Self-converse Mendelsohn designs with block size $6q$ *

Caikun Liu, Qingde Kang and Zhifen Guo

Institute of Mathematics, Hebei Normal University
Shijiazhuang 050016, P. R. China
qdkang@public.sj.he.cn

Abstract

A Mendelsohn design $MD(v,k,\lambda)$ is a pair $(X,\mathcal{B})$ where X is a v-set together with a collection $\mathcal{B}$ of cyclic k-tuples from X such that each ordered pair from X is contained in exactly λ cyclic k-tuples of $\mathcal{B}$. An $MD(v,k,\lambda)$ is said to be *self-converse*, denoted by $SCMD(v,k,\lambda)=(X,\mathcal{B},f)$, if there is an isomorphism f from $(X,\mathcal{B})$ to $(X,\mathcal{B}^{-1})$, where $\mathcal{B}^{-1}=\{\langle x_k,x_{k-1},...,x_2,x_1\rangle : \langle x_1,...,x_k\rangle \in \mathcal{B}\}$. The existence of $SCMD(v,3,\lambda)$, $SCMD(v,4,1)$ and $SCMD(v,4t+2,1)$ has been completely settled, where $2t+1$ is a prime power. In this paper, we investigate the existence of $SCMD(v,6q,1)$, where $gcd(q,6)=1$. In particular, when q is a prime power, the existence spectrum of $SCMD(v,6q,1)$ is solved, except possibly for two small subclasses. As well, our conclusion extends the existence results for $MD(v,k,1)$.

1 Introduction

Let X be a v-set and $3\le k\le v$. A *cyclic k-tuple* from X is a collection of k ordered pairs $(x_0,x_1),(x_1,x_2),\ ...,\ (x_{k-2},x_{k-1})$ and (x_{k-1},x_0), where $x_0,x_1,...,x_{k-1}$ are distinct elements of X. It is denoted by $\langle x_0,x_1,...,x_{k-1}\rangle$. A (v,k,λ)-*Mendelsohn design*, or $MD(v,k,\lambda)$, is a v-set together with a collection $\mathcal{B}$ of cyclic k-tuples (blocks) from X, such that each ordered pair (x,y) with $x\neq y\in X$ is contained in λ blocks of $\mathcal{B}$.

For an $MD(v,k,\lambda)=(X,\mathcal{B})$, define

$$\mathcal{B}^{-1}=\{\langle x_{k-1},x_{k-2},...,x_1,x_0\rangle : \langle x_0,x_1,...,x_{k-1}\rangle\in\mathcal{B}\}.$$

Obviously, $(X,\mathcal{B}^{-1})$ is also an $MD(v,k,\lambda)$, which is called the *converse* of $(X,\mathcal{B})$. If there exists an isomorphism f from $(X,\mathcal{B})$ to $(X,\mathcal{B}^{-1})$, then the $MD(v,k,\lambda)$ is

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called *self-converse* and this is denoted by $SCMD(v,k,\lambda) = (X, \mathcal{B}, f)$. For a block $B = \langle x_0, x_1, ..., x_{k-1}\rangle$, the block $f(B)^{-1} = \langle f(x_{k-1}), ..., f(x_1), f(x_0)\rangle$ is called the *f-converse* of B. To prove a system $(X, \mathcal{B}, f)$ is self-converse we only need to show that $f(B)^{-1} \in \mathcal{B}$ for any $B \in \mathcal{B}$. It is well known that a necessary condition for the existence of an $MD(v,k,\lambda)$ and $SCMD(v,k,\lambda)$ is

$$\lambda v(v-1) \equiv 0 \pmod{k}.$$

The known existence results for $MD(v,k,\lambda)$ and $SCMD(v,k,\lambda)$ can be summarized as follows.

Theorem 1.1 ([1]) *The above necessary condition for the existence of an $MD(v,k,\lambda)$ is also sufficient, if one of the following cases holds.*
(1) $k = 3$ *and* $(v,\lambda) \neq (6,1)$;
(2) $k = 4$ *and* $(v,\lambda) \neq (4, 2t+1)$ *for any integer* $t \geq 0$;
(3) $k = 6$ *and* $(v,\lambda) \neq (6,1)$;
(4) $k \in \{5,7,8,10,12,14\}$;
(5) $k \geq 7, v \equiv 0,1$ $(mod\ k)$.

Theorem 1.2 ([4],[5],[6])

(1) *There exists a simple $SCMD(v,3,\lambda)$ if and only if $\lambda v(v-1) \equiv 0$ $(mod\ 3)$, $\lambda \leq v-2$, $v \geq 3$ and $(v,\lambda) \neq (6,1),(6,3)$;*

(2) *There exists an $SCMD(v,4,1)$ if and only if $v \equiv 0,1$ $(mod\ 4)$ and $v > 5$;*

(3) *There exists an $SCMD(v,5,1)$ if and only if $v \equiv 0,1$ $(mod\ 5)$, $v \geq 5$ and $v \neq 6$;*

(4) *Let t be an odd integer and $t \geq 3$. There exists a self-converse $MD(v,2t,1)$ for $v \equiv$ 0 or 1 $(mod\ t)$ except for $v = 2t+1$ and $(v,t) = (6,3)$. In particular, when t is an odd prime power, the above condition for the existence of an $SCMD(v,2t,1)$ is also sufficient.*

In this paper, our main goal is to solve the existence problem for $SCMD(v,6q,1)$, where $gcd(q,6) = 1$. In particular, when q is a prime power, the existence spectrum of $SCMD(v,6q,1)$ is settled, except possibly for two small subclasses. As well, our conclusion extends the existence results for $MD(v,k,1)$. Our main results are:

Theorem 1.3 *Let q be positive integer with $gcd(q,6) = 1$. There exists an $SCMD(v,6q,1)$ for the following q and v.*
(1) $v \equiv 0,1$ $(mod\ 3q)$ *except for* $v = 6q+1$;
(2) $v \equiv q+1, 2q, 4q+1, 5q$ $(\ mod\ 6q)$ *and* $q \equiv 5$ $(\ mod\ 6)$;
(3) $v \equiv q, 2q+1, 4q, 5q+1$ $(\ mod\ 6q)$ *and* $q \equiv 1$ $(\ mod\ 6)$ *except possibly for*
  * $v = 7q$ *and* $q \equiv 1$ $(mod\ 12)$ *or* $q \equiv 7$ $(mod\ 48)$;
  * $v \equiv q$ $(mod\ 12q)$, $q \equiv 1$ $(mod\ 6)$ *where q is not a prime power.*

Theorem 1.4 *For prime power $q = p^m$ $(p > 3)$ and $v \neq 7q$ $(q \equiv 1$ $(mod\ 12)$ and $q \equiv 7$ $(mod\ 48))$, there exists $SCMD(v,6q,1)$ if and only if $v(v-1) \equiv 0$ $(mod\ 6q)$ except for $v = 6q+1$.*

Let $\lambda DK_{n_1,n_2,\cdots,n_h}$ be the complete multipartite directed graph with vertex set $X = \bigcup_{i=1}^{h} X_i$, where X_i $(1 \le i \le h)$ are disjoint sets with $|X_i| = n_i$ and where two vertices x and y from different sets X_i and X_j are joined by exactly λ arcs from x to y and λ arcs from y to x. A *holey Mendelsohn design*, briefly denoted by (v, k, λ)-*HMD*, is a trio $(X, \{X_i; 1 \le i \le h\}, \mathcal{A})$ where X is a v-set, $\mathcal{A}$ is a collection of cyclic k-tuples from X, which form an arc-disjoint decomposition of $\lambda DK_{n_1,\dots,n_h}$. Each X_i, $1 \le i \le h$, is called a *hole* (or *group*) of the design and the multiset $\{n_1, n_2, \dots, n_h\}$ is called the *type* of the design. Sometimes, we use an "exponential" notation to describe its type: a type $1^i2^j3^s$... denotes i occurrences of 1, j occurrences of 2, etc. If there exists an isomorphism f from $(X, \mathcal{A})$ to $(X, \mathcal{A}^{-1})$, then the (v, k, λ)-HMD is called a (v, k, λ)-$HSCMD = (X, \{X_i; 1 \le i \le h\}, \mathcal{A}, f)$. A (v, k, λ)-*HSCMD* of type $a_1^{m_1}a_2^{m_2}...a_s^{m_s}$ will be denoted by (v, k, λ)-*HSCMD*$(a_1^{m_1}...a_s^{m_s})$, in which $v = \sum_{i=1}^{s} m_i a_i$. When $\lambda = 1$, we shall briefly denote $SCMD(v, k, 1)$ and $(v, k, 1)$-$HSCMD(T)$ by k-$SCMD(v)$ and k-$HSCMD(T)$ respectively, where T represents the type of the $HSCMD$. A k-$HMD(1^{v-h}h^1)$ is also known as an *incomplete Mendelsohn design* and is denoted by k-$IMD(v, h)$. Similarly, a k-$HSCMD(1^{v-h}h^1)$ is known as an *incomplete self-converse MD*, and denoted by k-$ISCMD(v, h)$.

An *m-cycle system* of order v is a collection $CS(v, m)$ of undirected cycles with length m, whose (undirected) edges partition all edges of the complete graph K_v of order v. Obviously, if there exists a $CS(v, m) = (V, \mathcal{A})$, then an m-$SCMD(v) = (V, \mathcal{B}, f)$ exists. In fact, f can be the identity mapping and the block set $\mathcal{B}$ can be defined as

$$\{\langle a_1, \dots, a_m\rangle, \langle a_m, \dots, a_1\rangle; (a_1, \dots, a_m) \in \mathcal{A}\}.$$

2 Overall arrangement I

A necessary condition for the existence of a $6q$-$SCMD(v)$ is $v(v-1) \equiv 0 \pmod{6q}$. Let $gcd(q, 6) = 1$. By Theorem 1.2(4), there exists a $6q$-$SCMD(v)$ for $v \equiv 0, 1 \pmod{3q}$ and $v \ne 6q+1$. It is easy to see that, for general $6q$ $(gcd(q, 6) = 1)$, the following orders v satisfy the necessary condition $v(v-1) \equiv 0 \pmod{6q}$ besides $v \equiv 0, 1 \pmod{3q}$.

$q \equiv \pmod 6$	$v \equiv \pmod{6q}$			
1	q	$2q+1$	$4q$	$5q+1$
5	$q+1$	$2q$	$4q+1$	$5q$

These orders v (and corresponding q) are just the range considered in this paper. For prime power q, the range is exactly all admissible v for the existence of a $6q$-$SCMD(v)$.

Let $v = 6mq+h$, where m is a positive integer, $h \in \{2q, 2q+1, 4q, 4q+1, 5q, 5q+1\}$. Let

$V = X \cup Y$ and $X \cap Y = \emptyset$;

$$X = Z_m \times Z_{3q} \times Z_2;$$

$$Y = \begin{cases} Z_{\frac{h}{2}} \times Z_2 & (h \text{ even}) \\ (Z_{\frac{h-1}{2}} \times Z_2) \bigcup \{\infty\} & (h \text{ odd}) \end{cases}, \text{ when } h > 0;$$

$$f: \begin{cases} (i,j,k) \to (i,j,1-k) & (i,j,k) \in X \\ (a,b) \to (a,1-b) & (a,b) \in Y \backslash \{\infty\} \\ \infty \to \infty & \end{cases}.$$

Sometimes, we denote Y by $S = \{\infty_1, \infty_2, ..., \infty_h\}$ and denote the corresponding mapping by

$$g = \begin{cases} (i,j,k) \to (i,j,1-k), & (i,j,k) \in X \\ \infty_i \to \infty_{h+1-i}, \quad 1 \le i \le h & \end{cases}.$$

The mapping is uniform for all constructions throughout our paper. Below, especially in section 6, we will give the following results for different h:

(A) $6q$-$HSCMD((6q)^m) = (X, \{\{i\} \times Z_{3q} \times Z_2 : i \in Z_m\}, \mathcal{A}_m, f)$, where $m \ge 2$;
(B) $6q$-$SCMD(6q) = (\{i\} \times Z_{3q} \times Z_2, \mathcal{B}_i, f)$, $i \in Z_m$;
(C) $6q$-$HSCMD(h^1(6q)^1) = ((\{i\} \times Z_{3q} \times Z_2) \bigcup Y, \{\{i\} \times Z_{3q} \times Z_2, Y\}, \mathcal{C}_i, f)$, where $i \in Z_m$ and $3q \le h \le 6q$;
(D) $6q$-$SCMD(6q+h) = ((\{0\} \times Z_{3q} \times Z_2) \bigcup Y, \mathcal{D}, f)$;
(E) $6q$-$ISCMD(6q+h, h) = ((\{i\} \times Z_{3q} \times Z_2) \bigcup S, S, \Omega_i, g)$, where $i \in Z_m$ and $h < 3q$;
(F) $6q$-$SCMD(12q+h) = ((Z_2 \times Z_{3q} \times Z_2) \bigcup Y, \mathcal{F}, f)$.

