vertex induces a tournament. Throughout this paper all subscripts are taken modulo the corresponding number.

Local tournaments were introduced by Bang-Jensen [1] in 1990 and there exists extensive literature on this class of digraphs, e.g., the survey paper of Bang-Jensen and Gutin [2]. In particular, the Ph. D. theses of Y. Guo [4] and J. Huang [5] have been devoted to this subject. As a generalization of local tournaments, Bang-Jensen, Huang, and Prisner [3] studied the family of in-tournaments. But in-tournaments have, as yet, received little attention. Except for the above mentioned article of Bang-Jensen, Huang, Prisner [3], these digraphs have only been investigated by Tewes [7], [8], [9], and Tewes, Volkmann [10], [11]. It is the purpose of this paper to give more information about the properties of in-tournaments.

Very recently, we have proved [12] that every arc of a strongly connected tournament of order n (even every arc of a strongly connected n-partite tournament) is contained in a directed path of order $\lceil (n+3)/2 \rceil$. The following example shows that this is no longer valid for strongly connected in-tournaments.

Example 1.1 Let D consist of the cycle $x_1 x_2 \dots x_n x_1$ together with the arcs $x_1 x_i$ for $3 \leq i \leq n-1$. Then it is straightforward to verify that D is a strongly connected in-tournament of order n, and that the longest path through the arc $x_1 x_{n-1}$ is only of order three.

Definition 1.2 If the longest path through an arc uv consists of exactly m vertices, then we call uv an *m-path arc*.

In this paper we shall see that every strongly connected in-tournament of order $n \geq 4$ has at most one 3-path arc. More general, we shall prove that if a strongly connected in-tournament D of order n contains a k-path arc for every $3 \leq k \leq m \leq n-1$, then all other arcs of D belong to paths of order $m+1$. Also strongly connected in-tournaments without a 3-path arc but containing a 4-path arc, have only one

4-path arc, when the order is at least six. Furthermore, if a strongly connected in-tournament has a k-path arc for each $k = 3, 4, \ldots, m$ but no $(m+1)$-path arc, then it contains no $(m+2)$-path arc. In addition, we shall prove that every arc of a strongly connected in-tournament is contained in a path of order $k+2$, when $\max\{\delta^+, \delta^-\} \geq k$. Different examples will show that these results are best possible.

2. Preliminary results

The following known results play an important role in our investigations.

Theorem 2.1 (Rédei [6] 1934) Each tournament contains a Hamiltonian path.

Theorem 2.2 (Bang-Jensen, Huang, Prisner [3] 1993) An in-tournament has a Hamiltonian cycle if and only if it is strongly connected.

Theorem 2.3 (Bang-Jensen, Huang, Prisner [3] 1993) Let D be a strongly connected in-tournament and let S be a minimal separating set. Then there exists a unique order $D_1, D_2, \ldots, D_p$ of the strong components of $D - S$ such that there are no arcs from D_j to D_i for $j > i$, and for each $i = 1, 2, \ldots, p-1$ there exists a vertex $w_i \in V(D_i)$ such that $w_i \rightarrow D_{i+1}$. If in addition, xy is an arc from D_i to D_j for $i < j$, then $x \rightarrow (D_{i+1} \cup D_{i+2} \cup \ldots \cup D_j)$.

Theorem 2.4 (Bang-Jensen [1] 1990) Let D be a strongly connected local tournament and let S be a minimal separating set. Then there exists a unique order $D_1, D_2, \ldots, D_p$ of the strong components of $D - S$ such that there are no arcs from D_j to D_i for $j > i$, $D_i \rightarrow D_{i+1}$ for $i = 1, 2, \ldots, p-1$, and D_i is a tournament for $i = 1, 2, \ldots, p$.

The unique order $D_1, D_2, \ldots, D_q$ in Theorem 2.3 as well as in Theorem 2.4 is called the strong decomposition of $D - S$.

3. General results

Observation 3.1 Let uv be an arbitrary arc of a strongly connected in-tournament D. If $D - u$ or $D - v$ is strong, then D contains a Hamiltonian path starting with the arc uv or ending with the arc uv, respectively.

