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A sufficient condition for Hamiltonian cycles in bipartite tournaments*

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Abstract

A digraph T is said to satisfy the condition $W(k)$ if $d_T^-(u) + d_T^+(v) \geq k$ whenever uv is an arc of T. If a bipartite tournament T contains no factor, then its vertex set $V(X, Y)$ can be partitioned into four subsets P, Q, R and S such that $P \subseteq X$, $R = X \setminus P$ and $S = Y \setminus Q$ where $Q = N_T^+(P)$ and $|P| > |Q|$.

In this paper, we prove a new sufficient condition on degrees for a bipartite tournament to be Hamiltonian; that is, if an $n \times n$ bipartite tournament T satisfies the conditions: (i) $W(n-2)$, if $|Q| + |R| = n-1$; (ii) $W(n-3)$, if $|Q| + |R| \neq n-1$; then T is Hamiltonian, except for three exceptional graphs. This result is shown to be best possible in a sense.

1 Introduction

Throughout the paper we essentially use the terminology and notation of [1] and [9].

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Here $T(X, Y, E)$ denotes a bipartite tournament with bipartition (X, Y) and vertex-set $V(T) = X \cup Y$ and arc-set $E(T)$. If $|X| = m$ and $|Y| = n$, such a bipartite tournament is called an $m \times n$ bipartite tournament. For a vertex v of T and a subdigraph S of T, we define $N_s^-(v)$ and $N_s^+(v)$ to be the set of vertices of S which, respectively, dominate and are dominated by, the vertex v. Put

$$N_T^-(S) = \bigcup_{v \in S} N_T^-(v); \quad N_T^+(S) = \bigcup_{v \in S} N_T^+(v);$$

$$d_T^-(v) = |N_T^-(v)|; \quad d_T^+(v) = |N_T^+(v)|.$$

Let P be a subset of X and Q a subset of Y; $P \to Q$ (respectively, $Q \to P$) denotes $pq \in E(T)$ (respectively, $qp \in E(T)$) for all $p \in P$ and all $q \in Q$. If $P = \{x\}$ this becomes $x \to Q$. To simplify notation, we denote also $C_1 \to C_2$, $C_2 \to C_3$, $\cdots$, by $C_1 \to C_2 \to C_3 \to \cdots$. Moreover, a factor of T is a spanning subdigraph H of T such that $d_H^-(v) = d_H^+(v) = 1$ for all $v \in V(T)$. By $d_T(X, Y) = d(X, Y)$ we denote the number of arcs from X to Y, $i \cdot e \cdot$, $d(X, Y) =| \{xy \in E(D) : x \in X, y \in Y\} |$. Also T is said to be strong if for any two vertices u and v, there is a path from u to v and a path from v to u. A component of T is a maximal strong subdigraph.

By $T(r_1, r_2, r_3, r_4)$ we define the bipartite tournament with the four pairwise disjoint independent set of vertices B_1, B_2, B_3, B_4 with $| B_i |$=r_i, for i=1, 2, 3, 4 such that $B_1 \to B_2 \to B_3 \to B_4 \to B_1$.

The class $\tilde{T}(r_1, r_2, r_3, r_4)$ of bipartite tournaments originates from $T(r_1, r_2, r_3, r_4)$ by reversing some arcs between B_2 and B_3 or B_3 and B_4 such that $d(B_3, b_2) \le 1$ and $d(b_4, B_3) \le 1$ for every $b_2 \in B_2$ and $b_4 \in B_4$, or by reversing all the arcs.

The class $T^*(r_1, r_2, r_3, r_4)$ of bipartite tournaments originates from $T(r_1, r_2, r_3, r_4)$ by reversing some arcs between B_2 and B_3 or B_3 and B_4 such that $d(b_4, B_3) \le 1$ and $d_{B_3}^+(b_3) \le d_{B_4}^-(b_3)$ for every $b_4 \in B_4$ and $b_3 \in B_3$, $d_{B_3}^-(b_2) \le d_{B_4}^-(b_3)$ for every $b_2 \in B_2$ and $b_3 \in N_{B_3}^-(b_2)$.

