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A characterization of even order trees with domination number half their order minus one

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Abstract

We consider finite graphs G with vertex set $V(G)$. A subset $D \subseteq V(G)$ is a *dominating set* of the graph G, if every vertex $v \in V(G) - D$ is adjacent to at least one vertex in D. The *domination number* $\gamma(G)$ is the minimum cardinality among the dominating sets of G.

In this note, we characterize the trees T with an even number of vertices such that

$$\gamma(T) = \frac{|V(T)| - 2}{2}.$$

1 Terminology and introduction

We consider finite, undirected and simple graphs G with vertex set $V(G)$ and edge set $E(G)$. The number of vertices $|V(G)|$ of a graph G is called the *order* and is denoted by $n = n(G)$. The *neighborhood* $N(v) = N_G(v)$ of a vertex v consists of the vertices adjacent to v and $d(v) = d_G(v) = |N(v)|$ is the *degree* of v. A vertex of degree one is called a *leaf*. We denote with $L(G)$ the set of leaves of a graph G. If l is a leaf of G, we say the neighbor of l is the *support vertex* $s_G(l)$ of l. For a subset $S \subseteq V(G)$, we define by $G[S]$ the subgraph induced by S. If u and v are two vertices of G, then $d_G(u, v)$ denotes the distance between u and v in G.

Two vertices that are not adjacent in a graph G are said to be *independent*. A set I of vertices is *independent* if every two vertices of I are independent. A graph G without any cycle is called a *forest*, and a connected forest is a *tree*. For each vertex x in a graph G, we introduce a new vertex x' and join x and x' by an edge. The resulting graph is called the *corona* of G. A graph is said to be a *corona graph* if it is the corona of some graph.

A subset $D \subseteq V(G)$ is a *dominating set* of the graph G, if $|N_G(v) \cap D| \geq 1$ for every $v \in V(G) - D$. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality among the dominating sets of G. A dominating set of minimum cardinality in a graph G is called a $\gamma(G)$-set.

For a comprehensive treatment of domination in graphs, see the monographs by Haynes, Hedetniemi, and Slater [3], [4].

A well-known upper bound for the domination number of a graph was given by Ore [5] in 1962.

Proposition 1.1 ([5]) *If G is a graph without isolated vertices, then $\gamma(G) \leq n(G)/2$.*

In 1982, Payan and Xuong [6] and independently, in 1985, Fink, Jacobson, Kinch and Roberts [2] characterized the graphs achieving equality in Ore's bound.

Theorem 1.2 ([6], [2]) *Let G be a connected graph. Then $\gamma(G) = n(G)/2$ if and only if G is the corona graph of any connected graph J or G is isomorphic to the cycle C_4.*

In 1998, Randerath and Volkmann [7] and independently, in 2000, Xu, Cockayne, Haynes, Hedetniemi and Zhou [8] characterized the odd order graphs G for which $\gamma(G) = \lfloor n(G)/2 \rfloor$. The proof of this characterization is long and difficult (c.f. also [3], pp. 42–48). Thus we think that it is also hard to describe general even order graphs G with $\gamma(G) = \frac{n(G)-2}{2}$.

In this note, we characterize the trees T fulfilling $\gamma(T) = \frac{n(T)-2}{2}$. With this aim, we first show that an even order tree T with $\gamma(T) = \frac{n(T)-2}{2}$ contains two vertices u and v such that $T' = T - \{u, v\}$ is the corona graph of a forest. An example will demonstrate that this property is not valid for general graphs.

The next proposition by Bollobás and Cockayne [1] is the main tool for the proof of the structural property .