Proof. If $D - u$ is strong, then by Theorem 2.2, the in-tournament $D - u$ has a Hamiltonian cycle $vx_2x_3 \ldots x_{|V(D)|-1}v$. Therefore, the arc uv is the initial arc of the Hamiltonian path $uvx_2x_3 \ldots x_{|V(D)|-1}$ of D. Considering $D - v$ instead of $D - u$, we obtain analogously a Hamiltonian path with the terminal arc uv. □

Theorem 3.2 Let u be a vertex of a strongly connected local tournament D such that $D-u$ is not strong. If $D_1, D_2, \ldots, D_p$ is the strong decomposition of $D-u$, then the arcs from D_i to D_{i+1} for $1 \le i \le p-1$ and the arcs in D_i for $2 \le i \le p-1$ are contained in a Hamiltonian path.

Proof. In view of Theorem 2.2, each strong component D_i with at least three vertices has a Hamiltonian cycle $x_1^i x_2^i \ldots x_{|V(D_i)|}^i x_1^i$ for $1 \le i \le p$. Since D is strong, there exists a vertex, say x_1^1, in D_1 such that $u \to x_1^1$ and a vertex, say x_1^p, in D_p such that $x_1^p \to u$. By P_i we denote a Hamiltonian path of D_i for $1 \le i \le p$. Theorem 2.4 implies $D_i \to D_{i+1}$ for $i = 1, 2, \ldots, p-1$. In the following we always use this fact.
Case 1: Let $x_j^i x_k^{i+1}$ be an arc from D_i to D_{i+1} for $1 \le i \le p-1$.
Subcase 1.1: Let $p \ge 3$. If $i \ge 2$, then

$$ux_1^1 x_2^1 \ldots x_{|V(D_1)|}^1 P_2 \ldots P_{i-1} x_{j+1}^i x_{j+2}^i \ldots x_j^i x_k^{i+1} x_{k+1}^{i+1} \ldots x_{k-1}^{i+1} P_{i+2} \ldots P_p,$$

and if $i = 1$, then

$$x_{j+1}^1 x_{j+2}^1 \ldots x_j^1 x_k^2 x_{k+1}^2 \ldots x_{k-1}^2 P_3 \ldots P_{p-1} x_2^p x_3^p \ldots x_1^p u$$

is a Hamiltonian path of D through the arc $x_j^i x_k^{i+1}$.
Subcase 1.2: Let $p = 2$. If $u \to x_{j+1}^1$, then $ux_{j+1}^1 x_{j+2}^1 \ldots x_j^1 x_k^2 x_{k+1}^2 \ldots x_{k-1}^2$ is a desired Hamiltonian path. If u does not dominate x_{j+1}^1, then let $s \ge 2$ be the smallest integer such that $u \to x_{j+s}^1$. Then, because u and x_{j+s-1}^1 are negative neighbors of x_{j+s}^1, we conclude that $x_{j+s-1}^1 \to u$. But now $x_j^1 x_k^2$ is an arc of the Hamiltonian path

$$x_{j+1}^1 x_{j+2}^1 \ldots x_{j+s-1}^1 u x_{j+s}^1 \ldots x_j^1 x_k^2 x_{k+1}^2 \ldots x_{k-1}^2.$$

Case 2: Let $x_j^i x_k^i$ be an arc of the component D_i for $2 \le i \le p-1$. By Theorem 2.4, D_i is a tournament, and thus, $D_i' = D_i - \{x_j^i, x_k^i\}$ is also a tournament. According to Theorem 2.1, D_i' has a Hamiltonian path P_i'. Hence, we deduce that $x_j^i x_k^i$ is an arc of the Hamiltonian path

$$x_j^i x_k^i P_{i+1} P_{i+2} \ldots P_{p-1} x_2^p x_3^p \ldots x_1^p u x_1^1 x_2^1 \ldots x_{|V(D_1)|}^1 P_2 P_3 \ldots P_{i-1} P_i'. \quad \square$$

Example 3.3 Let T_5 be the tournament with the cycle $x_1x_2x_3x_4x_5x_1$ such that $x_1 \to \{x_3, x_4\}$, $x_2 \to \{x_4, x_5\}$, and $x_5 \to x_3$. Note that the arc x_1x_4 is not contained in a Hamiltonian path of T_5. Now let T_7 be the tournament consisting of T_5 and the two new vertices u and w such that $T_5 \to w \to u \to x_4$ and $\{x_1, x_2, x_3, x_5\} \to u$. Then, T_5 corresponds to the first component D_1 of $T_7 - u$, and it easy to see that the arc x_1x_4 is not contained in a Hamiltonian path of T_7.
Using the same method, it is a simple matter to construct strongly connected tournaments T of arbitrarily large order such that the strong components D_1 and D_p of $T-u$ have arcs which are not contained in a Hamiltonian path of T.

