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On the order of almost regular bipartite graphs without perfect matchings

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Abstract

A graph G is almost regular or more precisely is a $(d, d+1)$-graph, if the degree of each vertex of G is either d or $d+1$. Let $d \geq 2$ be an integer, and let G be a connected bipartite $(d, d+1)$-graph with partite sets X and Y such that $|X| = |Y|$. If the order of G is at most $4d+4$, then we show in this paper that G contains a perfect matching. Examples will demonstrate that the given bound on the order of G is best possible.

We shall assume that the reader is familiar with standard terminology on graphs (see, e.g., Chartrand and Lesniak [2]). In this paper, all graphs are finite and simple. The vertex set of a graph G is denoted by $V(G)$, and $n = n(G) = |V(G)|$ is called the *order* of G. The *neighborhood* $N_G(x)$ of a vertex x is the set of vertices adjacent with x, and the number $d_G(x) = |N_G(x)|$ is the *degree* of x in the graph G. If $d \leq d_G(x) \leq d+1$ for each vertex x in a graph G, then we speak of an *almost regular graph* or more precisely of a $(d, d+1)$-*graph*. If M is a matching in a graph G with the property that every vertex is incident with an edge of M, then M is a *perfect matching*. We denote by $K_{r,s}$ the complete bipartite graph with partite sets X and Y, where $|X| = r$ and $|Y| = s$. If G is a graph and $A \subseteq V(G)$, then we denote by $G[A]$ the subgraph induced by A and by $q(G-A)$ the number of odd components in the subgraph $G-A$.

As an extension of a theorem of Wallis [10] on regular graphs, Zhao [11] in 1991 proved the following result.

Theorem 1 (Zhao [11] 1991) *Let $d \geq 2$ be an integer, and let G be a $(d, d+1)$-graph without an odd component. If $|V(G)| \leq 3d+3$, then G has a perfect matching.*

For supplements, extensions or generalizations of Theorem 1, see the articles by Caccetta and Mardiyono [1], Volkmann [9] and Klinkenberg and Volkmann [3, 4, 5].

In this paper, we will prove an analogue to Zhao's theorem for bipartite graphs. The proof of our main theorem is based on Tutte's famous 1-factor theorem [7] (for a proof see e.g., [8]).

Theorem 2 (Tutte [7] 1947) *A nontrivial graph G has a perfect matching (or a 1-factor) if and only if $q(G-S) \leq |S|$ for every proper subset S of $V(G)$.*

Theorem 3 *Let $d \geq 2$ be an integer, and let G be a connected bipartite $(d, d+1)$-graph of order n with partite sets X and Y such that $|X| = |Y|$. If $n \leq 4d+4$, then G contains a perfect matching.*

Proof. Suppose to the contrary that G does not contain a perfect matching. Then, Theorem 2 implies that there exists a non-empty set $A \subset V(G)$ such that $q(G-A) \geq |A|+1$. Since n is even, the numbers $q(G-A)$ and $|A|$ of the same parity, and we deduce that

$$q(G-A) \geq |A| + 2. \tag{1}$$

We call an odd component of $G-A$ large if it has at least $2d+1$ vertices, and small otherwise. If we denote by α and β the number of large and small components, respectively, then we deduce from (1) that

$$\alpha + \beta = q(G-A) \geq |A| + 2. \tag{2}$$

If U is a small component of $G-A$ of minimum order, then we observe that

$$n \geq |A| + \alpha(2d+1) + \beta|V(U)|. \tag{3}$$

Since G is a bipartite $(d, d+1)$-graph, it is easy to verify that there are at least d edges of G joining each small component of $G-A$ with A. Using the hypothesis that G is connected, we deduce that

$$\alpha + d\beta \leq |A|(d+1). \tag{4}$$

Next we distinguish four cases.

Case 1: Assume that $\alpha \geq 3$. The hypothesis $n \leq 4d+4$ and (3) lead to the contradiction

$$4d + 4 \geq n \geq 3(2d+1).$$

Case 2: Assume that $\alpha = 2$. Inequality (2) yields $\beta \geq |A| \geq 1$, and thus we obtain by (3)

$$\begin{aligned} 4d + 4 \geq n & \geq |A| + 2(2d+1) + \beta|V(U)| \\ & \geq 4d + 2 + |A|(1 + |V(U)|) \end{aligned}$$

and therefore $|A| = |V(U)| = 1$. However, now the only vertex of the small component U has only one neighbor, a contradiction to $d \geq 2$.