Then $T(k, l, n-k, n-l) \subset \tilde{T}$, $T(k, l, n-k, n-l) \in T^*$.

A digraph T is said to satisfy the condition $O(r)$ if $d_T^+(u) + d_T^-(v) \ge r$ whenever uv is not an arc of T. T is said to satisfy the condition $W(r)$ if $d_T^-(u) + d_T^+(v) \ge r$ where uv is an arc of T.

Up to now, there are very few conditions that imply the existence of Hamiltonian cycles for bipartite tournaments. An obvious necessary condition for an $m \times n$ bipartite tournament to be Hamiltonian is $m = n$. Therefore, we are only interested in researching Hamiltonian properties in $n \times n$ bipartite tournaments. We recall now the well-known conditions for an $n \times n$ bipartite tournament to have Hamiltonian cycles.

The following results play an important role in the investigations of this section.

Theorem 1 (Jackson [2]). *If an $n \times n$ strong bipartite tournament T satisfies $O(n)$, then T is Hamiltonian.*

Theorem 2 (Wang [3]). *If an $n \times n$ bipartite tournament T satisfies $W(n-1)$, then T is Hamiltonian, unless T is isomorphic to $T(\frac{n+1}{2}, \frac{n+1}{2}, \frac{n-1}{2}, \frac{n-1}{2})$ when n is odd or $T(\frac{n}{2}, \frac{n-2}{2}, \frac{n}{2}, \frac{n+2}{2})$.*

An analogue to the results of Theorems 1 and 3 has been described in [7].

Theorem 3 (Zhang,Song,Wang [4]). *Let T be an $n \times n$ bipartite tournament with $n \geq 6$. If*

$$uv \in E(T) \Rightarrow d_T^-(u) + d_T^+(v) \geq n-2,$$

then T is Hamiltonian, unless T is isomorphic to $T(l+2, l, n-l-2, n-l)$ when $\frac{n-4}{2} \leq l \leq \frac{n}{2}$, or $T(n,n) \in T^*(l+1, l, n-l-1, n-l)$ with $\frac{n-5}{2} \leq l \leq \frac{n+1}{2}$.

2 Main results

Before proving Theorem 6, we need the following theorem.

Theorem 4 (Gutin [8] and Haggkvist, Manoussakis [5]). *A bipartite tournament T is Hamiltonian if and only if T is strong and contains a factor.*

Lemma 5 *If a bipartite tournament T contains no factor, then its vertex set $V(X,Y)$ can be partitioned into four subset P, Q, R and S such that $P \subseteq X$, $R = X \setminus P$ where $Q = N_T^+(P)$ and $\mid P \mid > \mid Q \mid$.*

Proof: Let T contains no cycle factor. Then we conclude from Proposition 3.11.6 (b) [see [9], p. 144] that there exists a subset $P \subseteq X$ such that $\mid P \mid > \mid N_T^+(P) \mid$ holds.

Theorem 6 *Let T be an $n \times n$ bipartite tournament with $n \geq 12$. In addition, T satisfies*

(i) $W(n-2)$, if $\mid Q \mid + \mid R \mid = n-1$;

(ii) $W(n-3)$, if $\mid Q \mid + \mid R \mid \neq n-1$;
then T is Hamiltonian, unless $T(n,n) \cong T(l+3, l, n-l-3, n-l)$ when $\frac{n-5}{2} \leq l \leq \frac{n}{2}$, or $T(n,n) \in T^*(l+2, l, n-l-2, n-l)$ when $\frac{n-7}{2} \leq l \leq \frac{n+1}{2}$, or $T(n,n) \in T^*(l+1, l, n-l-1, n-l)$ with $\frac{n-5}{2} \leq l \leq \frac{n+1}{2}$.

Proof of Theorem 6. Suppose that T is an $n \times n$ bipartite tournament satisfying the hypotheses of the theorem. We first establish two claims.

Claim 1. If $n \geq 12$, then T is strong.