Then, each of the following block sets will form a $6q$-$SCMD(v)$:

$$\mathcal{A}_m \cup \mathcal{D} \cup (\bigcup_{i \in Z_m^*} (B_i \cup C_i));$$

$$\mathcal{A}_m \cup \mathcal{D} \cup (\bigcup_{i \in Z_m^*} \Omega_i);$$

$$(\mathcal{A}_m \backslash \mathcal{A}_2) \cup \mathcal{F} \cup (\bigcup_{i \in Z_m^* \backslash \{1\}} \Omega_i), \text{ when } m > 2.$$

Here and below, Z_m denotes a residue class ring modulo m and $Z_m^* = Z_m \backslash \{0\}$.

Theorem 2.1 *The following designs exist:*

(1) $6q$-$HSCMD((6q)^m)$, $m \ge 2$;
(denote the design by $\mathcal{A}_m$, then $\mathcal{A}_2 \subset \mathcal{A}_m$ for $m > 2$)
(2) $6q$-$HSCMD(h^1(6q)^1)$, *where* $3q \le h < 6q$;
(3) $6q$-$SCMD(6q)$.

Proof By [6], the designs $2t$-$HSCMD((2t)^m)$, $m \ge 2$; $2t$-$HSCMD(h^1(2t)^1)$, $t \le h < 2t$; $2t$-$SCMD(2t)$ exist. We only need put $t = 3q$. In these self-converse designs given by [6], the mapping is the same as the uniform mapping defined by us. □

By the above description and Theorem 2.1, in order to complete $6q$-$SCMD(v)$, we only need construct

(D) and (E), when $v \equiv 2q \pmod{6q}$;
(E) and (F), when $v \equiv 2q+1 \pmod{6q}$;
(D), when $v \equiv 4q, 4q+1, 5q, 5q+1 \pmod{6q}$.

3 Overall arrangement II

For some h (such as $h = q$ or $q+1$), it is difficult to construct the desired design under the overall arrangement I. Now, let us consider an other arrangement.

Let $v = 12mq + h$, where $h \in \{q, q+1, 7q\}$ and $m > 0$ or $h \in \{7q+1\}$ and $m \geq 0$. Let

$$V = X \cup Y \text{ and } X \cap Y = \emptyset;$$

$$X = Z_m \times Z_{6q} \times Z_2;$$

$$Y = \begin{cases} Z_{\frac{h}{2}} \times Z_2 & (h \text{ even}) \\ (Z_{\frac{h-1}{2}} \times Z_2) \bigcup \{\infty\} & (h \text{ odd}) \end{cases}, \text{ when } h > 0.$$

Below, especially in section 6, we will give the following results for different h:

(A) $6q$-$HSCMD((12q)^m) = (X, \{\{i\} \times Z_{6q} \times Z_2 : i \in Z_m\}, \mathcal{A}, f)$, where $m \geq 2$;

(B) $6q$-$HSCMD(h^1(12q)^1) = ((\{i\} \times Z_{6q} \times Z_2) \bigcup Y, \{\{i\} \times Z_{6q} \times Z_2, Y\}, \mathcal{B}_i, f)$, where $i \in Z_m$ and $h \geq 3q$;

(C) $6q$-$SCMD(12q) = (\{i\} \times Z_{6q} \times Z_2, \mathcal{C}_i, f)$, where $i \in Z_m$;

(D) $6q$-$SCMD(12q + h) = (\{0\} \times Z_{6q} \times Z_2, \mathcal{D}, f)$;

(E) $6q$-$SCMD(h) = (Y, \mathcal{J}, f)$, where $h \geq 6q$;

(F) $6q$-$ISCMD(12q + h, h) = ((\{i\} \times Z_{6q} \times Z_2) \bigcup S, S, \Omega_i, g)$, where $i \in Z_m$ and $h < 3q$.

Then, each of the following block sets will form a $6q$-$SCMD(v)$:

$$\mathcal{A} \cup \mathcal{J} \cup (\bigcup_{i \in Z_m} (\mathcal{B}_i \cup \mathcal{C}_i));$$

$$\mathcal{A} \cup \mathcal{D} \cup (\bigcup_{i \in Z_m^*} (\mathcal{B}_i \cup \mathcal{C}_i));$$

$$(\mathcal{A} \cup \mathcal{D}) \cup (\bigcup_{i \in Z_m^*} \Omega_i).$$

Theorem 3.1 *The following designs exist:*

(1) $6q$-$HSCMD(12q)^m$, $m \geq 2$;

(2) $6q$-$SCMD(12q)$;

(3) $6q$-$HSCMD(h^1(12q)^1)$, $h \geq 3q$.

Proof By [6], the designs $2t$-$HSCMD((4t)^m)$ for $m \geq 2$, $2t$-$HSCMD(h^1(4t)^1)$ for $h \geq t$, and $2t$-$SCMD(4t)$ exist. We only need put $t = 3q$. In these self-converse designs given by [6], the mapping is same as the uniform mapping defined by us. □

By the above description and Theorem 3.1, in order to complete $6q$-$SCMD(v)$, we only need construct

(D), when $v \equiv 7q \pmod{12q}$;

(D) and (F), when $v \equiv q, q+1 \pmod{12q}$;

(E), when $v \equiv 7q + 1 \pmod{12q}$.

4 Notation and terminology

Consider the numbers and the differences in the set $Z_n \times Z_2$. In what follows, we will use the following notation and terminology, which was firstly introduced in [6].

(1) In $Z_n \times Z_2$, the number $(x, 0)$ is denoted by x_0 or x, the number $(x, 1)$ is denoted by x_1 or $\overline{x}$.

(2) The ordered pairs $(x_i, (x+d)_j)$ belong to the difference d_{ij}, where $x, d \in Z_n$, $i, j \in Z_2$ and $d \neq 0$ if $i = j$. A difference d_{ij} is said to be *pure* if $i = j$, or *mixed* if $i \neq j$. The difference d_{00} (or d_{11}) is called *0-pure* (or *1-pure*) and is denoted by d_0 (or d_1), respectively. Denote the set of all (pure and mixed) differences from $Z_n \times Z_2$ by $[Z_n \times Z_2]$.

(3) For integers a, b, k, $a < b$ and $k \geq 1$, $a \equiv b \pmod{k}$, define the integer intervals (as an ordered set under the natural ordering $<$):

$$[a, b]_k = (a, a+k, a+2k, \cdots, b),$$
$$[a, b]_k^{-1} = (b, b-k, \cdots, a+k, a).$$

The subscript k can be omitted when $k = 1$. For the numbers x_i in $Z_n \times Z_2$ and the differences d_{ij} in $[Z_n \times Z_2]$, the range of x and d is uniformly taken as $[-\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil]$, but $d \neq 0$ for the pure difference d_{ii}.

(4) Let $x_1, ..., x_m \in [Z_n \times Z_2]$. Call the ordered tuple $D = (x_1, ..., x_m)$ a *difference-tuple* on $[Z_n \times Z_2]$. If $x_1 = d_{ij}$, the corresponding number-tuple $(a_i, a_i + x_1, a_i + x_1 + x_2, ..., a_i + x_1 + ... + x_m)$ is denoted by $\widetilde{D}$ or $\widetilde{D}_a$, where $a \in Z_n$. Note that $b_i + d_{ij} = (b+d)_j$ and $b_i + d_{sj}$ is undefined if $s \neq i$, where $b, d \in Z_n$ and $i, j, s \in Z_2$. Usually, we write $a_i = head(\widetilde{D}_a)$, $a_i + x_1 + ... + x_m = tail(\widetilde{D}_a)$ and $\widetilde{D}_a = \widetilde{D}_0 + a$. For a difference-tuple D and corresponding number-tuple $\widetilde{D} = (y_0, y_1, ..., y_m)$, we have the unordered sets

$$\{\widetilde{D}\} = \{y_0, y_1, ..., y_m\}, \ \{\widetilde{D}\}_0 = \{x; (x, 0) \in \widetilde{D}\} \text{ and } \{\widetilde{D}\}_1 = \{x; (x, 1) \in \widetilde{D}\}.$$

For example, if $D = (2_{00}, (-1)_{01}, 2_{10}, (-4)_{00}, 1_{01}, 2_{11}, (-4)_{11})$ and $head(\widetilde{D}) = 0$, then $\widetilde{D} = (0, 2, \overline{1}, 3, -1, \overline{0}, \overline{2}, \overline{-2})$, $\{\widetilde{D}\}_0 = \{-1, 0, 2, 3\}$, $\{\widetilde{D}\}_1 = \{-2, 0, 1, 2\}$ and $tail(\widetilde{D}) = \overline{-2}$.

(5) Define two mappings $F(x)$ and $\overline{x}$ on $[Z_n \times Z_2]$ as follows.

$$F(d_{00}) = -d_{11}, \ F(d_{11}) = -d_{00}, \ F(d_{01}) = -d_{01}, \ F(d_{10}) = -d_{10},$$

$$\overline{d_{00}} = d_{11}, \ \overline{d_{11}} = d_{00}, \ \overline{d_{01}} = d_{10}, \ \overline{d_{10}} = d_{01},$$

where $-d_{ij} = (-d)_{ij}$. Let $D = (x_1, ..., x_m)$ be a difference-tuple. The following derived tuples are often useful:

$$-D = (-x_1, ..., -x_m), \quad D^{-1} = (x_m, ..., x_2, x_1),$$
$$F(D) = (F(x_1), F(x_2), ..., F(x_m)), \quad F^{-1}(D) = (F(D))^{-1}.$$

(6) Let $a, s \in Z_n$, $i, j, x, y \in Z_2$, $D = [a, a+s]$ be a difference-tuple on $[Z_n \times Z_2]$. Define

$$A_{ij}(D) = (a_{ij}, -(a+1)_{ji}, ..., (-1)^s (a+s)_{[ij]^s}),$$
$$-A_{ij}(D) = (-a_{ij}, (a+1)_{ji}, ..., (-1)^{s-1}(a+s)_{[ij]^s}) \text{ and}$$
$$A_{ij}(D^{-1}) = ((a+s)_{ij}, -(a+s-1)_{ji}, ..., (-1)^s a_{[ij]^s}),$$

where the subscript $[ij]^s$ denotes ij (if s even) or ji (if s odd). When $i = j$, these symbols are briefly denoted by $A_i(D), -A_i(D), A_i(D^{-1})$. As well, we define

$$MA_{ij}(D) = (a_{ij}, -(a+1)_{ji}, (a+2)_{ij}, ..., (-1)^s (a+s)_{[ij]^s},$$
$$(-1)^s (a+s)_{[ji]^s}, ..., -(a+1)_{ij}, a_{ji}).$$

Similarly, $MA_{ij}(D^{-1})$, $-MA_{ij}(D)$ can be defined also.

(7) A difference tuple $D = (x_1, ..., x_m)$ is called a *difference-path* on $[Z_n \times Z_2]$, denoted by $DP(D)$, if the following conditions are satisfied:

$$\begin{cases} \text{The numbers in } \widetilde{D}_0 \text{ are distinct;} \\ \text{If } x_s = d_{ij}, \text{ then } x_{s+1} = d'_{jk} \text{ for } 1 \le s \le m-1. \end{cases}$$

(8) A $DP(D) = (x_1, ..., x_m)$ is called a *difference-cycle* on $[Z_n \times Z_2]$, denoted by $DC(D)$, if the additional conditions are satisfied:

$$\begin{cases} d_1 + ... + d_m \equiv 0 \pmod n, \text{ where } x_s = (d_s)_{i_s j_s},\ d_s \in Z_n,\ i_s, j_s \in \{0,1\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{and } 1 \le s \le m; \\ \text{If } x_1 = d_{ij} \text{ then } x_m = d'_{si}. \end{cases}$$

A $DC(D)$ is said to be *complete*, denoted by $CDC(D)$, if the differences in D are distinct. A $CDC(D)$ corresponds to a block-orbit $dev(\widetilde{D}_0) = \{\widetilde{D}_0 + a; a \in Z_n\}$, which covers all ordered pairs $\{(a_i, a_i + d_{ij}); a \in Z_n, d_{ij} \in D\}$.

(9) In a $DC(d_1, ..., d_k)$, if $k = \lambda s$, $d_1, ..., d_s$ are distinct and $d_i = d_{i+s}, \forall 1 \le i \le k - s$, then this DC is called a λ-*partite* DC, denoted by λ-$DC(d_1, ..., d_s)$. It is not difficult to see that a difference-tuple $\lambda R = (d_1, ..., d_s, d_1, ..., d_s, ..., d_1, ..., d_s)$, where $R = (d_1, ..., d_s)$ is repeated λ times, forms a λ-*partite* DC if and only if $gcd(d_1 + ... + d_s, n) = \frac{n}{\lambda}$, $\lambda | n$ and $d_1, d_1 + d_2, ..., d_1 + d_2 + ... + d_s$ are not congruent modulo $\frac{n}{\lambda}$. A λ-$DC(d_1, ..., d_s)$ corresponds to a block-orbit $dev(\widetilde{N}_0) = \{\widetilde{N}_0 + a; 0 \le a \le \frac{n}{\lambda} - 1\}$. When $\lambda = 1$, the notation λ can be omitted and this DC is just a CDC.

(10) Let Q be a DP consisting of distinct differences in $[Z_n \times Z_2]$. If $Q \cap F(Q) = \emptyset, \widetilde{Q}_0 \cap f(\widetilde{Q}_0) = \emptyset$ and both head$(\widetilde{Q}_0)$ and tail$(\widetilde{Q}_0)$ belong to the same set $Z_n \times \{j\}$ for some $j \in Z_2$, then $(Q, 0_{j,1-j}, F^{-1}(Q), 0_{1-j,j})$ forms a *self-converse complete DC* on $[Z_n \times Z_2]$, which is denoted by $SDC(Q)$.