Remark 3.4 Example 3.3 shows that Theorem 3.2 is not valid for the arcs in D_1 or D_p, even for tournaments, in general. But if $u \to D_1$ or $D_p \to u$, then one can prove analogously to Case 2 that each arc in D_1 or D_p is contained in a Hamiltonian path, respectively.

Example 3.5 Let T_p be the transitive tournament with the vertex set $\{x_1, x_2, \ldots, x_p\}$ such that $x_i \to x_j$ for $1 \le i < j \le p$. Now let D be the strongly connected local tournament of order $p+1$ consisting of T_p, the new vertex u and the both arcs $x_p u$ and $u x_1$. Then, $D - u = T_p$ has the strong decomposition $D_1, D_2, \ldots, D_p$ such that $V(D_i) = \{x_i\}$ for $1 \le i \le p$, and we observe that no arc $x_i x_j$ with $j \ge i+2$ is contained in a Hamiltonian path.

In view of the Examples 3.3 and 3.5, we see that Theorem 3.2 is best possible.

Observation 3.6 Let uv be an arc of an in-tournament D. If $d^-(u) = m$, then D contains an $(m+2)$-path with the terminal arc uv.

Proof. It follows from the definition of an in-tournament that the induced subdigraph $D[N^-(u)]$ is a tournament. Thus, according to Theorem 2.1, there exists a Hamiltonian path $x_1 x_2 \ldots x_m$ of $D[N^-(u)]$. Consequently, $x_1 x_2 \ldots x_m uv$ is path of order $m+2$ in D with the terminal arc uv. □

Theorem 3.7 Let uv be an arc of a strong in-tournament D. If

$$\max\{d^-(u), d^+(v)\} = m,$$

then the arc uv is contained in a path of order $m+2$.

Proof. If $d^-(u) = m$, then we are done by Observation 3.6. Now assume that $d^+(v) = m$ and let $|V(D)| = n$. By Theorem 2.2, D has a Hamiltonian cycle, and hence the in-tournament $D - v$ contains a Hamiltonian path $x_1 x_2 \ldots x_{n-1}$. Let $u = x_k$ for some $1 \le k \le n-1$. If $k \ge m+1$, then $x_1 x_2 \ldots x_k v$ is a path of order $k+1 \ge m+2$ through the arc uv. If $k \le m$, then, because of $d^+(v) = m$, the vertex v has at least $m - (k-1)$ positive neighbors in the vertex set $\{x_{k+1}, x_{k+2}, \ldots, x_{n-1}\}$. If $j \ge k+1$ is the smallest index such that $v \to x_j$, then $j \le n - m + k - 1$. Therefore, the path $x_1 x_2 \ldots x_k v x_j x_{j+1} \ldots x_{n-1}$ through uv consists of at least $k+1+(n-1)-j+1 \ge n+k+1-(n-m+k-1) = m+2$ vertices. □

Corollary 3.8 Let D be a strongly connected in-tournament. If

$$\max\{\delta^+, \delta^-\} \ge m,$$

then every arc of D is contained in a path of order $m+2$.

The next example will demonstrate that Theorem 3.7 is best possible, even for the family of local tournaments.

Example 3.9 Let T_k be a strong tournament and let T_{m+1} be a transitive tournament with the vertex set $\{v, x_1, x_2, \ldots, x_m\}$ such that $x_i \to x_j$ for $1 \le i < j \le m$

and $v \to \{x_1, x_2, \ldots, x_m\}$. If the digraph D consists of the tournaments T_k and T_{m+1} and the vertex u such that $u \to (V(T_k) \cup \{v\})$, $\{x_1, x_2, \ldots x_m\} \to u$, and $T_k \to v$, then it is a simple matter to verify that D is a strongly connected local tournament with $d^+(v) = d^-(u) = m$ containing the $(m+2)$-path arc uv.

4. Strong in-tournaments containing a 3-path arc

First, we present a structure result of strongly connected in-tournaments containing a 3-path arc, which implies that only one such arc exists.