Proposition 1.3 ([1]) *Let G be a graph without isolated vertices, and let D be a $\gamma(G)$-set such that $|E(G[D])|$ is maximum. Then for all $d \in D$, there exists a vertex $f(d) \in (N[d] - N[D - \{d\}]) \cap (V(G) - D)$.*

2 Main result

Theorem 2.1 *Let T be a tree of even order $n \geq 4$. If $\gamma(T) = \frac{n-2}{2}$, then T contains two vertices u and v such that $T' = T - \{u, v\}$ is the corona graph of a forest.*

Proof. Assume that $n = 2q$ for an integer $q \geq 2$ and that $\gamma(T) = q - 1$. If $q = 2$, then T is isomorphic to the claw $K_{1,3}$ and we are done. Assume next that $q \geq 3$, and choose a $\gamma(T)$-set $D = \{a_1, a_2, \ldots, a_{q-1}\}$ of T such that $|E(T[D])|$ is maximum. In view of Proposition 1.3, for each $a_i \in D$, there exists a vertex $f_i = f(a_i) \in$

$(N[a_i] - N[D - \{a_i\}]) \cap (V(T) - D)$ for $1 \leq i \leq q-1$. Define $F = \{f_1, f_2, \ldots, f_{q-1}\}$, and let $\{x, y\} = V(T) - (D \cup F)$. Since D is a dominating set of T, the vertices x and y are adjacent to a vertex of D.

Case 1: Assume that x and y are adjacent to one vertex of D. Assume, without loss of generality, that x and y are adjacent to a_1.

First we will show that $T[F - \{f_1\}]$ is an empty graph. Suppose on the contrary that $T[F - \{f_1\}]$ contains an edge, say $f_2 f_3$. If a_2 and a_3 are leaves of T, then $D' = (D - \{a_2, a_3\}) \cup \{f_2, f_3\}$ is also a $\gamma(T)$-set with $|E(T[D'])| > |E(T[D])|$, a contradiction to the assumption that $|E(T[D])|$ is maximum. Next assume that a_2 or a_3, say a_2, is adjacent to a_i with $i = 1$ or $4 \leq i \leq q-1$. Then $D' = (D - \{a_2, a_3\}) \cup \{f_3\}$ is also a domination set of T with $|D'| < |D|$, a contradiction to the fact that D is a minimum dominating set. Finally, assume that a_2 or a_3 is only adjacent to x or y, say a_2 is adjacent to x and to no $a_i \in D$. It follows that $D' = (D - \{a_2, a_3\}) \cup \{x, f_3\}$ is also a $\gamma(T)$-set with $|E(T[D'])| > |E(T[D])|$, a contradiction to the assumption that $|E(T[D])|$ is maximum.

This implies that $T_1 = T - \{x, y, a_1, f_1\}$ is a corona graph of a forest, and f_i are leaves of T_1 for $2 \leq i \leq q-1$. If x or y or f_1, say f_1, is a leaf, then $T - \{x, y\}$ is a corona graph of a forest and we are done. Now we assume, without loss of generality, that f_1 is adjacent to f_2, x is adjacent to a_3 or f_3 and y is adjacent to a_4 or f_4.

Subcase 1.1: Assume that x is adjacent to a_3 and y is adjacent to a_4. Since T is a tree, we note that $\{a_1, a_2, a_3, a_4\}$ is an independent set. If there is an edge $a_2 a_i$ for $5 \leq i \leq q-1$, then $D' = (D - \{a_1, a_2\}) \cup \{f_1\}$ is a domination set, a contradiction. If there is an edge $a_1 a_i$ for $5 \leq i \leq q-1$, then $D' = (D - \{a_1, a_2\}) \cup \{f_2\}$ is a domination set, a contradiction. In the remaining case that there is no edge from $\{a_1, a_2\}$ to $\{a_3, a_4, \ldots, a_{q-1}\}$, we deduce that $D' = (D - \{a_1, a_2\}) \cup \{f_1, f_2\}$ is also a $\gamma(T)$-set with $|E(T[D'])| > |E(T[D])|$, a contradiction to the assumption that $|E(T[D])|$ is maximum.