(11) Let N be a DP consisting of distinct differences in $[Z_n \times Z_2]$. If $N \cap F(N) = \emptyset$ and both $head(\widetilde{N})$ and $tail(\widetilde{N})$ belong to the same set $Z_n \times \{j\}$ for some $j \in Z_2$, then $\langle \infty, \widetilde{N} \rangle$ forms a complete block-orbit. The corresponding CDC is denoted by $CDC_\infty(N)$.

(12) Let N be a DP consisting of distinct differences in $[Z_n \times Z_2]$, let $a_1, a_2, \cdots, a_{h-1}$ be distinct numbers in $(Z_n \times Z_2) \backslash \{\widetilde{N}_0\}$, if $N \cap F(N) = \emptyset$, a_i and a_{h-i} belong to the same set $Z_n \times \{j\}$ for some $j \in Z_2$, then $\langle \infty_1, a_1, \infty_2, a_2, ..., \infty_{h-1}, a_{h-1}, \infty_h, \widetilde{N}_0 \rangle$ forms a complete block-orbit. The corresponding CDC is denoted by $CDC_{\infty_{1,...,h}}(N)$.

5 Some typical DP and DC

Lemma 5.1 ([6]) (1) *Let a, m, d, k be positive integers, $d < m$ and $a + km \le \frac{n}{2}$. If $N = [a, a + km]_k$ or $[a, a + km]_k \backslash \{a + kd\}$, then $\pm A_{ij}(N)$ is a DP on $[Z_n \times Z_2]$.*
(2) *Suppose Q and N are both DP on $Z_n \times Z_2$, n even and $a, b \in Z_n$. If the members of Q (resp. N) are distinct 0-pure differences, $\widetilde{Q}_0 \cap (\widetilde{Q}_0 + \frac{n}{2}) = \emptyset, \widetilde{N}_0 \cap (\widetilde{N}_0 + \frac{n}{2}) = \emptyset$ and $a + b + \sum_{d \in Q \cup N} d \equiv \frac{n}{2} \pmod n$, then $(Q, a_{01}, \overline{N}, b_{10}, Q, a_{01}, \overline{N}, b_{10})$ forms a 2-partite DC on $[Z_n \times Z_2]$, denoted briefly by 2-$(Q, a_{01}, \overline{N}, b_{10})$.*

(3) *Let n be an even integer, P be a DP consisting of distinct differences of $[Z_n \times Z_2]$. If $P \cap F(P) = \emptyset$, $(\frac{n}{2})_{01}, (\frac{n}{2})_{10} \notin P$, $head(\widetilde{P}_0)$, $tail(\widetilde{P}_0) \in Z_n \times \{0\}$ and $\widetilde{P}_0 \cap f(\widetilde{P}_0 + \frac{n}{2}) = \emptyset$, then*

(a) $(P, (\frac{n}{2})_{01}, F^{-1}(P), (\frac{n}{2})_{10})$ *forms a complete DC on* $[Z_n \times Z_2]$, *denoted by* $FDC(P, (\frac{n}{2})_{01})$.

(b) *Let* $N = (P, (\frac{n}{2})_{01}, F^{-1}(P), a_{10})$, $a_{10} \notin P \cup F(P)$ *and* $a \notin \{\widetilde{P}_0\}_1 \cup (\{\widetilde{P}_0\}_0 + \frac{n}{2})$. *Then, the blocks* $\{\langle \infty, \widetilde{N}_i \rangle : 0 \le i \le \frac{n}{2} - 1\}$ *and their f-converse cover the differences* $d \in P \cup F(P) \cup \{(\frac{n}{2})_{01}\}$ *and a half of the number pairs with differences* a_{10} *and* $-a_{10}$. *The tuple N is denoted by* $FDC_\infty(N)$.

(4) *Let* $N = (A_0[a, a+r], (-1)^{r-1}A_{01}([\frac{n}{2} - s - 1, \frac{n}{2} - 1]))$, $2a + r + 2s + 2 < n$, $|b| \neq |c| \in [1, \frac{n}{2} - s - 2]$, $b, b + c \notin \{\widetilde{N}\}_0 \cup (\{\widetilde{N}\}_1 + \frac{n}{2})$ *and* $c \notin \{\widetilde{N}\}_1 \cup (\{\widetilde{N}\}_0 + \frac{n}{2})$, *where* $head(\widetilde{N}) = 0$. *If* $r \equiv s \pmod 2$, *then* $P = (\frac{n}{2}, N, (\frac{n}{2})_{[10]^r}, F^{-1}(N), \overline{\frac{n}{2}}, b_{10})$ *and* $Q = (-c_{01}, \overline{N}, (\frac{n}{2})_{[01]^r}, F^{-1}(\overline{N}), c_{01}, b_{10})$ *form a pair of* FDC_∞, *where* $[10]^r = 01$ *if r odd or* 10 *if r even.*

Lemma 5.2 *Let* $\lambda|(k, n)$, $i, j, s \in Z_2, c \in Z_n$ *and* $\lceil \frac{k}{2\lambda} \rceil \le \frac{n}{\lambda}$. *Then for the following* R, λR *forms a λ-partite DC if* $tail(\widetilde{R}_0) = \pm\frac{n}{\lambda}$.

(1) $R = (A_{ij}[a, a + \frac{k}{\lambda} - 2], c_{xy})$;

(2) $R = (A_{ij}([a, a + \frac{k}{\lambda} - 1] \backslash \{a + t\}), c_{xy})$.

where the subscript $xy = js$ (*or* $xy = is$) *if* $\frac{k}{\lambda}$ *even* (*or odd*).

Proof (1) By Lemma 5.1(1), $P = A_{ij}[a, a + \frac{k}{\lambda} - 2]$ is a DP and $\{\widetilde{P}_0\}_i = [-(\lceil \frac{k}{2\lambda} \rceil - 1), 0]$, $\{\widetilde{P}_0\}_j = [a, a + \lfloor \frac{k}{2\lambda} \rfloor - 1]$, and they are intervals with length $\lceil \frac{k}{2\lambda} \rceil$ and $\lfloor \frac{k}{2\lambda} \rfloor$ respectively. It is easy to see that $\lfloor \frac{k}{2\lambda} \rfloor \le \lceil \frac{k}{2\lambda} \rceil \le \frac{n}{\lambda}$, so they are not congruent modulo $\frac{n}{\lambda}$, thus λR indeed forms a DC.
(2) By Lemma 5.1(1), $P = A_{ij}([a, a + \frac{k}{\lambda} - 1] \backslash \{a + t\})$ forms a DP, and

$$\{\widetilde{P}_0\}_i = \begin{cases} [-(\lceil \frac{k}{2\lambda} \rceil - 1), 0] & (t \text{ even}) \\ [-\lceil \frac{k}{2\lambda} \rceil, 0] \backslash \{-\frac{t+1}{2}\} & (t \text{ odd}), \end{cases}$$

$$\{\widetilde{P}_0\}_j = \begin{cases} [a, a + \lfloor \frac{k}{2\lambda} \rfloor] \backslash \{a + \frac{t}{2}\} & (t \text{ even}) \\ [a, a + \lfloor \frac{k}{2\lambda} \rfloor - 1] & (t \text{ odd}). \end{cases}$$

Thus, by the definition of λ-*partite DC* (in (9) of section 4), it is easy to see that λR forms a DC. □

Lemma 5.3 *Let* n, m, a, b, c, d *be positive integers,* n *even,* $1 < d < m < \frac{n}{2}$ *and* $1 < b, c, a + m < \frac{n}{2}$. *Then for the following* R, $2R$ *forms a 2-partite DC.*

(1) $R = (\pm A_0([1, m] \backslash \{d\}), a_{01}, \pm A_1[1, c], b_{10})$, *where*

$$a + b = \frac{n}{2} - (-1)^{m-1}(\lceil \frac{m}{2} \rceil + \epsilon_d) sgnA - (-1)^c \lceil \frac{c}{2} \rceil sgnA;$$

(2) $R = (\pm A_0[a, a + m], b_{01}, c_{10})$, *where*

$$b + c = \begin{cases} \frac{n}{2} + (-1)^{m-1} \lceil \frac{m}{2} \rceil sgnA & (m \text{ odd}) \\ \frac{n}{2} + (-1)^{m-1} \lceil a + \frac{m}{2} \rceil sgnA & (m \text{ even}); \end{cases}$$

(3) $R = (\pm A_0([a, a + m] \backslash \{a + d\}), b_{01}, c_{10})$, *where*

$$b+c=\frac{n}{2}+\begin{cases}(-1)^{m-1}(\lceil\frac{m}{2}\rceil+\epsilon_d+a)sgnA & (m\ odd)\\ (-1)^{m-1}(\lceil\frac{m}{2}\rceil+\epsilon_d)sgnA & (m\ even)\end{cases}\quad and$$

$\epsilon_d = 0$ (*if d even*) *or* $(-1)^m$ (*if d odd*).

Proof (1) By Lemma 5.1(1), both $Q = \pm A_0([1,m]\backslash\{d\})$ and $P = \pm A_0[1,c]$ are DP, $head(\widetilde{Q}_0) = 0$, $tail(\widetilde{Q}_0) = (-1)^m(\lceil\frac{m}{2}\rceil + \epsilon_d)sgnA$, and $\widetilde{Q}_0$ is contained in a interval with length $m+1$ ($\leq \frac{n}{2}$), thus $\widetilde{Q}_0 \cap (\widetilde{Q}_0 + \frac{n}{2}) = \emptyset$. As well, $head(\widetilde{P}_0) = 0$, $tail(\widetilde{P}_0) = (-1)^{c-1}\lceil\frac{c}{2}\rceil sgnA$ and $\widetilde{P}_0$ is contained in a interval with length $c+1$ ($\leq \frac{n}{2}$), thus $\widetilde{P}_0 \cap (\widetilde{P}_0 + \frac{n}{2}) = \emptyset$. By Lemma 5.1(2), for the given value of $a+b$, $2R$ forms a 2-*partite* DC.

(2) By Lemma 5.1(1), $Q = \pm A_0[a, a+m]$ forms a DP, $\widetilde{Q}$ is contained in a interval with length $m+1(\leq \frac{n}{2})$, thus $(\widetilde{Q}_0 + \frac{n}{2}) \cap \widetilde{Q}_0 = \emptyset$, $head(\widetilde{Q}_0) = 0$ and

$$tail(\widetilde{Q}_0)=\begin{cases}-(a+\frac{m}{2})sgnA & (m \text{ even})\\ \lceil\frac{m}{2}\rceil sgnA & (m \text{ odd}).\end{cases}$$

Thus, for the given value of $b+c$, $2R$ forms a 2-*partite* DC.

(3) By Lemma 5.1(1), $Q = \pm A_0([a, a+m]\backslash\{a+d\})$ forms a DP, $head(\widetilde{Q}_0) = 0$ and

$$tail(\widetilde{Q}_0)=\begin{cases}(-1)^{m-1}(\lceil\frac{m}{2}\rceil+\epsilon_d+a)sgnA & (m \text{ odd})\\ (-1)^{m-1}(\lceil\frac{m}{2}\rceil+\epsilon_d)sgnA & (m \text{ even}).\end{cases}$$

Thus, for the given value of $b+c$, $2R$ forms a 2-*partite* DC. □

Lemma 5.4 *Let n be odd, a be even and $a < \frac{n-3}{2}$. Then the following difference-tuple N forms a SDC:*

(1) $N = (-(\frac{n-a+1}{2})_0, A_0[1,a], MA_{01}[a+1, \frac{n-1}{2}], -A_0[1, a-1]^{-1})$;

(2) $N = (-(\frac{n-a+3}{2})_0, A_0[1,a], MA_{01}[a+1, \frac{n-1}{2}], -A_{01}[a-1,a]^{-1}, -A_0[1,a-3]^{-1})$.

Proof (1) Since $\{\widetilde{N}\}_0 = [-\frac{n-1}{2}, \frac{a}{2}] \cup [\frac{n-a+1}{2}, \frac{n-1}{2}]$, $\{\widetilde{N}\}_1 = [\frac{a}{2}+1, \frac{n-a-1}{2}]$, $head(\widetilde{N}) = (\frac{n-a+1}{2})_0$ and $tail(\widetilde{N}) = -(\frac{n-1}{2})_0$, N satisfies the conditions of SDC.

(2) Since $\{\widetilde{N}\}_0 = [-\frac{n-1}{2}, \frac{a}{2}] \cup [\frac{n-a+3}{2}, \frac{n-1}{2}]$, $\{\widetilde{N}\}_1 = [\frac{a}{2}+1, \frac{n-a-1}{2}]$, $head(\widetilde{N}) = (\frac{n-a+3}{2})_0$ and $tail(\widetilde{N}) = -(\frac{n-1}{2})_0$, N satisfies the conditions of SDC. □

Lemma 5.5 *Let n be odd, b be even and $b < \frac{n-3}{2}$, then*

$$D = (A_0[1, \tfrac{n-1}{2}], (-1)^{\frac{n-3}{2}} A_0[b, \tfrac{n-1}{2} - 1]^{-1})$$

forms a CDC_∞.