Assume that T is not strong and has components C_1, C_2, $\cdots$, C_m with $m \geq 2$ such that $X(C_i) \to Y(C_j)$ and $Y(C_i) \to X(C_j)$ whenever $i \leq j$. Since C_1 is strong, we must have $|V(C_1)| \geq 4$, Otherwise $|V(C_1)| = 1$. We may assume $C_1 = \{x\} \subset X$; then $\{x\} \to Y$, for any $y \in$ Y we have $xy \in E$. In addition, in view of the hypotheses of the theorem and $d_T^-(x) = 0$, we deduce that

$$d_T^-(x) + d_T^+(y) = d_T^+(y) \geq n-3. \qquad (1)$$

Set $X_1 = N_T^+(Y)$, $X_2 = X \setminus X_1$; it follows from (1) that $\mid X_1 \mid \geq n-3$. Moreover,

$x \in X_2$ implies that $1 \leq| X_2 |\leq 3$. In addition,

$$\begin{aligned} n^2 &= \sum_{y\in Y} d_T^+(y) + \sum_{x\in X} d_T^+(x) \\ &= \sum_{y\in Y} d_T^+(y) + \sum_{x\in X_1} d_T^+(x) + \sum_{x\in X_2} d_T^+(x') \\ &\geq n(n-3) + n \mid X_2 \mid + \sum_{x\in X_1} d_T^+(x_1), \end{aligned}$$

we have

$$\sum_{x\in X_1} d_T^+(x_1) \leq n(3- \mid X_2 \mid). \qquad (2)$$

Let $d_T^+(x_0) = min\{d_T^+(x_1) \mid x \in X_1\}$, the (2) and $\mid X_1 \mid\geq n-3, 1 \leq\mid X_2 \mid\leq 3$ yield

$$(n-3)d_T^+(x_0) \leq \sum_{x\in X_1} d_T^+(x_1) \leq n,$$

then

$$d_T^+(x_0) \leq [\frac{n}{n-3}] = 1 \qquad (n \geq 12). \qquad (3)$$

The inequality (1) implies $d_T^-(y) \leq 3$. Let $yx_0 \in E$; in view of the hypotheses of the theorem and (1), (3), we deduce that

$$4 \geq d_T^-(y) + d_T^+(x_0) \geq n-3,$$

then $n \leq 7$; this is impossible. So, $\mid C_1 \mid\geq 4$. Similarly, $\mid C_m \mid\geq 4$.

Now, we can deduce $\sum_{v\in C_1} d_T^-(v) \leq \frac{|C_1|^2}{4}$. Hence, assume that there exists a vertex, say $u \in X(C_1)$, such that $d_T^-(u) \leq \frac{|C_1|}{4}$. If there is a vertex v in $Y(C_m)$ such that $d_T^+(v) \leq \frac{|C_m|}{4}$, then we have

$$d_T^-(u) + d_T^+(v) \leq \frac{\mid C_1 \mid + \mid C_m \mid}{4} \leq \frac{n}{2}. \qquad (4)$$

In particular, $uv \in E(T)$ implies

$$d_T^-(u) + d_T^+(v) \geq n-3. \qquad (5)$$

Combining this with (4) and (5), we can easily see $n \leq 6$ contradicting $n \geq 12$. Therefore we have $d_T^+(v) > \frac{|C_m|}{4}$ for every $v \in Y(C_m)$. Thus we can easily deduce that there is a vertex $w \in X(C_m)$ such that $d_T^+(w) < \frac{|C_m|}{4}$. Otherwise, suppose we have $d_T^+(w) \geq \frac{|C_m|}{4}$ for every $w \in X(C_m)$. Put $\mid X(C_m) \mid= p_1$, $\mid Y(C_m) \mid= p_2$; then $\mid C_m \mid= p_1 + p_2$, hence

$$\begin{aligned} p_1 \cdot p_2 &=\mid E(C_m) \mid= \sum_{r\in C_m} d_T^-(v) \\ &= \sum_{r\in Y(C_m)} d_T^+(v) + \sum_{\omega\in X(C_m)} d_T^+(w) \\ &> p_2 \cdot \tfrac{|C_M|}{4} + p_1 \cdot \tfrac{|C_m|}{4} = \tfrac{(p_1+p_2)^2}{4} \end{aligned}$$

which implies $(p_1 - p_2)^2 < 0$. This is impossible. Thus we have

$$d_T^-(u) + d_T^+(w) < \frac{| C_1 |}{4} + \frac{| C_m |}{4} \leq \frac{n}{2}. \tag{6}$$