Case 3: Assume that $\alpha = 1$. Inequality (2) yields $\beta \geq |A| + 1$, and thus we obtain by (4)

$$|A| \geq d + 1. \tag{5}$$

Applying (3), we arrive at

$$\begin{aligned} 4d + 4 \geq n &\geq |A| + \alpha(2d + 1) + \beta|V(U)| \\ &\geq d + 1 + 2d + 1 + \beta \\ &= 3d + 2 + \beta \end{aligned}$$

and thus

$$\beta \leq d + 2. \tag{6}$$

Using the hypothesis $n \leq 4d+4$, we altogether observe that $\beta = d+2 = |A|+1$, each small component consists of a single vertex, the large component is of order exactly $2d + 1$ and $n = 4d + 4$.

Since G is a connected $(d, d + 1)$-graph, there are at least $d^2 + 2d$ edges in G joining the small components of $G - A$ with A and at least one edge in G joining the large component of $G - A$ with A. In addition, there are at most $d^2 + 2d + 1$ edges in G joining A with the odd components of $G - A$. Consequently, all vertices in A are of degree $d + 1$, and the subgraph $G[A]$ induced by A is empty. Since there is only one edge, say uv, connecting the large component W of order $2d + 1$ with A, the large component W has a bipartition V', V'' such that $|V''| = |V'| + 1 = d + 1$. Without loss of generality, let u in W. Suppose that $u \in V'$. This implies that every vertex of V'' is connected with every vertex of V' in W, and we arrive at the contradiction $d_G(u) = d + 2$. Thus $u \in V''$, and now $X = V' \cup A$ and $Y = V(G) - (V' \cup A)$ is a bipartition of G with $|X| = 2d + 1$ and $|Y| = 2d + 3$. Since G is connected, this is a contradiction to the hypothesis that $|X| = |Y|$.

Case 4: Assume that $\alpha = 0$. Inequality (2) yields $\beta \geq |A| + 2$, and thus (4) leads to

$$|A| \geq 2d. \tag{7}$$

Applying the bound $\beta \geq |A| + 2$, we obtain

$$\beta \geq |A| + 2 \geq 2d + 2. \tag{8}$$

According to (3) and (7), we arrive at

$$4d + 4 \geq n \geq |A| + \alpha(2d + 1) + \beta|V(U)| \geq 2d + \beta \tag{9}$$

and thus

$$2d + 4 \geq \beta. \tag{10}$$

The inequalities (8) and (10) show that $2d + 2 \leq \beta \leq 2d + 4$.

Subcase 4.1: Assume that $\beta = 2d + 4$. In view of (9), it follows that $|A| = 2d$, and hence (4) yields the contradiction

$$d(2d + 4) = d\beta \leq |A|(d + 1) = 2d(d + 1).$$

Subcase 4.2: Assume that $\beta = 2d+3$. In view of (9), it follows that $|A| \leq 2d+1$. Because of $|A| \geq 2d$ and the fact that n is even, we deduce that $|A| = 2d+1$. As $n \leq 4d+4$, we conclude that all small components of $G - A$ are isolated vertices. Consequently, there are at least $2d^2 + 3d$ edges in G joining the small components of $G - A$ with A. In addition, there are at most $2d^2 + 3d + 1$ edges in G joining A with the odd components of $G - A$. Therefore, the subgraph $G[A]$ is empty. Thus $X = A$ and $Y = V(G) - A$ is a bipartition of G with $|X| = 2d+1$ and $|Y| = 2d+3$. Since G is connected, this is a contradiction to the hypothesis that $|X| = |Y|$.

Subcase 4.3: Assume that $\beta = 2d+2$. By (2) and (7), it follows that

$$2d + 2 = \beta \geq |A| + 2 \geq 2d + 2$$

and thus $|A| = 2d$. Hence there are at least $2d^2 + 2d$ edges in G joining the small components of $G - A$ with A, and there are at most $2d^2 + 2d$ edges in G joining A with the odd components of $G - A$. Therefore the subgraph $G[A]$ is empty.