Proof Obviously D forms a DP and satisfies the conditions of CDC_∞, since

$$\widetilde{D}_0=\begin{cases}[-\frac{n-b-1}{2}, \frac{n-b+1}{2}]\backslash\{\frac{n+3}{4}\} & (\frac{n-1}{2} \text{ even}) & tail(\widetilde{D}_0) = -(\frac{n-b-1}{2})_0\\ [-\frac{n-b+1}{2}, \frac{n-b-1}{2}]\backslash\{-\frac{n+1}{4}\} & (\frac{n-1}{2} \text{ odd}) & tail(\widetilde{D}_0) = -(\frac{n-b+1}{2})_0.\end{cases}$$ □

Lemma 5.6 *Let a be even, n, t be odd, $1 < s < t < \frac{n-1}{2} - a$ and $\frac{n+1}{2} \leq h \leq n$, then* $N = (A_0([a, \frac{n-3}{2}]\backslash\{a+s\}), (-1)^{\frac{n-1}{2}} A_0([a+t, \frac{n-1}{2}]\backslash\{a+s\})^{-1}, (a+s)_0, (-1)^{\frac{n+1}{2}}(\frac{n-1}{2})_0)$ *forms a $CDC_{\infty_{1,\ldots,h}}(N)$.*

Proof From the following table, it is easy to see that N forms a DP.

$\frac{n-1}{2}$	s	$\{\widetilde{N}\}$
even	even	$([\frac{a}{2}, \frac{n-a-t-2}{2}]\backslash\{\frac{a+s}{2}\}) \cup ([-\frac{n-a-t+2}{2}, -\frac{a}{2}]\backslash\{-\frac{n-a-s+1}{2}, -\frac{n+1}{4}\})$
	odd	$([\frac{a}{2}, \frac{n-a-t-2}{2}]\backslash\{\frac{n-a-s-2}{2}\}) \cup ([-\frac{n-a-t+2}{2}, -\frac{a}{2}]\backslash\{-\frac{s+a}{2}, -\frac{n+3}{4}\})$
odd	even	$([\frac{a}{2}, \frac{n-a-t}{2}]\backslash\{\frac{a+s}{2}, \frac{n+1}{4}\}) \cup ([-\frac{n-a-t}{2}, -\frac{a}{2}]\backslash\{-\frac{n-a-s-1}{2}\})$
	odd	$([\frac{a}{2}, \frac{n-a-t}{2}]\backslash\{\frac{n-a-s}{2}, \frac{n-3}{4}\}) \cup ([-\frac{n-a-t}{2}, -\frac{a}{2}]\backslash\{-\frac{s+a}{2}\})$

where $head(\widetilde{N}) = -(\frac{a}{2})_0$ and

$$tail(\widetilde{N})=\begin{cases} (-1)^{\frac{n-3}{2}}(\frac{n-3}{2})_0 & (s \text{ even}) \\ (-1)^{\frac{n-3}{2}}(\frac{n-1}{2})_0 & (s \text{ odd}). \end{cases}$$

Finally, in order to construct $CDC_{\infty_{1,\ldots,h}}(N)$, we take $\langle \infty_1, 1_1, \infty_2, 2_1, \ldots, \infty_{h-1}, (h-1)_1, \infty_h, \widetilde{N}_0 \rangle$ as the base block of a corresponding block-orbit. □

6 Constructions of $SCMD$

In all constructions of this section, we will use the notation DC of various kinds: $\lambda R, FDC, FDC_\infty, SDC, CDC, CDC_\infty, CDC_{\infty_{1,\ldots,h}}$ defined in the above section. Each DC represents one (or $\frac{1}{\lambda}$) block-orbit and their f-converse (except SDC, which is self-converse). Therefore, each DC will correspond to the following number of blocks from $Z_n \times Z_2$:

λR	FDC	FDC_∞	SDC	CDC	CDC_∞	$CDC_{\infty_{1,\ldots,h}}$
$\frac{n}{\lambda} \times 2$	$\frac{n}{2} \times 2$	$\frac{n}{2} \times 2$	$n \times 1$	$n \times 2$	$n \times 2$	$n \times 2$

Theorem 6.1 *There exists a* $6q$-$SCMD(14q+1)$ *for* $q \equiv 1 \pmod 6$.
Construction Let $q = 6t+1$ and $t \geq 1$. Construct a $(36t+6)$-$SCMD(84t+15) = (X, \mathcal{B})$ as follows. The point set is $X = (Z_{42t+7} \times Z_2) \cup \{\infty\}$, the block set $\mathcal{B}$ consists of three parts:

(I) $(6t+1)$-*partite DC*: $(6t+1)$-$DC(A_{01}(R_i))$, $0 \leq i \leq \lfloor \frac{5t}{2} \rfloor$,
$(6t+1)$-$DC(A_{10}(R_j))$, $0 \leq j \leq \lceil \frac{5t}{2} - 1 \rceil$,

where $R_0 = \{1, 2, 3, 4, 5, 10\}$, $R_i = \{6i, 6i+1, 6i+2, 6i+3, 6i+5, 6i+10\}$, $i \geq 1$;

(II) $CDC_\infty(A_0[1, 21t+3], (-1)^t A_0[6t+2, 21t+2]^{-1})$;

(III) $SDC((-1)^{t-1}(21t+3)_0, -A_0[1, 6t+1], P)$, where $P = MA_{01}([15t+3, 21t+3]\backslash\{15t+7\})$ for odd t or $(A_{01}([15t+1, 15t+5]\backslash\{15t+4\}),\ MA_{01}([15t+6, 21t+3]\backslash\{15t+10\}), (15t+10)_{01}, (15t)_{10})$ for even t.

Proof The number of $6q$-blocks in part (I)-(III) is $(42t+7) \times (2+1) + (5t+1) \times 7 \times 2 = 196t+35$, as expected. It is not difficult to see that all differences are contained in (I)-(III) exactly once. As for the correctness of each part, we can show it as follows.

(I) By Lemma 5.2 ($\lceil \frac{36t+6}{12t+2} \rceil \leq \frac{42t+7}{6t+1}$).

(II) By Lemma 5.5.

(III) We only verify the case: t even. Denote $R = -A_0[1, 6t+1]$ and $M = (A_{01}([15t+1, 15t+5]\backslash\{15t+4\}), MA_{01}([15t+6, 21t+3]\backslash\{15t+10\}))$. Then

$\{\widetilde{R}_0\}_0$=$[-(3t+1),3t]$, $tail(\widetilde{R}_0)=c=-(3t+1)_0$,
$\{\widetilde{M_c}\}_0$=$[-(9t+1),-(3t+2)]\backslash\{-(3t+3)\}$,
$\{\widetilde{M_c}\}_1$=$[12t,36t]\backslash\{12t+4,18t-2\}$,
$tail(\widetilde{M_c})$=$-(9t+1)_0$.

Furthermore, it is easy to see that the last two values of $\widetilde{P}_c$ are $(6t+9)_1$ and $-(21t-2)_0$. Obviously, the first number of the corresponding number-tuple of (III) is $(21t+3)_0$. By the list value, it is not difficult to see that the difference-tuple given by (III) satisfies the conditions of an SDC. □

Theorem 6.2 *There exists a* $6q$-$SCMD(10q)$ *for* $q\equiv 1\ (mod\ 6)$.

Construction

(Case 1) Let $q=12t+1$ and $t\geq 1$. Construct a $(72t+6)$-$SCMD(120t+10)=(X,\mathcal{B})$ as follows. The point set is $X=Z_{60t+5}\times Z_2$, the block set $\mathcal{B}$ consists of three parts:

(I) $(12t+1)$-*partite DC*: $(12t+1)$-$DC(A_{01}(R_i))$,
$(12t+1)$-$DC(A_{10}(R_i))$, $0\leq i\leq t-1$,

where $R_0=\{1,2,3,4,5,8\}$, $R_i=\{6i,6i+1,6i+3,6i+4,6i+5,6i+8\}$, $i\geq 1$;

(II) $CDC(P)$, where $P=(MA_{01}([6t,30t+2]\backslash\{6t+2\},(30t+2)_0,A_0[1,24t]^{-1},$
$-(18t)_0)$;

(III) $SDC(Q)$, where $Q=(A_0[18t+1,30t+2],-A_0[24t+1,30t+1]^{-1},$
$A_0[1,18t-1]^{-1})$.

(Case 2) Let $q=12t+7$ and $t\geq 0$. Construct a $(72t+42)$-$SCMD(120t+70)=(X,\mathcal{B})$ as follows.

When $t=0$, a 42-$SCMD(70)$ is given in Appendix 1. Below, suppose $t>0$. The point set is $X=Z_{60t+35}\times Z_2$, the block set $\mathcal{B}$ consists of three parts:

(I) $(12t+7)$-*partite DC*: $(12t+7)$-$DC(A_{01}(R_i))$, $0\leq i\leq t$,
$(12t+7)$-$DC(A_{10}(R_j))$, $0\leq j\leq t-1$,

where $R_0=\{1,2,3,4,5,8\}$, $R_i=\{6i,6i+1,6i+3,6i+4,6i+5,6i+8\}$, $i\geq 1$;

(II) $CDC(MA_{01}[6t+9,30t+17],-A_{01}([6t,6t+8]\backslash\{6t+2\})^{-1},(30t+17)_0,$
$A_0[2,24t+15]^{-1},-(18t+11)_0)$;

(III) $SDC(-A_0([1,30t+16]\backslash[18t+11,24t+15]),-A_0[18t+12,30t+17]^{-1},$
$-(6t+7)_{01},-(6t+6)_{10},1_0)$.

Proof We only verify case 1. The number of blocks is $(60t+5)\times 3+2t\times 5\times 2=200t+15$, as expected. The correctness of each orbit is shown as follows.

(I) By Lemma 5.2 ($\lceil\frac{72t+6}{24t+2}\rceil\leq\frac{60t+5}{12t+1}$).

(II) Since $\{\widetilde{P}\}_0=[-(24t+2),0]\cup[6t,30t]$, $\{\widetilde{P}\}_1=\{-(30t+2)\}\cup([6t,30t+1]\backslash\{6t+1\}$ and $head(\widetilde{P})=tail(\widetilde{P})=0_0$.

(III) Since $\widetilde{Q}_0=[-(30t+2),-(15t+3)]\cup[-(9t+1),0]\cup[18t+1,30t+2]$ and $tail(\widetilde{Q}_0)=-(24t+2)_0$. □

Theorem 6.3 *There exists a* $6q$-$SCMD(11q+1)$ *for* $q\equiv 1\ (mod\ 6)$.

Construction

(Case 1) Let $q=12t+7$ and $t\geq 0$. Construct a $(72t+42)$-$SCMD(132t+78)=(X,\mathcal{B})$ as follows.

When $t=0$, a 42-$SCMD(78)$ is given in Appendix 2. Below, suppose $t>0$. The point set is $X=Z_{66t+39}\times Z_2$, the block set $\mathcal{B}$ consists of three parts:

(I) 3-*partite DC*: $3\text{-}DC(A_{01}[1,24t+13],(32t+19)_{10})$;
(II) $CDC(M)$, where $M=(A_0[1,a],A_{01}([1,a]\backslash\{24t+13\})^{-1},$
$A_{01}[32t+20,33t+18]^{-1},-(24t+13)_{10},A_{01}[b,32t+18]^{-1},(22t+12)_0)$;
(III) $SDC(N)$, where $N=(P,A_{01}[24t+14,b-1],-(33t+19)_{10},$
$-A_0[22t+13,33t+19]^{-1})$,

$$(a,b)=\begin{cases}(33t+19,27t+15) & t \text{ even}\\(33t+18,27t+14) & t \text{ odd}\end{cases} \text{ and}$$

$$P=\begin{cases}-A_0[1,22t+11] & t \text{ even}\\((33t+19)_0,-A_0[1,22t+11]) & t \text{ odd}\end{cases}.$$

(Case 2) Let $q=12t+1$ and $t\geq 1$. Construct a $(72t+6)\text{-}SCMD(132t+12)=(X,\mathcal{B})$ as follows. The point set is $X=Z_{66t+6}\times Z_2$, the block set $\mathcal{B}$ consists of four parts:
(I) 6-*partite DC*: $6\text{-}DC(A_{01}[1,12t],-(5t+1)_0)$,
$6\text{-}DC(A_{10}[1,12t],-(5t+1)_1)$;
(II) 2-*partite DC*: $2\text{-}DC((-1)^{t-1}A_0([1,33t+2]\backslash\{5t+1\}),(12t+1)_{01},$
$(-1)^{t-1}A_1[1,3t],-c_{10})$;
(III) $FDC(MA_{01}([15t+1,33t+2]\backslash\{c\})^{-1},(33t+3)_{01})$;
(IV) $SDC(Q)$, where $Q=(-A_0[5t+2,33t+3]^{-1},c_{01},A_{10}[d,15t]^{-1},$
$-A_0[3t+1,5t]^{-1},-A_{01}[d+(-1)^{t-1},15t]^{-1})$ and

$$(c,d)=\begin{cases}(30t+2,12t+2) & t \text{ even}\\(30t+3,12t+1) & t \text{ odd}\end{cases}.$$