Subcase 1.2: Assume that x is adjacent to f_3 and y is adjacent to a_4. Since T is a tree, $\{a_1, a_2, a_3, a_4\}$ is an independent set. If there is an edge $a_3 a_i$ for $5 \leq i \leq q-1$, then $D' = (D - \{a_1, a_2, a_3\}) \cup \{x, f_2\}$ is also a domination set of T with $|D'| < |D|$, a contradiction. If there is an edge $a_2 a_i$ for $5 \leq i \leq q-1$, then $D' = (D - \{a_1, a_2, a_3\}) \cup \{f_1, f_3\}$ is a domination set, a contradiction. If there is an edge $a_1 a_i$ for $5 \leq i \leq q-1$, then $D' = (D - \{a_1, a_2, a_3\}) \cup \{f_2, f_3\}$ is a domination set, a contradiction. In the remaining case that there is no edge from $\{a_1, a_2, a_3\}$ to $\{a_4, a_5, \ldots, a_{q-1}\}$, it follows that $D' = (D - \{a_1, a_2, a_3\}) \cup \{f_1, f_2, f_3\}$ is also a $\gamma(T)$-set with $|E(T[D'])| > |E(T[D])|$, a contradiction.

Subcase 1.3: Assume that x is adjacent to f_3 and y is adjacent to f_4. Again we note that $\{a_1, a_2, a_3, a_4\}$ is an independent set. If there is an edge $a_4 a_i$ for $5 \leq i \leq q-1$, then $D' = (D - \{a_1, a_2, a_3, a_4\}) \cup \{y, f_2, f_3\}$ is a domination set of T, a contradiction. If there is an edge $a_3 a_i$ for $5 \leq i \leq q-1$, then $D' = (D - \{a_1, a_2, a_3, a_4\}) \cup \{x, f_2, f_4\}$ is a domination set of T, a contradiction. If there is an edge $a_2 a_i$ for $5 \leq i \leq q-1$, then $D' = (D - \{a_1, a_2, a_3, a_4\}) \cup \{f_1, f_3, f_4\}$ is a domination set, a contradiction. If there is an edge $a_1 a_i$ for $5 \leq i \leq q-1$, then $D' = (D - \{a_1, a_2, a_3, a_4\}) \cup \{f_2, f_3, f_4\}$ is a domination set, a contradiction. In the remaining case that there is no edge from $\{a_1, a_2, a_3, a_4\}$ to $\{a_5, a_6, \ldots, a_{q-1}\}$, we observe that $D' = (D - \{a_1, a_2, a_3, a_4\}) \cup \{f_1, f_2, f_3, f_4\}$ is also a $\gamma(T)$-set with $|E(T[D'])| > |E(T[D])|$, a contradiction.

Subcase 1.4: Assume that x is adjacent to a_3 and y is adjacent to f_4. This case is symmetric to Subcase 1.

Case 2: Assume that x is adjacent to a_1 and y is adjacent a_2. Analogously to Case 1, one can show that $T_2 = T - \{x, y, a_1, f_1, a_2, f_2\}$ is a corona graph of a forest such that $T[\{f_3, f_4, \ldots, f_{q-1}\}$ is the empty graph. If f_1 or x and f_2 or y are leaves of T, then we arrive easily at the desired structure. Thus we assume next that neither f_2 nor y is a leaf of T.

Subcase 2.1: Assume that f_2 is adjacent to f_i with $3 \le i \le q-1$. We assume, without loss of generality, that f_2 is adjacent to f_3.

Subcase 2.1.1: Assume that y is adjacent to f_i with $4 \le i \le q-1$, say $i = 4$. Since T is a tree, we notice that $\{a_2, a_3, a_4\}$ is an independent set. If there is an edge a_2a_i with $i \ne 2, 3, 4$, then $D' = (D - \{a_2, a_3, a_4\}) \cup \{f_3, f_4\}$ is also a domination set of T with $|D'| < |D|$, a contradiction. If there is an edge a_3a_i with $i \ne 2, 3, 4$, then $D' = (D - \{a_2, a_3, a_4\}) \cup \{f_2, f_4\}$ is a domination set of T, a contradiction. If there is an edge a_4a_i with $i \ne 2, 3, 4$, then $D' = (D - \{a_2, a_3, a_4\}) \cup \{y, f_3\}$ is a domination set of T, a contradiction. In the remaining case that there is no edge from $\{a_2, a_3, a_4\}$ to $\{a_1, a_5, a_6, \ldots, a_{q-1}\}$, we conclude that $D' = (D - \{a_2, a_3, a_4\}) \cup \{f_2, f_3, f_4\}$ is also a $\gamma(T)$-set with $|E(T[D'])| > |E(T[D])|$, a contradiction to the assumption that $|E(T[D])|$ is maximum.