If the small components of $G - A$ are isolated vertices, then we arrive a contradiction as above.

Otherwise, the hypothesis $n \leq 4d+4$ shows that there is exactly one small component of order three and that the remaining $2d+1$ small components are of order one. Hence there are at least $3d - 4 + d(2d+1) = 2d^2 + 4d - 4$ edges in G joining the small components of $G - A$ with A, and there are at most $2d^2 + 2d$ edges in G joining A with the odd components of $G - A$. This leads to a contradiction when $d \geq 3$. In the remaining case that $d = 2$, we obtain $|A| = 4$, $\beta = 6$ and $n = 12$. A straightforward calculation leads to the contradiction that G has a bipartition X, Y with $|X| = |Y| = 6$, and the proof of Theorem 3 is complete. □

The following family of examples will show that the bound presented in Theorem 3 is best possible.

Example 4 Let $d \geq 2$ be an integer, and let $K_{d+1,d+2}$ be the complete bipartite graph with the partite sets $\{x_1, x_2, \ldots, x_{d+2}\}$ and $\{y_1, y_2, \ldots, y_{d+1}\}$. If we delete in the graph $K_{d+1,d+2}$ the edges $x_1y_1, x_2y_2, \ldots, x_{d+1}y_{d+1}$ and $x_{d+2}y_{d+1}$, then we denote the resulting graph by H_1. In addition, let $K_{d+1,d+2}$ be the complete bipartite graph with the partite sets $\{u_1, u_2, \ldots, u_{d+2}\}$ and $\{v_1, v_2, \ldots, v_{d+1}\}$. If we delete the edges $u_1v_1, u_2v_2, \ldots, u_{d+1}v_{d+1}$ and $u_{d+2}v_{d+1}$, then we denote the resulting graph by H_2. Now let H be the disjoint union of H_1 and H_2 together with the edge $y_{d+1}v_{d+1}$. It is straightforward to verify that H is a connected bipartite $(d, d+1)$-graph of order $|V(H)| = 4d+6$ with a partition X, Y such that $|X| = |Y| = 2d+3$ without a perfect matching.

Corollary 5 *Let $d \geq 2$ be an integer, and let G be a bipartite $(d, d+1)$-graph of order n with partite sets X and Y such that $|X| = |Y|$. If $n \leq 4d+4$ and if G has no odd component, then G contains a perfect matching.*

Proof. Since G is a bipartite $(d, d+1)$-graph, each component of G has order at least $2d$. Hence G consists of at most two components when $d \geq 3$ and at most three components when $d = 2$. In the case that G is connected, the desired result follows from Theorem 3. If $d = 2$ and G has three components, then all components are isomorphic to $K_{2,2}$, and G contains a perfect matching. Assume next that G consists of exactly two components G_1 and G_2 such that, without loss of generality, $2d \leq n(G_1) \leq n(G_2) \leq 2d + 4$.

Case 1: Assume that $n(G_1) = 2d$. It follows that G_1 is isomorphic to $K_{d,d}$ and thus G_1 has a perfect matching. The hypothesis $|X| = |Y|$ implies that G_2 has a bipartition X_2, Y_2 with $|X_2| = |Y_2|$. Therefore, according to Theorem 3, the component G_2 has also a perfect matching, and we are done.

Case 2: Assume that $n(G_1) = n(G_2) = 2d + 2$. If G_1 and G_2 have partite sets X_1, Y_1 and X_2, Y_2 such that $|X_1| = |Y_1| = |X_2| = |Y_2| = d + 1$, then it follows from Theorem 3 that G_1 and G_2 have perfect matchings and so also G contains a perfect matching. In the remaining case, the components G_1 and G_2 have partite sets X_1, Y_1 and X_2, Y_2 such that, without loss of generality, $|X_1| = |X_2| = d$ and $|Y_1| = |Y_2| = d + 2$. However, since G is $(d, d + 1)$-graph, this is impossible, and the proof of Corollary 5 is complete. □

Note that the case $d = 1$ in Theorem 3 is trivial, since each $(1, 2)$-graph without an odd component has a perfect matching.

Finally notice that by a classical and well-known theorem of König [6], each d-regular bipartite graph contains a perfect matching for $d \geq 1$.

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