Proof The number of blocks is

$$\begin{cases}(22t+13)\times 2+(66t+39)\times(2+1)=242t+143 & \text{(case 1)}\\(11t+1)\times(2+2)+(66t+6)\times(1+1+1)=242t+22 & \text{(case 2)}\end{cases},$$

as expected. For case 1, the correctness of each orbit is shown as follows.
(I) By Lemma 5.2(1) ($\lceil\frac{72t+42}{6}\rceil\leq\frac{66t+39}{3}$).
(II) We only verify the case: t even. Since $\{\widetilde{M}\}_0=[-(\frac{33}{2}t+9),33t+19]\cup[-(33t+19),-(\frac{65}{2}t+21)]\cup[-(\frac{49}{2}t+14),-(22t+12)]$, $\{\widetilde{M}\}_1=[5t+3,\frac{15}{2}t+4]\cup[-(\frac{t}{2}+1),-2]\cup[-(33t+19),-(\frac{33}{2}t+10)]\backslash\{-(21t+13)\}$) and $head(\widetilde{M})=tail(\widetilde{M})=0_0$.
(III) We only verify the case: t even. Since $\{\widetilde{N_0}\}_0=[-(\frac{25}{2}t+6),11t+5]\cup[-(24t+14),-(\frac{37}{2}t+11)]\cup[\frac{29}{2}t+9,20t+12]$, $\{\widetilde{N_0}\}_1=[13+8,\frac{29}{2}t+8]$ and $tail(\widetilde{N_0})=(20t+12)_0$.
As for case 2, the correctness of each orbit is shown as follows.
(I) By Lemma 5.2(1) ($\lceil\frac{72t+6}{12}\rceil\leq\frac{66t+6}{6}$)).
(II) By Lemma 5.3(1).
(III) We only verify the case: t even. Let $R=MA_{01}([15t+1,33t+2]\backslash\{30t+2\})^{-1}$, then $\{\widetilde{R_0}\}_0=[-(27t+4),-(18t+4)]\cup[0,9t]$, $\{\widetilde{R_0}\}_1=[15t,33t+2]\backslash\{\frac{33}{2}t,\frac{63}{2}t+2\}$ and $tail(\widetilde{R_0})=-(18t+4)_0$. We see that $R\cap F(R)=\emptyset$, $\{\widetilde{R_0}\}\cap f(\{\widetilde{R_0}\}+33t+3)=\emptyset$.
(IV) We only verify the case: t even. Since $\{\widetilde{Q_0}\}_0=[-(33t+3),-(19t+3)]\cup[-(14t+1),0]\cup[\frac{49}{2}t+2,\frac{51}{2}t+1]\cup[27t+2,\frac{59}{2}t+2]$, $\{\widetilde{Q_0}\}_1=[\frac{27}{2}t+2,15+1]\cup[16t+1,\frac{35}{2}t]$ and $tail(\widetilde{Q_0})=(27t+2)_0$. □

Theorem 6.4 *There exists a $6q$-$SCMD(8q)$ for $q \equiv 5 \pmod 6$.*
Construction Let $q = 6t+5$ and $t \geq 0$. Construct a $(36t+30)$-$SCMD(48t+40) = (X, \mathcal{B})$ as follows. When $t = 0$, a 30-$SCMD(40)$ is given in Appendix 3. Below, suppose $t > 0$. The point set is $X = Z_{24t+20} \times Z_2$, the block set $\mathcal{B}$ consists of four parts:

(I) $(6t+5)$-*partite DC*: $(6t+5)\text{-}DC(A_{01}(R_i)),\ 0 \leq i \leq \lceil \frac{t}{2} \rceil,$
$(6t+5)\text{-}DC(A_{10}(R_i)),\ 0 \leq i \leq \lfloor \frac{t}{2} \rfloor - 1,$
$(6t+5)\text{-}DC(A_{01}(S_j)),\ 0 \leq j \leq \lfloor \frac{t}{2} \rfloor,$
$(6t+5)\text{-}DC(A_{10}(S_j)),\ 0 \leq j \leq \lceil \frac{t}{2} \rceil - 1,$

where $S_j = [12j+1, 12j+7] \backslash \{12j+6\}$, $R_i = [12i+6, 12i+12] \backslash \{12i+7\}$;

(II) $(12t+10)$-*partite DC*: $(12t+10)\text{-}DC(P, -1_0)$;

(III) $FDC(M, (12t+10)_{01})$, where $M = (A_{01}[6t+7, 12t+9]^{-1}, (12t+9)_{10}, -A_0[1, 12t+10]^{-1})$;

(IV) $SDC(N)$, where $N = (A_0[2, 12t+9], Q)$ and

$$(P, Q) = \begin{cases} (-A_{01}[6t+5, 6t+6]^{-1}, -A_{01}([6t+1, 12t+8] \backslash \{6t+5, 6t+7\})^{-1}) & t \text{ even} \\ (A_{01}[6t+2, 6t+3], (-A_{01}[6t+4, 12t+8]^{-1}, (6t)_{10})) & t \text{ odd} \end{cases}.$$

Proof The number of blocks is $(2t+1) \times 4 \times 2 + 2 \times 2 + (24t+20) \times 2 = 64t+52$, as expected. The correctness of each orbit is shown as follows.

(I) By Lemma 5.2 ($\lceil \frac{36t+30}{12t+10} \rceil \leq \frac{24t+20}{6t+5}$).

(II) By Lemma 5.2(1) ($\lceil \frac{36t+30}{24t+20} \rceil \leq \frac{24t+20}{12t+10}$).

(III) Since $\{\widetilde{M}_0\}_0 = [0, 3t+1] \cup [9t+7, 12t+10] \cup [-(12t+9), -(3t+3)]$, $\{\widetilde{M}_0\}_1 = [9t+8, 12t+9]$ and $tail(\widetilde{M}) = -(9t+8)_0$, we have $M \cap F(M) = \emptyset$, $\{\widetilde{M}_0\} \cap f(\{\widetilde{M}_0\} + 12t+10) = \emptyset$.

(IV) We only verify the case: t even. Since $\{\widetilde{N}_0\}_0 = [-(9t+8), 6t+5] \backslash \{-(9t+5), 1\}$, $\{\widetilde{N}_0\}_1 = [6t+8, 9t+11] \backslash \{9t+9\}$ and $tail\{\widetilde{N}_0\} = -(9t+8)_0$. □

Theorem 6.5 *There exists a $6q$-$SCMD(10q+1)$ for $q \equiv 5 \pmod 6$.*
Construction Let $q = 6t+5$ and $t \geq 0$. Construct a $(36t+30)$-$SCMD(60t+51) = (X, \mathcal{B})$ as follows. When $t = 0$, a 30-$SCMD(51)$ is given in Appendix 4. Below, suppose $t > 0$. The point set is $X = (Z_{30t+25} \times Z_2) \cup \{\infty\}$, the block set $\mathcal{B}$ consists of three parts:

(I) $6t+5$-*partite DC*: $(6t+5)\text{-}DC(A_{01}(R_i)),\ 0 \leq i \leq \lfloor \frac{t}{2} \rfloor,$
$(6t+5)\text{-}DC(A_{10}(R_j)),\ 0 \leq j \leq \lceil \frac{t}{2} \rceil - 1),$

where $R_0 = \{1, 2, 3, 4, 5, 8\}$, $R_i = \{6i, 6i+1, 6i+3, 6i+4, 6i+5, 6i+8\}$, $i \geq 1$;

(II) $SDC(M)$, where $M = ((-1)^{t-1}A_0[1, 15t+12], a_{01}, b_{10}, (-1)^t A_0[1, 3t])$;

(III) $CDC_\infty(N)$, where $N = (MA_{01}[c, 15t+12], P, -A_0[3t+1, 15t+12]^{-1})$,

$$(a, b, c) = \begin{cases} (3t+7, 3t+6, 3t+9) & t \text{ even} \\ (3t+3, 3t+4, 3t+6) & t \text{ odd} \end{cases}, \text{ and}$$

$$P = \begin{cases} -A_{01}([3t, 3t+4] \backslash \{3t+2\})^{-1} & t \text{ even} \\ -A_{01}[3t+3, 3t+4]^{-1} & t \text{ odd} \end{cases}.$$

Proof The number of blocks is $(t+1) \times 5 \times 2 + (30t+25) \times (2+1) = 100t+85$, as expected. The correctness of each orbit is shown as follows.

(I) By Lemma 5.2 ($\lceil\frac{36t+30}{12t+10}\rceil \le \frac{30t+25}{6t+5}$).

(II) We only verify the case: t even. Since $\{\widetilde{M}_0\}_0 = [-(\frac{15}{2}t+6), \frac{15}{2}t+6] \cup [12t+19, 15t+19]$, $\{\widetilde{M}_0\}_1 = \{\frac{21}{2}t+13\}$ and $tail\{\widetilde{M}_0\} = (12t+19)_0$.

(III) We only verify the case: t even. Since $\{\widetilde{N}_0\}_0 = [-(15t+12), 0] \cup [3t+5, 9t+10] \cup [12t+11, 15t+12]$, $\{\widetilde{N}_0\}_1 = [3t+9, 15t+12] \cup ([-(15t+12), -(15t+8)] \backslash \{-(15t+9)\}$ and $tail(\widetilde{N}_0) = (12t+11)_0$. □

Theorem 6.6 *There exists a $6q$-$SCMD(11q)$ for $q \equiv 5 \pmod 6$.*

Construction

(Case 1) Let $q = 12t+5$ and $t \ge 0$. Construct a $(72t+30)$-$SCMD(132t+55) = (X, \mathcal{B})$ as follows.

When $t = 0$, a 30-$SCMD(55)$ is given in Appendix 5. Below, suppose $t > 0$. The point set is $X = (Z_{66t+27} \times Z_2) \cup \{\infty\}$, the block set $\mathcal{B}$ consists of three parts:

(I) 3-*partite DC*: $3\text{-}DC(A_{01}[1, 24t+9], (32t+13)_{10})$;

(II) $SDC(M)$, where $M = (P, (-1)^{t-1}(32t+13)_{01}, Q)$;

(III) $CDC_\infty(N)$, where $N = (A_0[1, a], MA_{01}([24t+10, 33t+13] \backslash \{32t+13\}), A_{01}[b, 24t+9])$,

$$(a, b) = \begin{cases} (33t+12, 3t) & t \text{ even} \\ (32t+13, 3t+1) & t \text{ odd} \end{cases} \quad \text{and}$$

$$(P, Q) = \begin{cases} ((33t+13)_0, -A_0[1, 33t+13]), A_0[1, 3t-1]^{-1}) & t \text{ even} \\ (-A_0[1, 33t+13], -A_0[1, 3t]^{-1}) & t \text{ odd} \end{cases}.$$

(Case 2) Let $q = 12t+11$ and $t \ge 0$. Construct a $(72t+66)$-$SCMD(132t+121) = (X, \mathcal{B})$ as follows. The point set is $X = (Z_{66t+60} \times Z_2) \cup \{\infty\}$, the block set $\mathcal{B}$ consists of three parts:

(I) 6-*partite DC*: $6\text{-}DC(A_{01}[2, 12t+11], -(5t+5)_0)$,
$6\text{-}DC(A_{10}[2, 12t+11], -(5t+5)_1)$;

(II) $SDC(R)$, where $R = (A_{01}[12t+12, 30t+26], 1_{10}, A_{01}[12t+12, 30t+27]^{-1})$;

(III) $FDC_\infty((33t+30)_0, N, (33t+30)_{[01]^t}, F^{-1}(N), (33t+30)_1, (30t+27)_{10})$,
$FDC_\infty(-1_{01}, \overline{N}, (33t+30)_{[10]^t}, F^{-1}(\overline{N}), 1_{01}, (30t+27)_{10})$, where

$N = (A_0([1, 33t+29] \backslash \{5t+5\}), (-1)^t A_{01}[30t+28, 33t+29])$, $[01]^t$=01 (t even) or 10 (t odd).

Proof The number of blocks is

$$\begin{cases} (22t+9) \times 2 + (66t+27) \times (2+1) = 242t+99 & \text{(case 1)} \\ (11t+10) \times 4 + (66t+60) \times (2+1) = 242t+220 & \text{(case 2)} \end{cases},$$

as expected. For case 1, the correctness of each orbit is shown as follows.

(I) By Lemma 5.2(1) ($\lceil\frac{72t+30}{6}\rceil \le \frac{66t+27}{3}$).

(II) We only verify the case: t even. Since $\{\widetilde{M}\}_0 = [-(\frac{33}{2}t+7), \frac{33}{2}t+6] \cup [19t+7, \frac{41}{2}t+6] \cup \{-(33t+13)\}$, $\{\widetilde{M}\}_1 = [\frac{35}{2}t+7, 19t+6]$, $head(\widetilde{M}) = -(33t+13)_0$ and $tail(\widetilde{M}) = (19t+7)_0$.

(III) We only verify the case: t even. Since $\{\widetilde{N}_0\}_0 = ([-(33t+13), \frac{33}{2}t+6] \backslash \{-(\frac{41}{2}t+8)\}) \cup [30t+11, 33t+13]$, $\{\widetilde{N}_0\}_1 = [\frac{15}{2}t+4, \frac{33}{2}t+6] \cup [-(\frac{45}{2}t+11), -(12t+7)]$

and $tail(\widetilde{N}_0) = (30t+11)_0$.
As for case 2, the correctness of each orbit is shown as follows.