Furthermore, it follows from $| C_1 | \geq 4$ and $| C_m | \geq 4$ that there is a vertex $v \in Y(C_m)$ such that $uv, vw \in E$, we have $d_T^-(u) + d_T^+(v) \geq n - 3$ and $d_T^-(v) + d_T^+(w) \geq n - 3$, then we obtain

$$d_T^-(u) + d_T^+(w) \geq n - 6. \tag{7}$$

It follows from (6) and (7) that $n < 12$ contradicting $n \geq 12$.

Claim 2. Either T contains a factor, or else T is isomorphic to $T(l+3, l, n-l-3, n-l)$, or $T(n,n) \in T^*(l+2, l, n-l-2, n-l)$, or $T(n,n) \in T^*(l+1, l, n-l-1, n-l)$.

Suppose that T contains no factor. It follows from Lemma 5 that there exists a subset $P \subseteq X$ such that $| P |>| N_T^+(P) |$. Put $N_T^+(P) = Q$, $R = X \backslash P$, and $S = Y \backslash Q$. Then $S \neq \emptyset$ and $S \to P$. Consider the vertices p in P and s in S. We now see that $N_T^-(s) \subseteq R$ and $N_T^+(p) \subseteq Q$ and hence

$$d_T^-(s) + d_T^+(p) \leq | R | + | Q |, \tag{8}$$

$$| R | + | Q | < | R | + | P | = n. \tag{9}$$

Combining these with the fact that $sp \in E$, implying $d_T^-(s) + d_T^+(p) \geq n - 3$, we get $| Q | + | R | = n - 3$ or $n - 2$ or $n - 1$. Put $| Q | = l, | S | = n - l$.

Case 1. $| Q | + | R | = n - 3$. This implies $| R | = n - l - 3$, $| P | = l + 3$; it follows from (8) that $P \to Q$, $R \to S$. Consider the vertices $q \in Q$ and $r \in R$. If $rq \in E$, it follows from Theorem 6 that $n - 3 \leq d_T^-(r) + d_T^+(q) \leq | Q | -1 + | R | -1 = n - 5$, a contradiction. Therefore we must have $Q \to R$, that is, $T(n,n) \cong T(l+3, l, n-l-3, n-l)$. Considering the arcs pq and rs, we have $2n - 2l - 3 \geq n - 3$ and $2l + 3 \geq n - 3$, so $\frac{n-5}{2} \leq l \leq \frac{n}{2}$.

Case 2. $| Q | + | R | = n-2$. $i{\cdot}e{\cdot}$, $| R | = n-l-2$, $| P | = l+2$. Considering $sp \in E$, we have $d_T^-(s)+d_T^+(p) \geq n-3$. Furthermore, $d_T^-(s)+d_T^+(p) = | R | - d_R^+(s) + | Q | - d_Q^-(p)$, then

$$d_Q^-(p) + d_R^+(s) \leq 1. \tag{10}$$

If there is a vertex $s \in S$ such that $d_R^+(s) = 1$, then it follows from (10) that for any $p \in P$, we have $d_Q^-(p) = 0$ with $p \to Q$. Otherwise $R \to S$, since symmetry, we can assume $P \to Q$ and for any $s \in S$, then

$$d_R^+(s) \leq 1. \tag{11}$$

Put $R_1 = \{r \in R \mid d_S^-(r) > 0\}, R_2 = R \setminus R_1$.

Subcase 2.1. $R_1 = \emptyset$, $i \cdot e \cdot$, $R \to S$, we have $Q \to R$. So one can easily deduce that $T(n,n) \cong T(l+2, l, n-l-2, n-l) \in T^*(l+2, l, n-l-2, n-l)$. Considering the arc $pq \in E, rs \in E$, we have $2n - 2l - 2 \geq n - 3$ and $2l + 2 \geq n - 3$, and then $\frac{n-5}{2} \leq l \leq \frac{n+1}{2}$.