Subcase 2.1.2: Assume that y is adjacent to a_i with $4 \le i \le q-1$, say $i = 4$. Clearly, $\{a_2, a_3, a_4\}$ is an independent set. As in Subcase 2.1.1, one can show that a_2 and a_3 are not adjacent to a_i for $1 \le i \le q-1$. Thus $D' = (D - \{a_2, a_3\}) \cup \{f_2, f_3\}$ is also a $\gamma(T)$-set with $|E(T[D'])| > |E(T[D])|$, a contradiction.

Subcase 2.1.3: Assume that y is adjacent to a_1. Then we are in Case 1, and we are done.

Subcase 2.1.4: Assume that y is adjacent to f_1, and y is not adjacent to a_i or f_i for $4 \le i \le q-1$. If $d_T(x) = 1$, then $T - \{f_1, f_2\}$ is a corona graph of a forest. So we assume that $d_T(x) \ge 2$. Since T is a tree, this implies that x is adjacent to a_i or f_i for $i \ge 4$, say $i = 4$.

Subcase 2.1.4.1: Assume that x is adjacent to f_4. Then $\{a_1, a_2, a_3, a_4\}$ is an independent set, and similar to above, one can show that there is no edge from $\{a_1, a_2, a_3, a_4\}$ to $\{a_5, a_6, \ldots, a_{q-1}\}$. Thus $D' = (D - \{a_1, a_2, a_3, a_4\}) \cup \{f_1, y, f_3, f_4\}$ is also a $\gamma(T)$-set with $|E(T[D'])| > |E(T[D])|$, a contradiction.

Subcase 2.1.4.2: Assume that x is adjacent to a_4. Then $\{a_1, a_2, a_3\}$ is an independent set, and it is easy to verify that there is no edge from $\{a_1, a_2, a_3\}$ to $\{a_4, a_5, \ldots, a_{q-1}\}$. Thus $D' = (D - \{a_1, a_2, a_3\}) \cup \{f_1, y, f_3\}$ is also a $\gamma(T)$-set with $|E(T[D'])| > |E(T[D])|$, a contradiction.

Subcase 2.1.5: Assume that y is adjacent to x, and y is not adjacent to a_i or f_i for

$4 \leq i \leq q-1$. This case is similar to Subcase 2.1.4 and is therefore omitted.

Subcase 2.2: Assume that f_2 is adjacent to f_1, and that f_2 is not adjacent to f_i for $3 \leq i \leq q-1$.

Subcase 2.2.1: Assume that y is adjacent to f_i for $i \geq 3$, say $i = 3$. If $d_T(x) = 1$, then $T - \{f_1, y\}$ is a corona graph of a forest. So we assume that $d_T(x) \geq 2$. Since T is a tree, we observe that x is adjacent to a_i or f_i for $i \geq 4$, say $i = 4$.

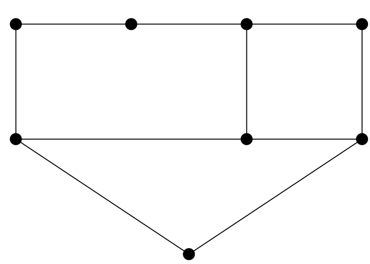
Subcase 2.2.1.1: Assume that x is adjacent to f_4. Then $\{a_1, a_2, a_3, a_4\}$ is an independent set, and there is no edge from $\{a_1, a_2, a_3, a_4\}$ to $\{a_5, a_6, \ldots, a_{q-1}\}$. Thus $D' = (D - \{a_1, a_2, a_3, a_4\}) \cup \{f_1, y, f_3, f_4\}$ is also a $\gamma(T)$-set with $|E(T[D'])| > |E(T[D])|$, a contradiction.

Subcase 2.2.1.2: Assume that x is adjacent to a_4. Then $\{a_1, a_2, a_3\}$ is an independent set, and there is no edge from $\{a_1, a_2, a_3\}$ to $\{a_4, a_5, \ldots, a_{q-1}\}$. Thus $D' = (D - \{a_1, a_2, a_3\}) \cup \{f_1, f_2, f_3\}$ is also a $\gamma(T)$-set with $|E(T[D'])| > |E(T[D])|$, a contradiction.