Theorem 4.1 Let D be a strongly connected in-tournament of order $n \geq 4$ containing a 3-path arc uv. Then, D has no further 3-path arc, $D - u$ is not strong, and the strong decomposition $D_1, D_2, \ldots, D_p$ of $D - u$ has the following properties. The strong component D_p consists of a single vertex, say w_p, such that $w_p \to u$, $V(D_{p-1}) = \{v\}$, $N^-(w_p) = \{v\}$, and $u \to (D_1 \cup D_2 \cup \ldots \cup D_{p-1})$.

Proof. From Observation 3.1 it follows that $D-u$ is not strong. If $D_1, D_2, \ldots, D_p$ are the strong components of $D-u$, then in view of Theorem 2.3, there are no arcs from D_j to D_i for $j > i$, and for each $i = 1, 2, \ldots, p-1$ there exists a vertex $w_i \in V(D_i)$ such that $w_i \to D_{i+1}$. Since D is strong, there is a vertex $w_p \in V(D_p)$ with $w_p \to u$. First, we show that $v \in V(D_j)$ implies $V(D_j) = \{v\}$. Because otherwise, the strong component D_j consists of at least three vertices, and according to Theorem 2.2, D_j has a Hamiltonian cycle, say $vx_1x_2 \ldots x_tv$ with $t \geq 2$. Then the arc uv belongs to the 4-path uvx_1x_2, a contradiction to the hypothesis that uv is a 3-path arc.
Since $w_p \in V(D_p)$ with $w_p \to u$, we conclude that $j \neq p$. Analogously, we can show that $V(D_p) = \{w_p\}$. Furthermore, if we assume that $j \leq p-2$, then $v = w_j \to D_{j+1}$, and hence, $uvw_{j+1}w_{j+2}$ is a 4-path containing the arc uv, a contradiction. Consequently, $j = p-1$, and therefore $V(D_{p-1}) = \{v\}$.
Next we note that there are no arcs xu and xw_p such that x is a vertex of $D_1 \cup D_2 \cup \ldots \cup D_{p-2}$, because otherwise, $xuvw_p$ and xw_puv would be a 4-path through uv, respectively. This implies, in particular that $N^-(w_p) = \{v\}$. The vertices u and w_{p-2} are negative neighbors of v, and thus they are adjacent. Since there is no arc from w_{p-2} to u, we deduce that $u \to w_{p-2}$. If D_{p-2} consists only of the single vertex w_{p-2}, then $u \to D_{p-2}$. In the other case we use the facts that D_{p-2} has a Hamiltonian cycle, that there is no arc from D_{p-2} to u, and $u \to w_{p-2}$, to verify that $u \to V(D_{p-2})$. If we continue this process, we finally arrive at $u \to (D_1 \cup D_2 \cup \ldots \cup D_{p-1})$. We notice that all other arcs of D do not influence the property that uv is a 3-path arc.
Finally, we show that all arcs different from uv are contained in a 4-path. If ux_i is an arc with $x_i \in V(D_i)$ for $1 \leq i \leq p-2$, then vw_pux_i is a 4-path through ux_i as well as through vw_p and w_pu. Each arc x_iy_i of D_i for $1 \leq i \leq p-2$ belongs to the 4-path $w_pux_iy_i$. In the case that x_iy_j is an arc from D_i to D_j for $1 \leq i < j \leq p-1$, we see that x_ix_j is an arc of the 4-path $w_pux_iy_j$. Since we have discussed all possible arcs, the proof is complete. □

Theorem 4.2 Let D be a strongly connected in-tournament of order $n \geq 5$ containing a 3-path arc but no 4-path arc. Then $n \geq 6$ and D contains no 5-path arc.