Subcase 2.2. $R_1 \neq \emptyset$, $R_2 \neq \emptyset$. We can conclude $Q \to R_2$, for otherwise there are vertices $r_2 \in R_2$, $q \in Q$ such that $r_2 q \in E$,and then $n-3 \leq d_T^-(r_2) + d_T^+(q) \leq \mid R \mid -1 + \mid Q \mid -1 = n-4$, which is impossible. In addition, if $d_Q^+(r_1) > 0$, there is a vertex $q \in Q$ such that $r_1 q \in E$, and then

$$n-3 \leq d_T^-(r_1) + d_T^+(q) = \mid Q \mid - d_Q^+(r_1) + d_S^-(r_1) + \mid R \mid - d_R^-(q).$$

We get $d_Q^+(r_1) + d_R^-(q) \leq d_S^-(r_1) + 1$. Combining this with the facts that $d_Q^+(r_1) \geq 1$ and $d_R^-(q) \geq 1$, we know

$$d_Q^+(r_1) \leq d_S^-(r_1), \quad d_R^-(q) \leq d_S^-(r_1). \tag{12}$$

If there is a vertex $r_1 \in R$ such that $d_Q^+(r_1) = 0$, then $d_Q^+(r_1) < d_S^-(r_1)$. It follows from (11) and (12) that $T(n,n) \in T^*(l+2, l, n-l-2, n-l)$. Considering the arcs pq and $r_2 s$, we have $2n-2l-2 \geq n-3$ and $2l+2 \geq n-3$, and then $\frac{n-5}{2} \leq l \leq \frac{n+1}{2}$.

Subcase 2.3. $R_2 = \emptyset$. It follows from $\mid R \mid = \mid S \mid -2$ and (11), that we can conclude $d_S^-(r) = 1$ for every $r \in R$ except for at most two vertices in R. Using Claim 1 and the hypothesis of the theorem, we obtain $\mid R \mid \geq 3$. So it is easy to see that there is a vertex $r_0 \in R$ such that $d_S^-(r_0) = 1$. That is to say, we have $s_0 \in S$ such that $s_0 r_0 \in E$. Similarly, as in Subcase (2.2), we can prove

$$d_Q^+(r) \leq d_S^-(r), \quad d_R^-(q) \leq d_S^-(r), \quad r \in R, q \in Q. \tag{13}$$

Considering the arc $s_0 r_0$, by the hypothesis of the theorem and (13), we have

$$n-3 \leq d_T^-(s_0) + d_T^+(r_0) \leq |R| - 1 + |S| - 1 + d_Q^+(r_0) \leq |R| + |S| - 2 + d_S^-(r_0) = 2n-2l-2,$$

so $l \leq \frac{n+1}{2}$. In addition, in view of the arc $r_0 s_1$, we have

$$n-3 \leq d_T^-(r_0) + d_T^+(s_1) \leq \mid P \mid + \mid Q \mid + 2 = 2l+4,$$

so $l \geq \frac{n-7}{2}$. We can easily see that $T(n,n) \in T^*(l+2, l, n-l-2, n-l)$.

Case 3. $\mid Q \mid + \mid R \mid = n-1$, $i \cdot e \cdot$, $\mid R \mid = n-l-1$, $\mid P \mid = l+1$. It follows from $sp \in E$ and the hypothesis of Theorem 6 that we have $d_T^-(s) + d_T^+(p) \geq n-2$. Moreover, $d_T^-(s) + d_T^+(p) = \mid R \mid - d_R^+(s) + \mid Q \mid - d_T^-(p)$, so then

$$d_Q^-(p) + d_R^+(s) \leq 1. \tag{14}$$

Put $R_1 = \{r \in R \mid d_S^-(r) > 0\}$, and $R_2 = R \setminus R_1$.