(I) By Lemma 5.2(1) ($\lceil \frac{72t+66}{12} \rceil \leq \frac{66t+60}{6}$).

(II) Since $\{\widetilde{R}_0\}_0 = [-(9t+7), 0] \cup [21t+20, 30t+28]$, $\{\widetilde{R}_0\}_1 = [-(24t+20), -(15t+13)] \cup [12t+12, 21t+19]$, and $tail(\widetilde{R}_0) = (30t+28)_0$.

(III) By Lemma 5.1(4). □

Theorem 6.7 *There exists a $6q$-$SCMD(7q+1)$ for $q \equiv 5 \pmod 6$.*

Construction

(Case 1) Let $q = 12t+5$ and $t \geq 0$. Construct a $(72t+30)$-$SCMD(84t+36) = (X, \mathcal{B})$ as follows.
When $t = 0$, a 30-$SCMD(36)$ is given in Appendix 6. Below, suppose $t > 0$. The point set is $X = Z_{42t+18} \times Z_2$, the block set $\mathcal{B}$ consists of three parts:

(I) 6-*partite DC*: 6-$DC(A_{01}[9t+5, 21t+8], -(t+1)_0)$;

(II) $FDC(N, (21t+9)_{01})$, where $N = (A_0[t+1, 21t+9], P, A_0[1,t]^{-1})$;

(III) $SDC(M)$, where $M = (Q, A_{01}([1, 21t+8] \backslash [9t+5, 15t+5])^{-1})$,

$$P = \begin{cases} (-A_{01}[2, 15t+5], 1_0) & t \text{ even} \\ -A_{01}[1, 15t+5] & t \text{ odd} \end{cases} \quad \text{and}$$

$$Q = \begin{cases} (1_{10}, -A_0([2, 21t+8] \backslash \{t+1\}) & t \text{ even} \\ -A_0([1, 21t+8 \backslash \{t+1\}) & t \text{ odd} \end{cases}.$$

(Case 2) Let $q = 12t+11$ and $t \geq 0$. Construct a $(72t+66)$-$SCMD(84t+78) = (X, \mathcal{B})$ as follows. The point set is $X = Z_{42t+39} \times Z_2$, the block set $\mathcal{B}$ consists of three parts:

(I) 3-*partite DC*: 3-$DC(A_{01}[2,a], -A_0[1,b]^{-1}, -(16t+15)_0)$,
3-$DC(A_{10}[2,a], -A_1[1,b]^{-1}, -(16t+15)_1)$;

(II) $SDC(D)$, where $D = (A_0[3t+4, 21t+19] \backslash \{16t+15\}), c_{01}$,
$A_1[b+1, 16t+14]^{-1}, -c_{10}, (-1)^{t-1} A_0[16t+16, 21t+19]^{-1}, R)$,

$$(a,b,c) = \begin{cases} (21t+19, 3t+3, 1) & t \text{ even} \\ (21t+18, 3t+4, -(42t+40)) & t \text{ odd} \end{cases} \quad \text{and}$$

$$R = \begin{cases} \emptyset & t \text{ even} \\ (-1_{01}, -1_{10}) & t \text{ odd} \end{cases}.$$

Proof The number of blocks is

$$\begin{cases} (7t+3) \times 2 + (42t+18) \times 2 = 98t+42 & \text{(case 1)} \\ (14t+13) \times 2 \times 2 + (42t+39) = 98t+91 & \text{(case 2)} \end{cases},$$

as expected. For case 1, the correctness of each orbit is shown as follows.

(I) By Lemma 5.2(1) ($\lceil \frac{72t+30}{12} \rceil \leq \frac{42t+18}{6}$).

(II) We only verify the case: t even. Since $\{\widetilde{N}_0\}_0 = [-(10t+4), 0] \cup [t+1, \frac{39}{2}t+8]$, $\{\widetilde{N}_0\}_1 = [\frac{7}{2}t+2, 11t+3]$ and $tail(\widetilde{N}_0) = (19t+8)_0$.

(III) We only verify the case: t even. Since $\{\widetilde{M}\}_0 = ([-(\frac{21}{2}t+4), 21t+9] \backslash \{-1, \frac{t}{2}\}) \cup [-(21t+8), -(18t+10)]$, $\{\widetilde{M}\}_1 = [-(18t+9), -(\frac{21}{2}t+6)] \cup \{-1\}$,

$head(\widetilde{M}) = -1_1$ and $tail(\widetilde{M}) = -(18t+9)_1$.
As for case 2, the correctness of each orbit is shown as follows.

(I) Let $S = (A_{01}[2, 21t+19], -A_0[1, 3t+3]^{-1})$, then $\{\widetilde{S}\}_0 = [-\frac{27}{2}t+12), 0]$, $\{\widetilde{S}\}_1 = [2, \frac{21}{2}t+10]$. Obviously they are not congruent modulo $14t+13$ and $tail(\widetilde{S}) - (16t+15) = 14t+13$.

(II) We only verify the case: t even. Since $\{\widetilde{D_0}\}_0 = [-(9t+8), 0] \cup ([\frac{t}{2}+1, 12t+11]\backslash\{3t+3\}) \cup [19t+18, 21t+19] \cup [-(21t+19), -(\frac{41}{2}t+19]$, $\{\widetilde{D_0}\}_1 = [-(\frac{41}{2}t+18), -(14t+13)] \cup [12t+12, \frac{37}{2}+17]$, and $tail(\widetilde{D_0}) = (19t+18)_0$. □

Theorem 6.8 *There exists a* $6q$-$SCMD(13q+1)$ *for* $q \equiv 5 \ (mod\ 6)$.

Construction

(Case 1) Let $q = 12t+5$ and $t \geq 0$. Construct a $(72t+30)$-$SCMD(156t+66) = (X, \mathcal{B})$ as follows.

When $t = 0$, a 30-$SCMD(66)$ is given in Appendix 7. Below, suppose $t > 0$. The point set is $X = Z_{78t+33} \times Z_2$, the block set $\mathcal{B}$ consists of three parts:

(I) 3-*partite DC*: 3-$DC(A_{01}[1, 24t+9], -(38t+16)_{10})$,
3-$DC(A_{10}[1, 24t+9], -(38t+16)_{01})$;

(II) $SDC(M)$, where $M = (A_0[a, 39t+14]^{-1}, -b_0, -(39t+15)_0,$
$A_0[3t+3, a-2]^{-1}, 1_0)$;

(III) $CDC(N)$, where $N = (MA_{01}([24t+10, 39t+16]\backslash\{38t+16\}),$
$-A_0[2, 39t+16]^{-1}, (39t+16)_0, A_0[3, 3t+2]^{-1}, c_0, -(a-1)_0)$ and

$$(a, b, c) = \begin{cases} (6t+4, 1, 2) & t \text{ even} \\ (6t+5, 2, -1) & t \text{ odd} \end{cases}.$$

(Case 2) Let $q = 12t+11$ and $t \geq 0$. Construct a $(72t+66)$-$SCMD(156t+144) = (X, \mathcal{B})$ as follows. The point set is $X = Z_{78t+72} \times Z_2$, the block set $\mathcal{B}$ consists of four parts:

(I) 6-*partite DC*: 6-$DC(A_{01}[2, 12t+11], -(7t+7)_0)$;

(II) 2-*partite DC*: 2-$DC(A_0([3t+4, 39t+35]\backslash\{7t+7\}), a_{01}, b_{10})$,
2-$DC(-A_0([3t+4, 39t+35]\backslash\{7t+7\}), -b_{01}, -a_{10})$;

(III) $FDC(P, (39t+36)_{01})$, where $P = (MA_{01}[21t+20, 39t+35]^{-1})$;

(IV) $SDC(Q)$, where $Q = (A_0[1, 3t+3], -MA_{01}([12t+12, 21t+19]\backslash\{18t+16\}),$
$A_{01}[2, 12t+11]^{-1}, (39t+36)_0, (7t+7)_0, -A_0[1, 3t+3]^{-1})$ and

$$(a, b) = \begin{cases} (1, 18t+16) & t \text{ even} \\ (18t+16, 1) & t \text{ odd} \end{cases}.$$

Proof The number of blocks is

$$\begin{cases} (26t+11) \times 2 \times 2 + (78t+33) \times (2+1) = 338t+143 & \text{(case 1)} \\ (13t+12) \times 2 + (78t+72) \times (2+1+1) = 338t+312 & \text{(case 2)} \end{cases},$$

as expected. For case 1, the correctness of each orbit is shown as follows.

(I) By Lemma 5.2(1) ($\lceil \frac{72t+30}{6} \rceil \leq \frac{78t+33}{3}$).

(II) We only verify the case: t even. Since $\widetilde{N_0} = [0, \frac{33}{2}t+5] \cup [\frac{45}{2}t+8, 39t+14] \cup [-(\frac{33}{2}t+7), -(15t+6)] \cup [-(12t+4), -(\frac{21}{2}t+5)]$ and $tail(\widetilde{N_0}) = -(15t+6)_0$.

(III) We only verify the case: t even. Since $\{\widetilde{M}\}_0 = ([-(39t+16), 0]\backslash\{-(\frac{69}{2}t+14)\}) \cup [24t+11, 39t+16] \cup ([\frac{9}{2}t+1, \frac{15}{2}t+3]\backslash\{6t+2\})$, $\{\widetilde{M}\}_1 = \{-(39t+16)\} \cup ([24t+10, 39t+16]\backslash\{31t+13\})$ and $head(\widetilde{M}) = tail\{\widetilde{M}\} = 0_0$.

As for case 2, the correctness of each orbit is shown as follows.

(I) By Lemma 5.2(1) ($\lceil\frac{72t+66}{12}\rceil \leq \frac{78t+72}{6}$).

(II) By Lemma 5.3(3).

(III) Since $\{\widetilde{P_0}\}_0 = [0, 18t+16], \{\widetilde{P_0}\}_1 = [-(21t+19), -(12t+12)] \cup [30t+28, 39t+35]$ and $tail\{\widetilde{P_0}\} = (18t+16)_0$.

(IV) We only verify the case: t even. Since $\{\widetilde{Q_0}\}_0 = [-(36t+33), -(\frac{51}{2}t+25] \cup [-(\frac{3}{2}t+1), 6t+5] \cup [\frac{35}{2}t+15, \frac{41}{2}t+18] \cup \{\frac{27}{2}t+11\}$, $\{\widetilde{Q_0}\}_1 = [-(\frac{51}{2}t+23), -(\frac{21}{2}t+10)]\backslash\{-(\frac{27}{2}t+12, -(\frac{33}{2}t+16)\}$ and $tail(\widetilde{Q_0}) = (19t+16)_0$. □

Theorem 6.9 *There exists a $6q$-$SCMD(13q)$ for $q \equiv 1$ (mod 6).*

Construction

(Case 1) Let $q = 12t+7$ and $t \geq 0$. Construct a $(72t+42)$-$SCMD(156t+91) = (X, \mathcal{B})$ as follows. The point set is $X = (Z_{78t+45} \times Z_2) \cup \{\infty\}$, the block set $\mathcal{B}$ consists of three parts:

(I) 3-*partite DC*: $3\text{-}DC(A_{01}[1, 24t+13], -(38t+22)_{10})$,
$3\text{-}DC(A_{10}[1, 24t+13], -(38t+22)_{01})$;

(II) $CDC_\infty(A_0[1, 39t+22], (-1)^{t-1}A_0[6t+4, 39t+21]^{-1})$;

(III) $SDC(P)$, where $P = ((-1)^t(39t+22)_0, -A_0[1, 6t+3],$
$MA_{01}([24t+14, 39t+22]\backslash\{38t+22\})$.

(Case 2) Let $q = 12t+1$ and $t \geq 1$. Construct a $(72t+6)$-$SCMD(156t+13) = (X, \mathcal{B})$ as follows. The point set is $X = (Z_{78t+6} \times Z_2) \cup \{\infty\}$, the block set $\mathcal{B}$ consists of three parts:

(I) 3-*partite DC*: $3\text{-}DC(A_{01}[2, 24t+2], -(38t+4)_{10})$,
$3\text{-}DC(A_{10}[2, 24t+2], -(38t+4)_{01})$;

(II) $SDC(Q)$, where $Q = (A_0[1, 4t], (24t+3)_{01}, -MA_{10}[24t+4, 38t+3], 1_{10},$
$A_0[1, 4t]^{-1})$;

(III) $FDC_\infty((39t+3)_0, N, (39t+3)_{01}, F^{-1}(N), (39t+3)_1, (24t+3)_{10})$,
$FDC_\infty(-1_{01}, \overline{N}, (39t+3)_{10}, F^{-1}(\overline{N}), 1_{01}, (24t+3)_{10})$,

where $N = (A_0[4t+1, 39t+2], (-1)^t A_{01}[38t+5, 39t+2])$.

Proof The number of blocks is

$$\begin{cases} (26t+15) \times 2 \times 2 + (78t+45) \times (2+1) = 338t+195 & \text{(case 1)} \\ (26t+2) \times 2 \times 2 + (78t+6) \times (1+1+1) = 338t+26 & \text{(case 2)} \end{cases},$$

as expected. For case 1, the correctness of each orbit is shown as follows.