Proof. Let uv be the 3-path arc of D, and let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D-u$. Then, Theorem 4.1 implies $V(D_p) = \{w_p\}$, $w_p \to u$, $V(D_{p-1}) = \{v\}$, $N^-(w_p) = \{v\}$, and $u \to (D_1 \cup D_2 \cup \ldots \cup D_{p-1})$. Since D contains no 4-path arc, we deduce that $|V(D_{p-2})| \geq 3$ and thus, $n \geq 6$. By Theorem 2.2, there exists a Hamiltonian cycle $b_1 b_2 \ldots b_t b_1$ of D_{p-2} with $t \geq 3$, and in view of Theorem 2.3, we assume without loss of generality that $w_{p-2} = b_1 \to v$.
First, we show that all arcs, different from uv, of the subdigraph induced by the vertices $u, v, w_p, b_1, b_2, \ldots, b_t$ are contained in a 6-path. The path $vw_pub_ib_{i+1}\ldots b_{i-1}$ for $1 \leq i \leq t$ shows that the arcs ub_i, vw_p, and w_pu are contained in a path of order at least 6. If $b_i \to v$ for any $1 \leq i \leq t$, then b_iv is an arc of the path $b_ivw_pub_{i+1}b_{i+2}\ldots b_{i-1}$ which is of order at least 6. Each arc b_ib_j with $i, j \neq 1$ belongs to the 6-path $b_1vw_pub_ib_j$. An arc b_ib_1 with $i \geq 3$ is contained in the 6-path $b_ib_1vw_pub_2$, and an arc b_1b_i with $i \neq t$ is contained in the 6-path $vw_pub_1b_ib_{i+1}$.
Using in particular the fact that $u \to (D_1 \cup D_2 \cup \ldots \cup D_{p-1})$, we now prove that the other arcs are also contained in a 6-path, when $p \geq 4$. Each arc ux_i with $x_i \in V(D_i)$ for $1 \leq i \leq p-3$ belongs to the 6-path $b_tb_1vw_pux_i$. If x_iy_i is an arc of D_i for $1 \leq i \leq p-3$, then it is contained in the 6-path $b_1vw_pux_iy_i$. Finally, let x_iy_j be an arc from D_i to D_j for $1 \leq i \leq p-3$ and $i < j \leq p-1$. If $j \leq p-3$, then $b_1vw_pux_iy_j$ is a 6-path through the arc x_iy_j. In the case $j = p-2$, we observe that $y_j = b_s$ for any $1 \leq s \leq t$, and obviously, $vw_pux_ib_sb_{s+1}$ is such a desired 6-path. In the remaining case $j = p-1$, we have $y_j = v$. Since $i \leq p-3$, we observe that $x_ivw_pub_1b_2$ is a 6-path with the initial arc x_iy_j. Consequently, D contains no 5-path arc, and the proof is complete. □

Next we will show that Theorem 4.2 is sharp in the sense that there exist in-tournaments containing a 3-path arc, without 4 or 5-path arcs, however with 6-path arcs.

Example 4.3 Let D be consists of the cycle $C = b_1b_2\ldots b_nb_1$, the arcs b_1b_i for $3 \leq i \leq n-1$, and the vertices u, v and w_3 such that $b_1 \to v \to w_3 \to u \to (V(C) \cup \{v\})$. Then, it is straightforward to verify that D is a strongly connected in-tournament of order $n+3$ with the 3-path arc uv, without a 4 or a 5-path arc, but D contains the 6-path arc b_1b_{n-1}.

Example 4.4 Let D be consists of the cycle $C = b_1b_2b_3b_1$, the vertices u, v, w_1 and w_4 such that $w_1 \to C$ and $b_1 \to v \to w_4 \to u \to (V(C) \cup \{v, w_1\})$. Then, D is a strongly connected in-tournament of order 7 containing the 3-path arc uv, without a 4 or a 5-path arc, but with the two 6-path arcs ub_1 and ub_3.

Example 4.5 Let D be consists of the cycle $C = b_1b_2b_3b_1$, the vertices u, v, w_4, and an arbitrary tournament T_1 such that $b_1 \to v \to w_4 \to u \to (V(C) \cup V(T_1) \cup \{v\})$ and $T_1 \to C$. Then, D is a strongly connected in-tournament with the 3-path arc

uv, without a 4 or a 5-path arc, but D contains the 6-path arc ub_1.

Theorem 4.6 Let D be a strongly connected in-tournament of order $n \geq 5$ containing the 3-path arc uv. If $D_1, D_2, \ldots, D_p$ is the strong decomposition of $D - u$, and if D contains an k-path arc for each $4 \leq k \leq m \leq n-1$, then $V(D_{p+2-k}) = \{w_{p+2-k}\}$, $N^-(w_{p+3-k}) = \{u, w_{p+2-k}\}$, and uw_{p+2-k} is the unique k-path arc for $4 \leq k \leq m$.