Subcase 3.1. $R_1 = \emptyset$, $i \cdot e \cdot$, $R \to S$; we have $Q \to R$, then $T(n,n) \cong T(l+1, l, n-l-1, n-l) \in T^*(l+1, l, n-l-1, n-l)$. Considering $pq \in E$ and $rs \in E$, we have $2n-2l-1 \geq n-2$ and $2l+1 \geq n-2$, and then $\frac{n-3}{2} \leq l \leq \frac{n+1}{2}$.

Subcase 3.2. $R_1 \neq \emptyset$, $R_2 \neq \emptyset$. We can conclude that $Q \to R_2$. Otherwise there are vertices $r_2 \in R_2$, $q \in Q$ such that $r_2 q \in E$, and then $n-2 \leq d_T^-(r_2) + d_T^+(q) \leq \mid R \mid -1 + \mid Q \mid -1 = n-3$, which is impossible. In addition, if $d_Q^+(r_1) > 0$, there is a vertex $q \in Q$ such that $r_1 q \in E$, and then

$$n-2 \leq d_T^-(r_1) + d_T^+(q) = \mid Q \mid - d_Q^+(r_1) + d_S^-(r_1) + \mid R \mid - d_R^-(q),$$

so we get $d^+_Q(r_1) + d^-_R(q) \le d^-_S(r_1) + 1$. Similarly, as in Subcase (2.2), we know

$$d^+_Q(r_1) \le d^-_S(r_1), \quad d^-_R(q) \le d^-_S(r_1). \tag{15}$$

If there is a vertex $r_1 \in R$ such that $d^+_Q(r_1) = 0$, then $d^+_Q(r_1) < d^-_S(r_1)$. Hence we have $T(n,n) \in T^*(l+1, l, n-l-1, n-l)$. Considering pq and r_2s, we have $\frac{n-3}{2} \le l \le \frac{n+1}{2}$.

Subcase 3.3. $R_2 = \emptyset$. From $|R| = |S| - 1$ and $d^+_R(s) \le 1$, we can conclude that $d^-_S(r) = 1$ for every $r \in R$ except for at most one vertex in R. Similarly, as in Subcase (2.3), we can prove

$$d^+_Q(r) \le d^-_S(r), \quad d^-_R(q) \le d^-_S(r), \quad r \in R,\ q \in Q. \tag{16}$$

We can easily see that $T(n,n) \in T^*(l+1, l, n-l-1, n-l)$; here $\frac{n-5}{2} \le l \le \frac{n}{2}$.

This proves Claim 2.

The proof of the theorem is now complete.

Remark 1. Observe that a bipartite tournament T satisfies $W(r)$ for each arc uv if and only if T satisfies $O(r)$ for each non-arc uv with u and v being in different partite sets. Thus, $W(r)$ is weaker than $O(r)$.

Obviously, Theorem 6 improves Theorem 3. Now, we prove another condition, ensuring an $n \times n$ bipartite tournament is Hamiltonian, except for three described cases.

Corollary 7. *Let T be an $n \times n$ bipartite tournament with $n \ge 12$. In addition, if T satisfies*

(i) $O(n-2)$, if $|Q| + |R| = n - 1$;

(ii) $O(n-3)$, if $|Q| + |R| \ne n - 1$;
then T is Hamiltonian, unless $T(n,n) \cong T(\frac{n+3}{2}, \frac{n-3}{2}, \frac{n-3}{2}, \frac{n+3}{2})$, or $T(n,n) \in \tilde{T}(l+2, l, n-l-2, n-l)$; here $\frac{n-5}{2} \le l \le \frac{n-1}{2}$ or $T(n,n) \in \tilde{T}(l+1, l, n-l-1, n-l)$; here $\frac{n-3}{2} \le l \le \frac{n}{2}$.

Proof. The condition $O(n-3)$ includes the condition $W(n-3)$. Moreover, by Theorem 6 we can get $T(l+3, l, n-l-3, n-l)$, where $\frac{n-5}{2} \le l \le \frac{n}{2}$, $T^*(l+2, l, n-l-2, n-l)$, where $\frac{n-7}{2} \le l \le \frac{n+1}{2}$, $T^*(l+1, l, n-l-1, n-l)$, here $\frac{n-5}{2} \le l \le \frac{n+1}{2}$. Then we can exclude the above which does not satisfy $O(n-3)$.

