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On almost 1-extendable graphs

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Abstract

A graph G is 1-extendable or almost 1-extendable if every edge is contained in a perfect or almost perfect matching of G, respectively. Let $d \geq 3$ be an integer, and let G be a graph of order n with exactly one odd component such that the degree of each vertex is either d or $d+1$. If G is not almost 1-extendable, then we prove that $n \geq 2d+5$. In the special case that $d \geq 4$ is even and G is a d-regular graph, we obtain the better bound $n \geq 3d+5$. Examples will show that the given bounds are best possible.

We shall assume that the reader is familiar with standard terminology on graphs (see, e.g., Chartrand and Lesniak [2]). In this paper, all graphs are finite and simple. The vertex set of a graph G is denoted by $V(G)$. The *neighborhood* $N_G(x) = N(x)$ of a vertex x is the set of vertices adjacent with x, and the number $d_G(x) = d(x) = |N(x)|$ is the *degree* of x in the graph G. If $d \leq d_G(x) \leq d+k$ for each vertex x in a graph G, then we speak of a *close to regular graph* or more precisely of a $(d, d+k)$-*graph*. If $k = 0$, then the graph is *d-regular*. If X is a subset of the vertex set of a graph G, then $G[X]$ is the subgraph induced by X. A *perfect matching* (*almost perfect matching*) of a graph G, is a matching M in G with the property that every vertex (with exactly one exception) is incident with an edge of M. We denote by K_n the complete graph of order n and by $K_{r,s}$ the complete bipartite graph with partite sets A and B, where $|A| = r$ and $|B| = s$. If G is a graph and $A \subseteq V(G)$, then we denote by $q(G-A)$ the number of odd components in the subgraph $G-A$.

A graph G is p-*extendable* if it contains a set of p independent edges and every set of p independent edges can be extended to a perfect matching. In 1980, Plummer [9] studied the properties of p-extendable graphs. We call a graph G *almost* p-*extendable* if it contains a set of p independent edges and every set of p independent edges can be extended to an almost perfect matching.

In 1975, Little, Grant and Holton [7] characterized the family of 1-extendable graphs as follows.

Theorem 1 (Little, Grant, Holton [7] 1975) *A graph G of even order is* 1-*extendable if and only if for any $A \subseteq V(G)$*

(1) $q(G-A) \le |A|$ *and*

(2) $q(G-A) = |A|$ *implies that* $G[A]$ *is an empty graph.*

As an application of Theorem 1, Volkmann [11] has shown recently that if a d-regular graph with $d \ge 3$ of order n without odd components is not 1-extendable, then $n \ge 2d+4$. In this note we present an analogue for almost 1-extendable graphs. For similar results, we refer the reader to Wallis [12], Zhao [13], Cacetta and Mardiyono [1], Volkmann [10] and Klinkenberg and Volkmann [4], [5], [6].

As a generalization of Theorem 1, Liu and Yu [8] have given a characterization of so called (p,q)-extendable graphs. For our proofs we mainly use the following special case of this chracterization for $p=1$ and $2q=n-1$.

Theorem 2 (Liu, Yu [8] 2001) *A graph G of odd order is almost* 1-*extendable if and only if for any $A \subseteq V(G)$*

(1) $q(G-A) \le |A|+1$ *and*

(2) $q(G-A) = |A|+1$ *implies that* $G[A]$ *is an empty graph.*

Theorem 3 *Let $d \ge 4$ be an even integer, and let G be a d-regular graph of order n with exactly one odd component. If G is not almost* 1-*extendable, then $n \ge 3d+5$.*

Proof. Suppose to the contrary that there exists a d-regular graph G of order $n \le 3d+3$ with exactly one odd component which is not almost 1-extendable. Then it follows from the hypothesis and Theorem 2 that there exists a non-empty set $A \subseteq V(G)$ such that $q(G-A) \ge |A|+2$ or $q(G-A) = |A|+1$ and $G[A]$ contains an edge.