Proof. We proceed by induction on m, using the structure of D, described in Theorem 4.1 and parts of the proof of Theorem 4.2.
Let $m = 4$. Suppose first that $|V(D_{p-2})| \geq 3$. Then, by the proof of Theorem 4.2, we see that all arcs different from uv of the subdigraph induced by the vertices u, v, w_p and the vertex set $V(D_{p-2})$ are contained in a 6-path. But since D contains 4-path arc, we deduce that $p \geq 4$.
Next we prove that all arcs different from uv and ux with $x \in V(D_{p-2})$ belong to a 5-path of D, independently from the order of D_{p-2}. Every arc ux_i with $x_i \in V(D_i)$ is contained in the path $w_{p-2}vw_pux_i$ for $1 \leq i \leq p-3$. Thus, vw_p and w_pu are also arcs of a 5-path. Every arc x_iy_i of D_i is contained in the path $vw_pux_iy_i$ for $1 \leq i \leq p-2$. Now let x_ix_j be an arc from D_i to D_j for $1 \leq i < j \leq p-1$. If $j \leq p-2$, then the 5-path $vw_pux_ix_j$ has the terminal arc x_ix_j. If $j = p-1$, then $x_j = v$, and x_iv belongs to the 5-path $x_ivw_pux_s$ with $s \neq i, p-1, p$ and $x_s \in V(D_s)$.
All together we see there exists at most a 4-path arc in D, if $p \geq 4$ and if D_{p-2} consists of the single vertex w_{p-2}. But in this case, certainly, uw_{p-2} is the only 4-path arc of D, when $N^-(v) = N^-(w_{p-1}) = \{u, w_{p-2}\}$.
Now let $5 \leq m \leq n-1$ and assume that D contains a k-path arc for each $4 \leq k \leq m$. Then, D contains a k-path arc for each $4 \leq k \leq m-1$, and by the induction hypothesis $V(D_{p+2-k}) = \{w_{p+2-k}\}$, $N^-(w_{p+3-k}) = \{u, w_{p+2-k}\}$, and uw_{p+2-k} is the unique k-path arc for $4 \leq k \leq m-1$. Analogously to the case $m = 4$, one can prove that $|V(D_{p+2-m})| \geq 3$ is not possible, and thus $p \geq m$, and that all arcs different from uv, uw_{p+2-k} for $4 \leq k \leq m-1$ and ux with $x \in V(D_{p+2-m})$ are contained in an $(m+1)$-path, independently from the order of D_{p+2-m}. Consequently, D_{p+2-m} consists of the single vertex w_{p+2-m}. In addition, from the hypothesis that D has an m-path arc, it follows that $N^-(w_{p+3-m}) = \{u, w_{p+2-m}\}$ and this implies that uw_{p-2} is the only m-path arc of D. □

Using Theorem 4.6, it is no problem to obtain the next result, analogously to Theorem 4.2.

Theorem 4.7 Let D be a strongly connected in-tournament of order $n \geq m+2$ containing a k-path arc for each $k = 3, 4, \ldots, m$ but no $(m+1)$-path arc. Then $n \geq m+3$ and D contains no $(m+2)$-path arc.

Theorem 4.8 Let D be a strongly connected local tournament of order $n \geq 4$ with the 3-path arc uv. Then all arcs of D which are not incident with u are contained in a Hamiltonian path.

Proof. By Observation 3.1, we assume without loss of generality that $D-u$ is not strong. If $D_1, D_2, \ldots, D_p$ is the strong decomposition of $D-u$, then by Theorem 4.1, $V(D_p)=\{w_p\}$, $w_p \to u$, $V(D_{p-1})=\{v\}$, $N^-(w_p)=\{v\}$, and $u \to (D_1 \cup D_2 \cup \ldots \cup D_{p-1})$. Furthermore, Theorem 2.4 implies $D_i \to D_{i+1}$ for $i=1,2,\ldots,p-1$. In the following let $x_1^i x_2^i \ldots x_{|V(D_i)|}^i x_1^i$ be a Hamiltonian cycle of the strong component D_i for $1 \le i \le p-2$, when $|V(D_i)| \ge 3$, and define by P_i a Hamiltonian path of D_i. By Theorem 3.2 and Remark 3.4, every arc from D_i to D_{i+1} for $1 \le i \le p-1$, and each arc of the component D_i for $1 \le i \le p-2$ is contained in a Hamiltonian path of D. Now, let $x_j^i x_k^t$ be an arc from D_i to D_t for $1 \le i \le p-2$ and $i+2 \le t \le p-1$. Then, because of $D_i \to D_{i+1}$ for $i=1,2,\ldots,p-1$, we deduce that