Subcase 2.2.2: Assume that y is adjacent to a_i for $i \geq 3$, say $i = 3$. This case is similar to Subcase 2.2.1 and is therefore omitted.

Subcase 2.3: Assume that f_2 is adjacent to x, and f_2 is not adjacent to f_i for $1 \leq i \leq q-1$. If $d_T(f_1) = 1$, then $T - \{x, y\}$ is a corona graph of a forest. If $d_T(f_1) \geq 2$, then f_1 is adjacent to a vertex f_i for $2 \leq i \leq q-1$. However, this situation is discussed before, and the proof is complete. □

The graph G of order 8 and $\gamma(G) = 3$ in the figure demonstrates that Theorem 2.1 is not valid in general.



Theorem 2.2 *Let T be a tree of even order $n \geq 4$. Then $\gamma(T) = \frac{n-2}{2}$ if and only if T is not the corona of a tree and there are two vertices $u, v \in V(T)$ such that $T' = T - \{u, v\}$ is the corona of a forest and, if Q is the subforest of T' consisting of all components of order 2, either*

(1) u as well as v have at most one neighbor in $L(T' - V(Q))$, or

(2) u is a leaf, there are exactly two vertices $l_1, l_2 \in L(T' - V(Q))$ adjacent to v and either

(i) $us_{T'}(l_1) \in E(T)$, or

(ii) $ul_1 \in E(T)$, *or*

(iii) $d_T(l_1) = 2$ *and there is a leaf* l *contained in the same component of* T' *as* l_1 *with* $ul \in E(T)$, $d_{T'}(l, l_1) = 3$.

Proof. Let T be a tree of even order $n \geq 4$ with $\gamma(T) = \frac{n-2}{2}$. Then, Theorem 1.2 yields directly that T is not the corona of a tree. Further, from Theorem 2.1, we know that T contains two vertices u and v such that $T' := T - \{u, v\}$ is the corona of a forest. Let S' be the set of support vertices of $T' - V(Q)$ and S'' the set of vertices of Q which are adjacent either to u or to v. Then the set $S := S' \cup S''$ contains $\frac{n-2}{2}$ vertices. Now we distinguish two cases.

Case 1: Suppose that $uv \in E(T)$. Let $l_1, l_2, \ldots, l_r$ be the leaves of $L(T' - V(Q))$ which are adjacent to v. As T is a tree, obviously these leaves belong pairwise to different components of $T' - V(Q)$. Then $(S - \{s_{T'}(l_1), s_{T'}(l_2), \ldots, s_{T'}(l_r)\}) \cup \{v\}$ is a dominating set of T. Since $|S| = \frac{n-2}{2}$ and $\gamma(T) = \frac{n-2}{2}$, it follows that $r \leq 1$. Analogously, u is adjacent to at most one leaf of $T' - V(Q)$. Thus, we obtain case (1).

Case 2: Suppose that $uv \notin E(T)$. If u and v have at most one neighbor in $L(T' - V(Q))$, we have case (1). Thus, assume that v has neighbors $l_1, l_2, \ldots, l_r \in L(T' - V(Q))$ for an $r \geq 2$. If $r \geq 3$, then $(S - \{s_{T'}(l_1), s_{T'}(l_2), \ldots, s_{T'}(l_r)\}) \cup \{u, v\}$ is a dominating set with less than $\frac{n-2}{2}$ vertices, a contradiction. Thus, we obtain that $r = 2$.