We call an odd component of $G-A$ large if it has more than d vertices and small otherwise. We denote by α and β the number of large and small components of $G-A$, respectively. Since G is a d-regular graph with exactly one odd component, it is easy to see that there are at least d edges in G joining each small component of $G-A$ with A and at least two edges per large component with one possible exception. The d-regularity of G therefore implies

$$2(\alpha-1) + d\beta \le d|A|. \tag{1}$$

Case 1. Assume that $q(G-A) \ge |A|+2$. Since n is odd, the numbers $q(G-A)$ and $|A|$ are of different parity, and we deduce that

$$\alpha+\beta = q(G-A) \ge |A|+3. \tag{2}$$

Inequality (1) yields $\beta \leq |A|$ and thus (2) leads to $\alpha \geq 3$. Applying the assumption $n \leq 3d+3$, and using the fact that $A \neq \emptyset$, we obtain the contradiction

$$3d+3 \geq n \geq |A| + \alpha(d+1) + \beta \geq |A| + 3(d+1) > 3d+3. \tag{3}$$

Case 2: Assume that $\alpha + \beta = q(G-A) = |A|+1$ and that $G[A]$ contains an edge. This implies that $|A| \geq 2$ and instead of (1), we deduce that

$$2(\alpha - 1) + d\beta \leq d|A| - 2. \tag{4}$$

If $\alpha \geq 3$, then we arrive a contradiction as in (3). In the case $\alpha = 0$, we have $\beta = |A|+1$, a contradiction to (1). If $\alpha = 1$, then $\beta = |A|$, a contradiction to (4).

It remains the case that $\alpha = 2$ and thus there exists at least one small component in $G-A$. If U is a small component of minimum order in $G-A$, then we observe that

$$|V(U)| \geq d - |A| + 1. \tag{5}$$

Now our assumption $n \leq 3d+3$ leads to

$$3d+3 \geq n \geq |A| + 2(d+1) + (|A|-1)|V(U)|. \tag{6}$$

Subcase 2.1: Assume that $|A| \geq d$. Then the hypothesis $d \geq 4$ and (6) yield the contradiction

$$\begin{aligned} 3d+3 \geq n &\geq 3d+2+(|A|-1)|V(U)| \\ &\geq 3d+2+d-1 = 4d+1 \\ &> 3d+3. \end{aligned}$$

Subcase 2.2: Assume that $d-1 \geq |A| \geq 3$. Then (5) and (6) lead to the contradiction

$$\begin{aligned} 3d+3 \geq n &\geq |A| + 2(d+1) + (|A|-1)|V(U)| \\ &\geq |A| + 2(d+1) + (|A|-1)(d-|A|+1) \\ &\geq |A| + 2(d+1) + 2(d-|A|+1) \\ &\geq |A| + 2(d+1) + (d-|A|+1) + 2 \\ &= 3d+5. \end{aligned}$$

Subcase 2.3: Assume that $|A| = 2$. Now (5) implies $|V(U)| \geq d-1$ and thus $|V(U)| = d-1$. Hence there are at least $2d-2$ edges joining U with A, and we finally arrive at the contradiction

$$2(\alpha-1) + 2(d-1) = 2d \leq d|A| - 2 = 2d-2.$$

□

Remark 4 It is obvious that each 2-regular graph with exactly one odd component is almost 1-extendable.

Example 5 Let $d \geq 4$ be an even integer. In addition, let H be a complete Graph K_2 with vertex set u, v, let H_1 be a complete graph K_{d+1} without a matching $M_1 = \{x_1x_2, x_3x_4, \ldots, x_{d-3}x_{d-2}\}$, let H_2 be a complete graph K_{d+1} without a matching $M_2 = \{y_1y_2, y_3y_4, \ldots, y_{d-3}y_{d-2}\}$, and let H_3 be a complete graph K_{d+1} without an edge ab. Now we define the graph G of order $3d+5$ as the disjoint union of H, H_1, H_2 and H_3 together with the edges au, bv, ux_i as well as vy_i for $1 \leq i \leq d-2$. The resulting graph G is d-regular, however, the edge uv is not contained in an almost perfect matching of G. This example shows that Theorem 3 is best possible.