Case 1. Clearly, we observe that $l = \frac{n-5}{2}, \frac{n-4}{2}, \frac{n-2}{2}, \frac{n-1}{2}$, and $\frac{n}{2}$ do not satisfy $O(n-3)$. Therefore we only have $T(\frac{n+3}{2}, \frac{n-3}{2}, \frac{n-3}{2}, \frac{n+3}{2})$ which satisfy $O(n-3)$ in $T(l+3, l, n-l-3, n-l)$; here $\frac{n-5}{2} \le l \le \frac{n}{2}$.

Case 2. We easily find that $l = \frac{n-7}{2}, \frac{n-6}{2}, \frac{n}{2}$, and $\frac{n+1}{2}$ do not satisfy $O(n-3)$ in $T^*(l+2, l, n-l-2, n-l)$. Assume $T(n,n) \in T^*(l+2, l, n-l-2, n-l)$ and that it satisfies $O(n-3)$. If there exists a vertex, say $b_2 \in B_2$, with the property that $d^-_{B_3}(b_2) \ge 2$, we deduce that $d^-_{B_3}(b_4) = 1$. So we have

$$d^+_T(b_2) + d^-_T(b_4) \le \mid B_3 \mid -2 + \mid B_3 \mid -1 = 2n - 2l - 4 - 3 = 2n - 2l - 7. \tag{a}$$

In addition, the arc $b_2b_4 \notin E$, so we obtain

$$d_T^+(b_2) + d_T^-(b_4) \geq n - 3. \tag{b}$$

Combining (a) with (b) implies that $l \leq \frac{n-4}{2}$. Now considering the vertices $b_1 \in B_1$, $b_3 \in B_3$, it follows from $b_1b_3 \notin E$ that

$$d_T^+(b_1) + d_T^-(b_3) \geq n - 3. \tag{c}$$

Moreover, it is easy to see that

$$d_T^+(b_1) + d_T^-(b_3) \leq 2 \mid B_2 \mid + d_{B_4}^-(b_3) = n - 4 + d_{B_4}^-(b_3). \tag{d}$$

It follows from (c) and (d) that $d_{B_4}^-(b_3) \geq 1$. According to the arbitrariness of b_3, and since $d_{B_3}^+(b_4) \leq 1$ for all $b_4 \in B_4$, we conclude that there exist at most two vertices $b_3', \tilde{b_3} \in B_3$ which satisfy $d_{B_4}^-(b_3') \geq 2$ and $d_{B_4}^-(\tilde{b_3}) \geq 2$. Furthermore, we get $d_{B_4}^-(b_3) = 1$ except for $b_3', \tilde{b_3}$ (here $b_3 \in B_3$). In addition, in view of the hypotheses $d_{B_3}^-(b_2) \geq 2$, there exists a vertex, say $\breve{b_3} \in N_{B_3}^-(b_2)$, with the property that $d_{B_4}^-(\breve{b_3}) = 1$, for every $b_1 \in B_1$, we have

$$d_T^+(b_1) + d_T^-(\breve{b_3}) \leq \mid B_2 \mid + \mid B_2 \mid -1 + 1 \leq n - 4.$$

It follows from $b_1\breve{b_3} \notin E$ that $d_T^+(b_1) + d_T^-(\breve{b_3}) \geq n - 3$. This is impossible. Thus we have $d_{B_3}^-(b_2) \leq 1$, that is, $T(n,n) \in \tilde{T}(l+2, l, n-l-2, n-l)$, where $\frac{n-5}{2} \leq l \leq \frac{n-1}{2}$.