Subcase 2.1: Suppose that u has a neighbor $s \in S$. If $s \notin \{s_{T'}(l_1), s_{T'}(l_2)\}$, then $(S - \{s_{T'}(l_1), s_{T'}(l_2)\}) \cup \{v\}$ is a dominating set with $|S| - 1 = \frac{n-2}{2} - 1$ vertices, a contradiction to the hypothesis. As T is a tree, we thus can assume, without loss of generality, that $\{s_{T'}(l_1)\} = N(u) \cap S$. Hence, u is either a leaf of T or it is adjacent to a leaf of a component of $T' - V(Q)$ different from the ones containing l_1 and l_2. If the latter holds, say u is adjacent to a leaf $l \in L(T' - V(Q)) - \{l_1, l_2\}$, then $(S - \{s_{T'}(l_1), s_{T'}(l_2), s_{T'}(l)\}) \cup \{u, v\}$ is a dominating set of T with less vertices than $\frac{n-2}{2}$, which is not possible. Therefore, u is a leaf and we obtain case (2)(i).

Subcase 2.2: Assume that u has no neighbors in S. Since T is connected, there is a leaf l of T' which is adjacent to u. Let s be the support vertex of l in T'. If l belongs to a component of $T' - V(Q)$ different from the ones containing l_1, l_2, then $S - \{s_{T'}(l_1), s_{T'}(l_2), s_{T'}(l)\}) \cup \{u, v\}$ is a dominating set of T with $\frac{n-2}{2} - 1$ vertices, which is a contradiction. Hence, u is a leaf and its support vertex l belongs to, say, the component of $T' - V(Q)$ to which l_1 belongs. If $l = l_1$, then T satisfies (2)(ii). Thus, suppose that $d_T(l, l_1) \geq 3$. If $d_T(l, l_1) > 3$, then $S - \{s_{T'}(l_1), s_{T'}(l_2), s_{T'}(l)\} \cup \{v, l\}$ is a dominating set of T with $\frac{n-2}{n} - 1$ vertices, which is not possible. Hence, $d_T(l, l_1) = 3$. Suppose that $d(s_{T'}(l_1)) > 2$. Then $s_{T'}(l_1)$ has a neighbor in S different from $s_{T'}(l)$ and we obtain again that $S - \{s_{T'}(l_1), s_{T'}(l_2), s_{T'}(l)\} \cup \{v, l\}$ is a dominating set of T' with $\frac{n-2}{2} - 1$ vertices. Hence, l_1 and $s_{T'}(l)$ are the only neighbors of $s_{T'}(l_1)$, which implies that T satisfies (2)(iii).

We prove the converse. Suppose that T is a tree as described in the theorem. Since T is not the corona of a tree and n is even, we deduce with Theorem 1.2 that

$\gamma(T) \le \frac{n-2}{2}$. Let D be a minimum dominating set of T. Again, let S' be the set of support vertices of $T' - V(Q)$, S'' the set of vertices of Q which are adjacent either to u or to v and $S := S' \cup S''$. Again, S contains $\frac{n-2}{2}$ vertices.

Let T have property (1). Suppose first that $uv \in E(T)$. If neither u nor v are adjacent to a leaf of $T' - V(Q)$, then we can suppose, without loss of generality, that $S \subseteq D$ and hence $\gamma(T) \ge \frac{n-2}{2}$. If u is adjacent to a leaf l of $T' - V(Q)$, then we can assume that $S - \{s_{T'}(l)\} \subseteq D$. But $D \ne S - \{s_{T'}(l)\}$, otherwise l and u would not be dominated. Hence $\gamma(T) \ge |S - \{s_{T'}(l)\}| + 1 = \frac{n-2}{2}$. If u and v are adjacent to leaves l_u and, respectively, l_v of $T' - V(Q)$, then we may assume that $S - \{s_{T'}(l_u), s_{T'}(l_v)\} \subseteq D$. Again, in order to dominate the remaining vertices, we need in this case at least 2 more vertices and thus $\gamma(T) \ge |S - \{s_{T'}(l_u), s_{T'}(l_v)\}| + 2 = \frac{n-2}{2}$. Therefore, we obtain in all these cases $\gamma(T) = \frac{n-2}{2}$.