Corollary 6 (Volkmann [11]) *Let $d \geq 4$ be an even integer, and let G be a d-regular graph of order n without odd components. If G is not 1-extendable, then $n \geq 2d+4$.*

Proof. Suppose to the contrary that $n \leq 2d+2$. The disjoint union $H = G \cup K_{d+1}$ is a d-regular graph with exactly one odd component, and H is not almost 1-extendable. Because of $|V(H)| \leq 3d+3$, this is a contradiction to Theorem 3. □

Theorem 7 *Let $d \geq 3$ be an integer, and let G be a $(d, d+1)$-graph with exactly one odd component. If G is not almost 1-extendable, then $n \geq 2d+5$.*

Proof. Suppose to the contrary that there exists a $(d, d+1)$-graph G of order $n \leq 2d+3$ with exactly one odd component which is not almost 1-extendable. Then it follows from the hypothesis and Theorem 2 that there exists a non-empty set $A \subseteq V(G)$ such that $q(G-A) \geq |A|+2$ or $q(G-A) = |A|+1$ and $G[A]$ contains an edge. Note that this implies $A \neq \emptyset$.

We call an odd component of $G-A$ large if it has more than d vertices and small otherwise. We denote by α and β the number of large and small components of $G-A$, respectively. Since G is a $(d, d+1)$-graph with exactly one odd component, it is easy to see that there are at least d edges in G joining each small component of $G-A$ with A and at least one edge per large component with one possible exception. This implies that

$$\alpha - 1 + d\beta \leq (d+1)|A|. \tag{7}$$

Applying the assumption $n \leq 2d+3$, we obtain

$$2d+3 \geq n \geq |A| + \alpha(d+1) + \beta. \tag{8}$$

Case 1. Assume that $q(G-A) \geq |A|+2$. Since n is odd, the numbers $q(G-A)$ and $|A|$ are of different parity, and we deduce that

$$\alpha + \beta = q(G-A) \geq |A| + 3. \tag{9}$$

If $\alpha \geq 3$, then (8) immediately leads to a contradiction. If $\alpha = 2$, then (9) implies $\beta \geq 2$, and (8) yields the contradiction

$$2d+3 \geq n \geq |A| + 2(d+1) + \beta \geq 2d+5.$$

If $\alpha = 1$, then it follows from (9) that $\beta \geq |A| + 2 \geq 3$ and thus (7) leads to the inequality $|A| \geq 2d$. Applying (8), we obtain the contradiction

$$2d + 3 \geq n \geq |A| + (d + 1) + \beta \geq 3d + 4.$$

If $\alpha = 0$, then it follows from (9) that $\beta \geq |A| + 3 \geq 4$ and thus by (7), the bound $|A| \geq 3d - 1$. In view of (8), we arrive at the contradiction

$$2d + 3 \geq n \geq |A| + \beta \geq 3d + 3.$$

Case 2: Assume that

$$\alpha + \beta = q(G - A) = |A| + 1 \tag{10}$$

and that $G[A]$ contains an edge. This implies that $|A| \geq 2$.

If $\alpha \geq 2$, then (8) immediately leads to a contradiction. If $\alpha = 0$, then (10) implies $\beta = |A| + 1$, and we have $d\beta \leq (d + 1)|A| - 2$. It follows that $|A| \geq d + 2$, and hence (8) yields the contradiction

$$2d + 3 \geq n \geq |A| + \beta = 2|A| + 1 \geq 2d + 5.$$

It remains the case that $\alpha = 1$, and thus there exist $\beta = |A| \geq 2$ small components in $G - A$. If U is a small component of minimum order in $G - A$, then we observe that

$$|V(U)| \geq d - |A| + 1. \tag{11}$$

Now our assumption $n \leq 2d + 3$ leads to

$$2d + 3 \geq n \geq |A| + (d + 1) + |A||V(U)|. \tag{12}$$

Subcase 2.1: Assume that $|A| \geq d$. Since $d \geq 3$, inequality (12) yields the contradiction

$$2d + 3 \geq n \geq |A| + d + 1 + |A||V(U)| \geq 3d + 1.$$

Subcase 2.2: Assume that $d - 1 \geq |A| \geq 2$. Then (11) and (12) lead to the contradiction

$$\begin{aligned} 2d + 3 \geq n &\geq |A| + (d + 1) + |A||V(U)| \\ &\geq |A| + (d + 1) + |A|(d - |A| + 1) \\ &\geq |A| + (d + 1) + 2(d - |A| + 1) \\ &\geq |A| + (d + 1) + (d - |A| + 1) + 2 \\ &= 2d + 4. \end{aligned}$$