$$P_1 P_2 \ldots P_{i-1} x_{j+1}^i x_{j+2}^i \ldots x_j^i x_k^t x_{k+1}^t \ldots x_{k-1}^t P_{t+1} \ldots v w_p u P_{i+1} \ldots P_{t-1}$$

is a Hamiltonian path through $x_j^i x_k^t$, and this completes the proof. □

Obviously, in Theorem 4.8, the arc $w_p u$ and all arcs from u to D_1 are also contained in a Hamiltonian path, even in a Hamiltonian cycle. Example 4.3 shows that Theorem 4.8 is no longer valid for in-tournaments in general.

5. Strong in-tournaments without a 3-path arc

Next we describe the structure of strongly connected in-tournaments containing a 4-path arc but no 3-path arc. We shall see that such in-tournaments have only one 4-path arc, when the order is at least six.

Theorem 5.1 Let D be a strongly connected in-tournament of order $n \ge 6$ containing a 4-path arc uv but no 3-path arc. If $D_1, D_2, \ldots, D_p$ is the strong decomposition of $D-u$, then $p \ge 4$, D_p consists of a single vertex, say w_p such that $w_p \to u$, $V(D_{p-1})=\{w_{p-1}\}$, $u \to (D_1 \cup D_2 \cup \ldots \cup D_{p-2})$, $v \in V(D_{p-1}) \cup V(D_{p-2})$, and D has no further 4-path arc. In addition:
If $v \in V(D_{p-1})$, then $V(D_{p-1})=\{v\}$, $V(D_1)=\{w_1\}$, and $N^-(w_p)=\{w_1, v\}$.
If $v \in V(D_{p-2})$, then $V(D_{p-2})=\{v\}$ and there are no arcs from D_j to the vertices w_{p-1} or w_p for $1 \le j \le p-3$. Furthermore, if $u \to w_{p-1}$, then $v \to w_p$.

Proof. From Observation 3.1 it follows that $D-u$ is not strong. Since D is strong, there exists a vertex $w_p \in V(D_p)$ with $w_p \to u$.
Suppose first that $v \in V(D_p)$. Since the vertex $w_p \ne v$ is also in D_p, the strong component D_p consists of at least three vertices, and according to Theorem 2.2, D_p has a Hamiltonian cycle, say $v x_1 x_2 \ldots x_t v$, with $t \ge 2$. If $t \ge 3$, then $u v x_1 x_2 x_3$ is 5-path through uv, a contradiction. Thus, $t=2$. The vertices u and x_2 are adjacent, since they are negative neighbors of v. If $x_2 \to u$, then $w_{p-1} x_1 x_2 u v$ is a 5-path, a contradiction. Consequently, $u \to x_2$ and $w_p = x_1 \to u$. But now it follows easily from the hypothesis $n \ge 6$ that uv is not a 4-path arc, a contradiction.
Second, let $v \in D_{p-1}$. If $|V(D_{p-1})| \ge 3$, then there exists a Hamiltonian cycle $v x_1 x_2 \ldots x_t v$ of D_{p-1}, and $w_p u v x_1 x_2$ is a 5-path through uv, a contradiction. This implies $V(D_{p-1})=\{v\}$, and similarly we find that $V(D_p)=\{w_p\}$. Since uv is a

4-path arc, there exist an arc wu or ww_p with $w \in V(D_j)$ for $1 \leq j \leq p-2$. In both cases we deduce that $w \in V(D_1)$ and $|V(D_1)| = 1$. This is a contradiction if wu is an arc of D, because there is also an arc from u to D_1. In the other case we see that $w = w_1$, $N^-(w_p) = \{w_1, v\}$, and $p \geq 4$. Analogously to the proof of Theorem 4.1, we obtain $u \to (D_1 \cup D_2 \cup \ldots \cup D_{p-2})$, and this implies that D has no further 4-path arc.