(I) By Lemma 5.2(1) ($\lceil\frac{72t+42}{6}\rceil \leq \frac{78t+45}{3}$).

(II) By Lemma 5.5.

(III) We only verify the case: t even. Since $\{\widetilde{P}\}_0 = [-(18t+10), 3t+1] \cup \{-(39t+22)\}$, $\{\widetilde{P}\}_1 = [21t+12, 36t+21]\backslash\{28t+16, 29t+17\}$, $head(\widetilde{P}) = -(39t+22)_0$ and

$tail(\widetilde{P}) = -(18t+10)_0$.

As for case 2, the correctness of each orbit is shown as follows.

(I) By Lemma 5.2(1).

(II) Since $\{\widetilde{Q}_0\}_0 = [-9t, 2t] \cup [-(18t-1), -11t] \cup [-(39t+2), -(38t+2)] \cup [36t+4, 39t+3]$, $\{\widetilde{Q}_0\}_1 = [22t+3, 36t+3]$ and $tail(\widetilde{Q}_0) = (38t+4)_0$.

(III) By Lemma 5.1(4). □

Theorem 6.10 *There exists a $6q$-$SCMD(19q)$ for $q \equiv 1 \pmod 6$.*

Construction

(Case 1) Let $q = 12t+1$ and $t \geq 1$. Construct a $(72t+6)$-$SCMD(228t+19) = (X, \mathcal{B})$ as follows. The point set is $X = (Z_{114t+9} \times Z_2) \cup \{\infty\}$, the block set $\mathcal{B}$ consists of four parts:

(I) 3-*partite DC*: $3\text{-}DC(A_{01}[1, 24t+1], -(50t+4)_{10})$,
$3\text{-}DC(A_{10}[1, 24t+1], -(50t+4)_{01})$;

(II) $CDC(N)$, where $N = (MA_{01}([24t+2, 57t+4] \backslash \{50t+4\}), -A_0[1, 6t+1]^{-1}, (36t+3)_0)$;

(III) $CDC_\infty(R)$, where $R = ((-1)^{t-1}(57t+4)_0, A_0[6t+2, 57t+4], (-1)^t A_0[36t+4, 57t+3]^{-1})$;

(IV) $SDC(A_0[1, 36t+2])$.

(Case 2) Let $q = 12t+7$ and $t \geq 0$. Construct a $(72t+42)$-$SCMD(228t+133) = (X, \mathcal{B})$ as follows. When $t = 0$, a 42-$SCMD(133)$ is given in Appendix 8. Below, suppose $t > 0$. The point set is $X = (Z_{114t+66} \times Z_2) \cup \{\infty\}$, the block set $\mathcal{B}$ consists of five parts:

(I) 3-*partite DC*: $3\text{-}DC(A_{01}[1, 24t+13], -(50t+29)_{10})$,
$3\text{-}DC(A_{10}[1, 24t+13], -(50t+29)_{01})$;

(II) 2-*partite DC*: $2\text{-}DC(A_0[1, 36t+19], -(51t+29)_{01}, -(24t+14)_{10})$;

(III) $FDC(-A_0[6t+7, 42t+26], (57t+33)_{01})$;

(IV) $CDC_\infty(M)$, where $M = ((24t+14)_{01}, -MA_{10}([24t+15, 57t+32] \backslash \{50t+29, 51t+29\}), -(51t+29)_{10}, -A_0[1, 6t+6])$;

(V) $SDC(Q)$, where $Q = (-A_0[36t+20, 57t+33], (-1)^t A_0[42t+27, 57t+32]^{-1})$.

Proof The number of blocks is

$$\begin{cases} (38t+3) \times 2 \times 2 + (114t+9) \times (2+2+1) = 722t+57 & \text{(case 1)} \\ (38t+22) \times 2 \times 2 + (114t+66) \times (2+2+1) = 722t+418 & \text{(case 2)} \end{cases},$$

as expected. For case 1, the correctness of each orbit is shown as follows.

(I) By Lemma 5.2(1) ($\lceil \frac{72t+6}{6} \rceil \leq \frac{114t+9}{3}$).

(II) Since $\{\widetilde{N}\}_0 = [-(39t+3), 0]$, $\{\widetilde{N}\}_1 = ([24t+2, 57t+4] \backslash \{37t+3, 44t+4\}) \cup \{-(57t+4)\}$ and $head(\widetilde{N}) = tail(\widetilde{N}) = 0_0$.

(III) We only verify the case: t even. Since $\{\widetilde{R}\} = ([-(\frac{73}{2}t+2), 0] \backslash \{-(\frac{51}{2}t+2)\}) \cup [6t+2, 42t+3] \cup \{57t+4\}$, $head(\widetilde{R}) = (57t+4)_0$ and $tail(\widetilde{R}) = (42t+3)_0$.

(IV) It is trivial by Lemma 5.1(1).

As for case 2, the correctness of each orbit is shown as follows.

(I) By Lemma 5.2(1) ($\lceil \frac{72t+42}{6} \rceil \leq \frac{114t+66}{3}$).

(II) By Lemma 5.3(2).

(III) It is trivial by Lemma 5.1(1).

(IV) Since $\{\widetilde{M}\}_0 = [-(9t+7), -(3t+1)] \cup ([-(45t+26), -(12t+7)] \backslash \{-(25t+15), -(\frac{57}{2}t+17), -(32t+19)\})$, $\{\widetilde{M}\}_1 = [12t+7, 45t+25] \backslash \{\frac{51}{2}t+14, \frac{63}{2}t+18\}$, $head(\widetilde{M}) = -(12t+7)_0$ and $tail(\widetilde{M}) = -(3t+1)_0$.

(V) Since $\widetilde{Q}_0 = [-(54t+29), -(36t+20)] \cup [0, 18t+10]$ and $tail(\widetilde{Q}) = (18t+10)_0$.

□

7 Constructions of $ISCMD$

Theorem 7.1 *There exists a* $6q$-$ISCMD(6q+2q+1, 2q+1)$ *for* $q \equiv 1 \ (mod\ 6)$.

Construction

(Case 1) Let $q = 12t+1$ and $t \geq 1$. Construct a $(72t+6)$-$ISCMD(96t+9, 24t+3) = (X, \mathcal{B})$ as follows. The point set is $X = (Z_{36t+3} \times Z_2) \cup \{\infty_1, ..., \infty_{24t+3}\}$, the block set $\mathcal{B}$ consists of three parts:

(I) $(36t+3)$-*partite DC*: $(36t+3)$-$DC(A_{01}[i, i+1])$, $i \in [1, 6t+1]_2$,
$(36t+3)$-$DC(A_{10}[j, j+1])$, $j \in [1, 6t-1]_2$;

(II) $SDC(-(15t+2)_0, A_0[1, 6t+2], MA_{01}[6t+3, 18t+1], -A_{01}[6t+1, 6t+2]^{-1}, -A_0[1, 6t-1]^{-1})$;

(III) $CDC_{\infty_1,...,_{24t+3}}(A_0([6t, 18t] \backslash \{15t+2\}), -A_0([6t+3, 18t+1] \backslash \{15t+2\})^{-1}, (15t+2)_0, (18t+1)_0)$.

(Case 2) Let $q = 12t+7$ and $t \geq 0$. Construct a $(72t+42)$-$ISCMD(96t+57, 24t+15) = (X, \mathcal{B})$ as follows. The point set is $X = (Z_{36t+21} \times Z_2) \cup \{\infty_1, ..., \infty_{24t+15}\}$, the block set $\mathcal{B}$ consists of three parts:

(I) $(36t+21)$-*partite DC*: $(36t+21)$-$DC(A_{01}[i, i+1])$, $i \in [1, 6t+3]_2$,
$(36t+21)$-$DC(A_{10}[i, i+1])$, $i \in [1, 6t+3]_2$;

(II) $SDC(-(15t+9)_0, A_0[1, 6t+4], MA_{01}[6t+5, 18t+10], -A_0[1, 6t+3]^{-1})$;

(III) $CDC_{\infty_1,...,_{24t+15}}(A_0([6t+4, 18t+9] \backslash \{15t+9\}), A_0([6t+5, 18t+10] \backslash \{15t+9\})^{-1}, (15t+9)_0, -(18t+10)_0)$.

Proof The number of blocks is

$$\begin{cases} (36t+3) \times (2+1) + (6t+1) \times 2 = 120t+11 & \text{(case 1)} \\ (36t+21) \times (2+1) + (6t+4) \times 2 = 120t+71 & \text{(case 2)} \end{cases},$$

as expected. The correctness of each orbit is shown as follows.

(I) It is trivial.

(II) By Lemma 5.4(1)(2).

(III) By Lemma 5.6.

□

Theorem 7.2 *There exists a* $6q$-$ISCMD(6q+2q, 2q)$ *for* $q \equiv 5 \ (mod\ 6)$.

Construction

(Case 1) Let $q = 12t+5$ and $t \geq 0$. Construct a $(72t+30)$-$ISCMD(96t+40, 24t+10) = (X, \mathcal{B})$ as follows. When $t = 0$, a 30-$ISCMD(40, 10)$ is given in Appendix 9. Below, suppose $t > 0$. The point set is $X = (Z_{36t+15} \times Z_2) \cup \{\infty_1, ..., \infty_{24t+10}\}$, the block set $\mathcal{B}$ consists of three parts:

(I) $(36t+15)$-*partite DC*: $(36t+15)$-$DC(A_{01}[i, i+1])$, $i \in [1, 6t+1]_2$,

$(36t+15)$-$DC(A_{10}[i,i+1])$, $i \in [1, 6t+1]_2$;

(II) $SDC(-(15t+7)_0, A_0[1,6t+2], MA_{01}[6t+3, 18t+7], -A_0[1,6t+1]^{-1})$;

(III) $CDC_{\infty_{1,\dots,24t+10}}(A_0([6t+2,18t+6]\backslash\{15t+7\}),$
$-A_0([6t+3,18t+7]\backslash\{15t+7\})^{-1}, (15t+7)_0, (18t+7)_0)$.

(Case 2) Let $q = 12t+7$ and $t \geq 0$. Construct a $(72t+66)$-$ISCMD(96t+88, 24t+22) = (X, \mathcal{B})$ as follows. The point set is $X = (Z_{36t+33} \times Z_2) \cup \{\infty_1, \dots, \infty_{24t+22}\}$, the block set $\mathcal{B}$ consists of three parts:

(I) $(36t+33)$-*partite DC*: $(36t+33)$-$DC(A_{01}[i,i+1])$, $i \in [1, 6t+5]_2$,
$(36t+33)$-$DC(A_{10}[j,j+1])$, $j \in [1, 6t+3]_2$;

(II) $SDC(-(15t+15)_0, A_0[1,6t+6], MA_{01}[6t+7, 18t+16],$
$-A_{01}[6t+5, 6t+6]^{-1}, -A_0[1,6t+3]^{-1})$;

(III) $CDC_{\infty_{1,\dots,24t+22}}(A_0([6t+4,18t+15]\backslash\{15t+15\}),$
$A_0([6t+7,18t+16]\backslash\{15t+15\})^{-1}, (15t+15)_0, -(18t+16)_0)$.

Proof The number of blocks is

$$\begin{cases} (36t+15)\times(2+1)+(6t+2)\times 2 = 120t+49 & \text{(case 1)} \\ (36t+33)\times(2+1)+(6t+5)\times 2 = 120t+109 & \text{(case 2)} \end{cases},$$

as expected. The correctness of each orbit is shown as follows.

(I) It is trivial.

(II) By Lemma 5.4(1)(2).

(III) By Lemma 5.6. □

Lemma 7.3 *Let q be prime power and $q \equiv 1$ (mod 6); then there are at least $\frac{q+1}{2}$ integers d_j such that $q \leq d_j \leq 3q$ and $gcd(d_j, 6q) = 1$ for each j.*

Proof Let $q = p^n$. Since $\phi(6p^n) = 2p^{n-1}(p-1)$, there are $p^{n-1}(p-1)$ integers w_j such that $1 \leq w_j \leq 3q$ and $gcd(w_j, 6q) = 1$ for each j.
Let $S = \{d|\ gcd(d, 6q) = 1,\ q \leq d \leq 3q\}$. Note that $\phi(p^n) = p^{n-1}(p-1)$, so we have

$$\begin{aligned} |S| = &\ p^{n-1}(p-1) - [\phi(p^n) - (\lfloor \tfrac{p^n}{2} \rfloor + \lfloor \tfrac{p^n}{3} \rfloor) + (\lfloor \tfrac{p^n}{6} \rfloor + \lfloor \tfrac{p^n}{2p} \rfloor + \lfloor \tfrac{p^n}{3p} \rfloor) - \lfloor \tfrac{p^n}{6p} \rfloor] \\ = &\ \lfloor \tfrac{p^n}{2} \rfloor + \lfloor \tfrac{p^n}{3} \rfloor - \lfloor \tfrac{p^n}{6} \rfloor - \lfloor \tfrac{p^n}{2p} \rfloor - \lfloor \tfrac{p^n}{3p} \rfloor + \lfloor \tfrac{p^n}{6p} \rfloor \\ = &\ \tfrac{p^n-1}{2} + \tfrac{p^n-1}{3} - \tfrac{p^n-1}{6} - (\lfloor \tfrac{p^{n-1}}{2} \rfloor + \lfloor \tfrac{p^{n-1}}{3} \rfloor - \lfloor \tfrac{p^{n-1}}{6} \rfloor) \\ = &\ \tfrac{2}{3}(p^n-1) - (\lfloor \tfrac{p^{n-1}}{2} \rfloor + \lfloor \tfrac{p^{n-1}}{3} \rfloor - \lfloor \tfrac{p^{n-1}}{6} \rfloor). \end{aligned}$$

Obviously, $p^{n-1} \equiv 1$ or 5 (mod 6) when $p^n \equiv 1$ (mod 6).