□

Remark 8 The graph G consisting of the disjoint union of a triangle and a diamond shows that Theorem 7 is not valid for $d = 2$ in general. However, it remains true for $d = 2$ when G is connected.

Example 9 Let $d \geq 3$ be an integer, and let H be a bipartite graph with the partite sets A and B with $|A| = d+2$ and $|B| = d+3$ such that $d_H(x) = d$ for all $x \in B$ and $d \leq d_H(y) \leq d+1$ for $y \in A$. These conditions imply that there are exactly two vertices u and v in A such that $d_H(u) = d_H(v) = d$. If we add the edge uv to H, then we obtain a $(d, d+1)$-graph G with the property that the edge uv is not contained in an almost perfect matching of G. Thus G is not almost 1-extendable. This example shows that Theorem 7 is best possible.

Observation 10 Let $d, k \geq 2$ be integers, and let G be a $(d, d+k)$-graph of order n with exactly one odd component. If $n \leq 2d-1$, then G is almost 1-extendable.

Proof. Let uv be an arbitray edge of G, and define the graph $H = G - \{u, v\}$. Then H is a $(d-2, d+k)$ graph of odd order such that $n(H) \leq 2d-3$. By the classical theorem of Dirac [3], H has a Hamiltonian path. Consequently, the edge uv is contained in an almost perfect matching of G. This implies that G is almost 1-extendable. □

Example 11 Let $d \geq 2$ be an integer, and let H be the complete bipartite graph $K_{d,d+1}$ with the partite sets A and B with $|A| = d$ and $|B| = d+1$. If we add the edge uv to H with $u, v \in A$, then we obtain a $(d, d+2)$-graph G with the property that the edge uv is not contained in an almost perfect matching of G. Thus G is not almost 1-extendable. This example shows that Observation 10 is best possible.

References

[1] L. Cacetta and S. Mardiyono, On the existence of almost-regular-graphs without one-factors, *Australas. J. Combin.* **9** (1993), 243–260.

[2] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, 3rd Edition, Chapman and Hall, London, 1996.

[3] G.A. Dirac, Some theorems on abstract graphs, *Proc. London Math. Soc.* (3) **2** (1952), 69–81.

[4] S. Klinkenberg and L. Volkmann, On the order of almost regular graphs without a matching of given size, *Australas. J. Combin.* **34** (2006), 187–194.

[5] S. Klinkenberg and L. Volkmann, On the order of close to regular graphs without a matching of given size, *Ars Combin.* **85** (2007), 98–106.

[6] S. Klinkenberg and L. Volkmann, On the order of certain close to regular graphs without a matching of given size, *Czecheslovak Math. J.* **57** (2007), 907–918.

[7] C.H. Little, D.D. Grant and D.A. Holton, On defect-d matchings in graphs, *Discrete Math.* **13** (1975), 41–54.

[8] G. Liu and Q.L. Yu, Generalizations of matching extension in graphs, *Discrete Math.* **231** (2001), 311–320.

[9] M.D. Plummer, On n-extendable graphs, *Discrete Math.* **31** (1980), 201–210.

[10] L. Volkmann, On the size of odd order graphs with no almost perfect matching, *Australas. J. Combin.* **29** (2004), 119–126.

[11] L. Volkmann, The smallest regular graphs which are not 1-extendable, *Ars. Combin.* (to appear).

[12] W.D. Wallis, The smallest regular graphs without one-factors, *Ars Combin.* **11** (1981), 295–300.

[13] C. Zhao, The disjoint 1-factors of $(d, d+1)$-graphs, *J. Combin. Math. Combin. Comput.* **9** (1991), 195–198.

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