Suppose third that $v \in V(D_j)$ for any $j \leq p-2$. The cases $j \leq p-3$ or $j = p-2$ and $|V(D_{p-2})| \geq 3$ lead to a contradiction, and thus, $V(D_{p-2}) = \{v\}$. This implies immediately $V(D_p) = \{w_p\}$, $V(D_{p-1}) = \{w_{p-1}\}$, and $p \geq 4$. Next we note that there are no arcs $x_i w_{p-1}$ or $x_i w_p$ with $x_i \in V(D_i)$ for $1 \leq i \leq p-3$, because otherwise $x_i w_{p-1} w_p uv$ or $x_i w_p uv w_{p-1}$ would be 5-paths through uv. Obviously, there is no arc from D_j to u for $j \leq p-3$, and hence, analogously to the proof of Theorem 4.1, we obtain $u \to (D_1 \cup D_2 \cup \ldots \cup D_{p-2})$. If $u \to w_{p-1}$, then it follows that $v \to w_p$, because otherwise uw_{p-1} would be a 3-path arc. With help of the hypothesis $n \geq 6$, it is straightforward to verify that there is no further 4-path arc in D. □

Remark 5.2 For $n = 5$ there exist exactly three non isomorphic strongly connected in-tournaments containing a 4-path arc but no 3-path arc. Let $C = uw_1w_2w_3w_4u$ be a 5-cycle.
If T_5 is the tournament consisting of C such that $u \to \{w_2, w_3\}$, $w_1 \to \{w_3, w_4\}$, and $w_4 \to w_2$, then T_5 contains the the unique 4-path arc uw_3.
If D_5 is the in-tournament consisting of C such that $u \to w_2$ and $w_3 \to u$, then D_5 has even the two 4-path arcs uw_2 and w_3u. If we add in D_5 the arc w_2w_4, then we obtain an in-tournament with the unique 4-path arc uw_2.

With help of Theorem 3.2 and Theorem 5.1, one can prove the next result, analogously to Theorem 4.8.

Theorem 5.3 Let D be a strongly connected local tournament of order $n \geq 6$ with the 4-path arc uv but without a 3-path arc. Then all arcs of D which are not incident with u are contained in a Hamiltonian path, with exception of the arc vw_p, when $v = w_{p-2}$ and u and w_{p-1} are not adjacent. But in this situation the arc vw_p is contained in an $(n-1)$-path.

Our next example shows that Theorem 5.3 is not valid for strong in-tournaments in general.

Example 5.4 Let D be consists of the cycle $C = b_1b_2 \ldots b_nb_1$, the arcs b_1b_i for $3 \leq i \leq n-1$, and the vertices u, v, w_2 and w_3 such that $b_1 \to v \to w_2 \to w_3 \to u \to (V(C) \cup \{v\})$ and $v \to w_3$. Then, D is a strongly connected in-tournament of order $n+4$ without a 3-path arc containing the 4-path arc uv. We observe that D has the $(n+3)$-path arc vw_3 and the 7-path arc b_1b_{n-1}, so that b_1b_{n-1} is not contained in a Hamiltonian path, when $n \geq 4$.

We also have a corresponding result to the Theorems 4.1 and 5.1, when uv is a

5-path arc and D contains neither a 3-path arc nor a 4-path arc. Since the description of such in-tournaments is long and not very transparent, we omit it here. But especially, we have found the following uniquenes theorem.

Theorem 5.5 Let D be a strong in-tournament of order $n \geq 8$ containing a 5-path arc but neither a 3-path arc nor a 4-path arc. Then, D has exactly one 5-path arc.

Next we present an example that demonstrates that the condition $n \geq 8$ in Theorem 5.5 is necessary.

Example 5.6 Let $m \geq 5$ be an integer, and let the strongly connected in-tournament D consists of the cycle $x_1x_2 \ldots x_{m-2}x_{m-1}y_1y_2 \ldots y_{m-2}x_1$ such that $x_1 \to \{x_3, x_4, \ldots, x_{m-1}\}$ and $x_{m-1} \to \{y_2, y_3, \ldots, y_{m-2}\}$. Then, D is of order $2m - 3$ with the two m-path arcs x_1x_{m-1} and $x_{m-1}y_{m-2}$.

Theorems 4.1, 5.1, 5.5, and Example 5.6 leads us to the following conjecture.

Conjecture 5.7 Let $m \geq 6$ be an integer, and let D be a strongly connected in-tournament of order $n \geq 2m - 2$. If D has an m-path arc but no k-path arc for $3 \leq k \leq m - 1$, then there exists exactly one m-path arc.

Example 5.6 shows that the condition $n \geq 2m - 2$ in Conjecture 5.7 would be best possible.

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(Received 10/3/99)