If $p^{n-1} \equiv 1$ (mod 6), then

$$\begin{aligned} |S| - \tfrac{p^n+1}{2} = &\ \tfrac{2}{3}(p^n-1) - (\tfrac{p^{n-1}-1}{2} + \tfrac{p^{n-1}-1}{3} - \tfrac{p^{n-1}-1}{6}) - \tfrac{p^n+1}{2} \\ = &\ \tfrac{2}{3}(p^n-1) - \tfrac{2}{3}(p^{n-1}-1) - \tfrac{p^n+1}{2} \\ = &\ \tfrac{1}{6}[p^{n-1}(p-4)-3] \geq 0; \end{aligned}$$

If $p^{n-1} \equiv 5 \pmod{6}$, then

$$|S| - \frac{p^n+1}{2} = \frac{2}{3}(p^n - 1) - (\frac{p^{n-1}-1}{2} + \frac{p^{n-1}-2}{3} - \frac{p^{n-1}-5}{6}) - \frac{p^n+1}{2}$$

$$= \frac{1}{6}[p^{n-1}(p-4) - 5] \geq 0$$

Therefore, $|S| \geq \frac{p^n+1}{2}$ in both cases. The conclusion holds. □

Theorem 7.4 *There exists a* $6q$-$ISCMD(12q+q,q)$, *where* q *is prime power and* $q \equiv 1 \ (mod\ 6)$.

Construction Let $q = 6t+1$ and $t \geq 1$. Construct a $(36t+6)$-$ISCMD(78t+13, 6t+1) = (X, \mathcal{B})$ as follows. The point set is $X = (Z_{36t+6} \times Z_2) \cup \{\infty_1, ..., \infty_{6t+1}\}$, the block set $\mathcal{B}$ consists of five parts:

(I) 6-*partite DC*: $6\text{-}DC(A_{01}[1,6t], (9t+1)_0)$;

(II) $6q\text{-}DC(d_j)$ and $6q\text{-}DC(-d_j)$, $1 \leq j \leq 3t$, where $q \leq d_j \leq 3q$, $d_j \neq 9t+1$, $gcd(d_j, 6q) = 1$ and $1 \leq j \leq 3t$;

(III) $FDC(A_0[1, 6t-1], -(9t+1)_0, A_{01}[6t+1, 18t+2], (18t+3)_{01})$;

(IV) $SDC(A_{01}[1, 18t+2]^{-1})$;

(V) $CDC_{\infty_{1,...,6t+1}}(-A_0([1, 18t+3]\backslash\{S\}), (-1)^t A_0([6t, 18t+2]\backslash\{S\})^{-1})$, where $S = \{d_1, d_2, \cdots, d_{3t}, 9t+1\}$.

Proof The number of blocks is $(6t+1) \times 2 + 6t \times 2 + (36t+6) \times 4 = 168t + 26$, as expected. The correctness of each orbit is shown as follows.

(I) By Lemma 5.2(1) ($\lceil \frac{36t+6}{12} \leq \frac{36t+6}{6}$).

(II) By Lemma 5.7.

(III) Let $P = (A[1, 6t-1], -(9t+1), A_{01}[6t+1, 18t+2])$, then $\{\widetilde{P_0}\}_0 = [-(3t-1), 3t] \cup [-(12t+2), -(6t+1)]$, $\{\widetilde{P_0}\}_1 = [0, 6t]$, and $tail(\widetilde{P}) = -(12t+2)_0$. Obviously $P \cap F(P) = \emptyset$, and it is easy to see $\widetilde{P_0} \cap f(\widetilde{P_0} + 18t + 3) = \emptyset$.

(IV) By Lemma 5.1(1), $N = A_{01}[1, 18t+2]^{-1}$ forms a DP.

(V) Let $N = (-A_0([1, 18t+3]]\backslash\{S\}), (-1)^t A_0([6t, 18t+2]\backslash\{S\})^{-1})$, then $\widetilde{N_0} = (0, -1, 1, -2, \cdots) = (c_1, b_1, c_2, b_2, \cdots)$. Obviously the sequences c_i and b_i are monotone increasing and decreasing respectively, and they are mutually distinct, so N forms a DP. Finally, when constructing $CDC_{\infty_{1,...,6t+1}}$, we take $\langle \infty_1, 1_1, \infty_2, 2_2, ..., (6t+1)_1, \infty_{6t+1}, \widetilde{N_0} \rangle$ as the base block of the corresponding block-orbit. □

Theorem 7.5 *There exists a* $6q$-$ISCMD(12q+q+1, q+1)$ *for* $q \equiv 5 \ (mod\ 6)$.

Construction Let $q = 6t+5$ and $t \geq 0$. Construct a $(36t+30)$-$ISCMD(78t+66, 6t+6) = (X, \mathcal{B})$ as follows. The point set is $X = (Z_{36t+30} \times Z_2) \cup \{\infty_1, ..., \infty_{6t+6}\}$, the block set $\mathcal{B}$ consists of five parts:

(I) $3\text{-}DC(A_0[1, 12t+9], (18t+15)_0)$;

(II) $6q\text{-}DC(-1)$;

(III) $FDC(A_{01}[1, 18t+14], (18t+15)_{01})$;

(IV) $SDC(A_{01}[1, 18t+14]^{-1})$;

(V) $CDC_{\infty_{1,...,6t+6}}(-(18t+14)_0, A_0[2, 18t+14], A_0[12t+10, 18t+13]^{-1})$.

Proof The number of blocks is $(12t+10) \times 2 + 1 \times 2 + (36t+30) \times 4 = 168t + 142$, as expected. The correctness of each orbit is shown as follows.

(I) By Lemma 5.1(1), $D = A_0[1, 12t+9]$ forms a DP, and $\widetilde{D}_0 = [-(6t+4), 6t+5]$ is a interval with length $12t+10$, so they are not congruent modulo $12t+10$.

(II) It is trivial.

(III) It is trivial by Lemma 5.1(1).

(IV) It is trivial by Lemma 5.1(1).

(V) Let $N = (-(18t+14)_0, A_0[2, 18t+14]], A_0[12t+10, 18t+13]^{-1})$, $\widetilde{N}_0 = ([-(12t+10), 12t+10]\backslash\{1, -(9t+7), -(9t+8)\}) \cup \{18t+14\}$. Obviously N forms a DP. Finally, when constructing $CDC_{\infty_{1,\dots,6t+6}}$, we take $\langle \infty_1, 1_1, \infty_2, 2_2, \dots, (6t+6)_1, \infty_{6t+6}, \widetilde{N}_0 \rangle$ as the base block of the corresponding block-orbit. □

8 The proof of Theorem 1.3 and 1.4

By [7] and all the Theorems in sections 6 and 7, we have the following table (the block size is $6q$):

$q \equiv$ (mod 6)	$v \equiv$(mod -)	$SCMD(v)$	$ISCMD(v,h)$	$CS(v,6q,1)$	Theorems
1	$2q+1$ $(6q)$	$14q+1$	$(8q+1, 2q+1)$	$8q+1$	6. 1 7. 1 [7]
1	$4q$ $(6q)$	$10q$			6. 2
1	$5q+1$ $(6q)$	$11q+1$			6. 3
5	$2q$ $(6q)$	$8q$	$(8q, 2q)$		6. 4 7. 2
5	$4q+1$ $(6q)$	$10q+1$			6. 5
5	$5q$ $(6q)$	$11q$			6. 6
5	$7q+1$ $(12q)$	$7q+1$			6. 7
5	$q+1$ $(12q)$	$13q+1$	$(13q+1, q+1)$		6. 8 7. 5
1	q $(12q)$	$13q$	$*$ $(13q, q)$		6. 9 7. 4
1	$7q$ $(12q)$	$19q$		$*$ $7q$	6.10 [7]

The proof of Theorem 1.3 is trivial by section 2, 3 and the above table. Theorem 1.4 is a consequence of Theorem 1.3. □

The conclusion of Theorem 1.3 extends the existence results for $MD(v, k, 1)$ as well (refer to Theorem 1.1). Two possible exceptions in Theorem 1.3 correspond to the two "$*$"s in the table. For the first "$*$", the construction of Theorem 7.4, i.e. $6q$-$ISCMD(13q, q)$, holds only for odd prime powers $q = p^m$ $(p \geq 3)$. For the second "$*$", the existence of a $CS(7q, 6q, 1)$ has not been completely settled. These two parts are still open.

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Appendix

1. 42-$SCMD(70)$
The point set is $X = Z_{35} \times Z_2$, the block set:
(I) 7-*partite DC*: 7-$DC(A_{01}[1,5], -8_{10})$;
(II) $SDC(-A_0[1,17]\backslash\{11,12\}, -16_0, 12_0, 7_{01}, -6_{10}, 1_0)$;
(III) $CDC(MA_{01}[9,17], -A_{01}[1,8]^{-1}, 17_0, A_0[2,15]^{-1}, -11_0)$.
2. 42-$SCMD(78)$
The point set is $X = Z_{39} \times Z_2$, the block set:
(I) 3-*partite DC*: 3-$DC(A_{01}[1,13], 19_{10})$;
(II) $SDC(-19_0, -A_0[1,17], -A_{01}[14,15]^{-1})$;
(III) $CDC(A_0[1,19], A_{01}[1,18]^{-1}, A_{01}[16,19]^{-1}, 18_0)$.
3. 30-$SCMD(40)$
The point set is $X = Z_{20} \times Z_2$, the block set:
(I) 5-*partite DC*: 5-$DC(A_{01}[1,5], 1_{10})$;
(II) 10-*partite DC*: 10-$DC(A_{01}[2,3], -1_0)$;
(III) $FDC(10_0, A_0[1,9], -A_{01}[4,5], -A_{01}[8,9], 10_{01})$;
(IV) $SDC(A_0[2,9], -9_{01}, A_{10}[6,8], -A_{01}[6,7])$.
4. 30-$SCMD(51)$
The point set is $X = (Z_{25} \times Z_2) \cup \{\infty\}$, the block set:
(I) 5-*partite DC*: 5-$DC(A_{01}[1,5], 7_{01})$;
(II) $SDC(-A_0[1,12], -A_{01}[1,2])$;

(III) $CDC_\infty(30_0, A_0[1,12], MA_{01}[8,12], -A_{01}[6,7]^{-1}, A_{01}[4,6])$.

5. 30-$SCMD(55)$

The point set is $X = (Z_{27} \times Z_2) \cup \{\infty\}$, the block set:

(I) 3-*partite* DC: 3-$DC(A_{01}[1,9], 13_{10})$;

(II) $SDC(13_0, -A_0[1,13])$;

(III) $CDC_\infty(A_0[1,12], MA_{01}[10,12], (13)_{01}, 9_{10}, A_{01}[4,8]^{-1}, A_{10}[1,3])$.

6. 30-$SCMD(36)$

The point set is $X = Z_{18} \times Z_2$, the block set:

(I) 6-*partite* DC: 6-$DC(A_{01}[1,4], -1_0)$;

(II) $FDC(A_0[1,8], A_{01}[2,7])$;

(III) $SDC(8_{01}, -1_{10}, A_{01}[5,8], -A_0[2,9]^{-1})$.

7. 30-$SCMD(66)$

The point set is $X = Z_{33} \times Z_2$, the block set:

(I) 3-*partite* DC: 3-$DC(A_{01}[1,9], -16_{10})$ and 3-$DC(A_{10}[1,9], -16_{01})$;

(II) $SDC(-15_0, -1_0, A_0[4,14], 1_0)$;

(III) $CDC(MA_{01}[10,15], -A_0[2,16]^{-1}, 16_0, 2_0, -3_0)$.

8. 42-$SCMD(133)$

The point set is $X = (Z_{66} \times Z_2) \cup \{\infty\}$, the block set:

(I) 3*partite* DC: 3-$DC(A_{01}[1,13], -29_{10})$ and 3-DC$(A_{10}[1,13], -29_{01})$;

(II) 2-$DC(A_0[1,19], -28_{01}, -15_{10})$;

(III) $FDC(-A_0[7,26], 33_{01})$;

(IV) $CDC_\infty(MA_{01}([14,32]\backslash\{15,28\}, 15_{01}, 28_{10}, -A_0[1,6])$;

(V) $SDC(-A_0[20,33], A_0[27,32]^{-1})$.

9. 30-$ISCMD(40,10)$

The point set is $X = (Z_{15} \times Z_2) \cup \{\infty_1, ..., \infty_{10}\}$, the block set:

(I) 15-*partite* DC: 15-$DC(1_{01}, -2_{10})$ and 15-$DC(1_{10}, -2_{01})$;

(II) $SDC(-7_0, A_0[1,2], MA_{01}[3,7], -1_0)$;

(III) $CDC_{\infty_{1,...,10}}(A_0[2,6], A_0[3,7]^{-1})$.

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