Suppose now that $uv \notin E(T)$. If neither u nor v are adjacent to a leaf of $T' - V(Q)$, then, again, we can suppose that $S \subseteq D$ and hence $\gamma(T) \ge \frac{n-2}{2}$. Suppose that u is adjacent to exactly a leaf l of $T' - V(Q)$ and v to none. Then we can assume that $S - \{s_{T'}(l)\} \subseteq D$. If v has a neighbor in $S - \{s_{T'}(l)\}$, since l is not dominated by $S - \{s_{T'}(l)\}$, we need at least one more vertex and hence $\gamma(T) \ge |S - \{s_{T'}(l)\}| + 1 = \frac{n-2}{2}$. If v is only adjacent to $s_{T'}(l)$, then u cannot be a leaf, otherwise T would be the corona of a tree. Thus, u has a neighbor $s \in S - \{s_{T'}(l)\}$ and, since every vertex in S is adjacent to a leaf of T, without loss of generality, we can suppose that $S \subseteq D$ and hence $\gamma(T) \ge \frac{n-2}{2}$. If u and v are adjacent to leaves l_u and, respectively, l_v of $T' - V(Q)$, we may assume that $S - \{s_{T'}(l_u), s_{T'}(l_v)\}$ is contained in D. In the case that $l_u = l_v$, we need one more vertex in order to dominate the remaining vertices, and in the case that $l_u \ne l_v$, we need two. Hence, in both cases we obtain that $\gamma(T) \ge |S - \{s_{T'}(l_u), s_{T'}(l_v)\}| + |\{l_u, l_v\}| = \frac{n-2}{2}$. All in all, it follows that $\gamma(T) = \frac{n-2}{2}$.

Suppose that T has property (2)(i), that is, there are exactly two leaves $l_1, l_2 \in L(T' - V(Q))$ adjacent to v, u is a leaf and $us_{T'}(l_1) \in E(T)$. Then, without loss of generality, $S - \{s_{T'}(l_2)\} \subseteq D$ and, to dominate the rest, we need at least one more vertex. Thus $\gamma(T) \ge |S - \{s_{T'}(l_2)\}| + 1 = \frac{n-2}{2}$ and hence $\gamma(T) = \frac{n-2}{2}$.

Next, assume that T has property (2)(ii), that is, there are exactly two leaves $l_1, l_2 \in L(T' - V(Q))$ adjacent to v, u is a leaf and $ul_1 \in E(T)$. Then we may assume that $S - \{s_{T'}(l_1), s_{T'}(l_2)\} \subseteq D$ and, in order to dominate the remaining vertices, at least 2 more vertices are required, which implies that $\gamma(T) \ge |S - \{s_{T'}(l_1), s_{T'}(l_2)\}| + 2 = \frac{n-2}{2}$ and hence $\gamma(T) = \frac{n-2}{2}$.

Finally, let T have property (2)(iii). Without loss of generality, we can assume that $S - \{s_{T'}(l_1), s_{T'}(l_2), s_{T'}(l_r)\}) \cup \{l\} \subseteq D$. Since, in order to dominate the vertices v,l_1,l_2 and $s_{T'}(l_1)$, two more vertices are required, we obtain that $|D| \ge \frac{n-2}{2}$ and hence $\gamma(T) = \frac{n-2}{2}$ and we are done. $\square$

References

[1] B. Bollobás and E.J. Cockayne, Graph-theoretic parameters concerning domination, independence, and irredundance, *J. Graph Theory* **3** (1979), 241–249.

[2] J.F. Fink, M.S. Jacobson, L. Kinch and J. Roberts, On graphs having domination number half their order, *Period. Math. Hungar.* **16** (1985), 287–293.

[3] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York (1998).

[4] T.W. Haynes, S.T. Hedetniemi and P.J. Slater (eds.), *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York (1998).

[5] O. Ore, *Theory of graphs*, Amer. Math. Soc. Colloq. Publ. **38** (Amer. Math. Soc., Providence, R1), 1962.

[6] C. Payan and N.H. Xuong, Domination-balanced graphs, *J. Graph Theory* **6** (1982), 23–32.

[7] B. Randerath and L. Volkmann, Characterizations of graphs with equal domination and covering number, *Discrete Math.* **191** (1998), 159–169.

[8] B. Xu, E.J. Cockayne, T.W. Haynes, S.T. Hedetniemi and S. Zhou, Extremal graphs for inequalities involving domination parameters, *Discrete Math.* **216** (2000), 1–10.

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