Case 3. In this case, we easily find when $l = \frac{n-5}{2}, \frac{n-4}{2}$ and $\frac{n+1}{2}$, that l does not satisfy $O(n-2)$ in $T^*(l+1, l, n-l-1, n-l)$. Assume $T(n,n) \in T^*(l+1, l, n-l-1, n-l)$ and that it satisfies $O(n-2)$, If there exists a vertex, say $b_2 \in B_2$, with the property that $d_{B_3}^-(b_2) \geq 2$, then we deduce that $d_{B_3}^-(b_4) = 1$. So we have

$$d_T^+(b_2) + d_T^-(b_4) \leq |B_3| - 2 + |B_3| - 1 = 2n - 2l - 5. \tag{e}$$

In addition, $b_2b_4 \notin E$, and we can obtain

$$d_T^+(b_2) + d_T^-(b_4) \geq n - 2. \tag{f}$$

Combining (e) with (f), we know $l \leq \frac{n-3}{2}$. Now in view of the vertices $b_1 \in B_1$ and $b_3 \in B_3$, it follows from $b_1b_3 \notin E$ that

$$d_T^+(b_1) + d_T^-(b_3) \geq n - 2. \tag{m}$$

Moreover, it is easy to see that

$$d_T^+(b_1) + d_T^-(b_3) \leq 2|B_2| + d_{B_4}^-(b_3) = n - 3 + d_{B_4}^-(b_3). \tag{n}$$

It follows from (m) and (n) that $d_{B_4}^-(b_3) \geq 1$. According to the arbitrariness of b_3 and $d_{B_3}^+(b_4) \leq 1$ for all $b_4 \in B_4$, we conclude that there exists at most one vertex b_3' satisfying $d_{B_4}^-(b_3') \geq 2$; moreover, we obtain $d_{B_4}^-(b_3) = 1$ for all b_3 except for

b_3'. In addition, in view of the hypotheses $d^-_{B_3}(b_2) \geq 2$, there exists a vertex, say $\breve{b_3} \in N^-_{B_3}(b_2)$, with the property $d^-_{B_4}(\breve{b_3}) = 1$, for every $b_1 \in B_1$, we have

$$d^+_T(b_1) + d^-_T(\breve{b_3}) \leq \mid B_2 \mid + \mid B_2 \mid -1 + 1 \leq n - 3.$$

It follows from $b_1\breve{b_3} \notin E$ that

$$d^+_T(b_1) + d^-_T(\breve{b_3}) \geq n - 2.$$

This is impossible. Thus we have $d^-_{B_3}(b_2) \leq 1$, that is, $T(n,n) \in \tilde{T}(l+1, l, n-l-1, n-l)$, here $\frac{n-3}{2} \leq l \leq \frac{n}{2}$.

Corollary 8. *If an $n \times n$ bipartite tournament T satisfies $O(n-2)$ and $n \geq 12$, then T is Hamiltonian, unless $T(n,n) \cong T(\frac{n+2}{2}, \frac{n-2}{2}, \frac{n-2}{2}, \frac{n+2}{2})$, or $T(n,n) \in \tilde{T}(l+1, l, n-l-1, n-l)$; here $\frac{n-3}{2} \leq l \leq \frac{n}{2}$.*

Proof. The condition $O(n-3)$ includes the condition $O(n-2)$, so using Corollary 7 we can easily get that $T(\frac{n+3}{2}, \frac{n-3}{2}, \frac{n-3}{2}, \frac{n+3}{2})$, $\tilde{T}(l+2, l, n-l-2, n-l)$, where $\frac{n-5}{2} \leq l \leq \frac{n-1}{2}$, and $T(n,n) \in \tilde{T}(l+1, l, n-l-1, n-l)$, where $\frac{n-3}{2} \leq l \leq \frac{n+1}{2}$. Then we can exclude the above case which does not satisfy $O(n-2)$.

Case 1. Clearly, we observe that $T(\frac{n+3}{2}, \frac{n-3}{2}, \frac{n-3}{2}, \frac{n+3}{2})$ do not satisfy $O(n-2)$.

Case 2. We only have $T(\frac{n+2}{2}, \frac{n-2}{2}, \frac{n-2}{2}, \frac{n+2}{2})$ which satisfy $O(n-2)$ in $T(l+2, l, n-l-2, n-l)$.

Case 3. Clearly, using Corollary 8, this is true.

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