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# The minimum number of vertices of graphs containing two monochromatic triangles for any edge 2-coloring

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**Abstract**

We show that the minimum number of vertices of $K_6$-free graphs containing (at least) two monochromatic triangles for any edge 2-coloring is ten, giving a concrete (minimal) graph on ten vertices with such a property. Moreover, we show the uniqueness of the graph of all $K_6$-free graphs on (at most) ten vertices.

## 1 Introduction

Ramsey theory, initiated by Ramsey [22], is one of the most important areas of combinatorics. Ramsey theory studies how many elements of some structure there need to be to guarantee that a particular property on the structure holds. (See [9] as a classical textbook.) The simplest problem in graph theoretic Ramsey theory is to ask for the minimum number of vertices of complete graphs, say, $K_n$, such that there is one monochromatic triangle (i.e., $K_3$) for any edge 2-coloring. The answer to this question is six, that is, $K_6$. In fact, there are at least two monochromatic triangles in $K_6$ for any 2-coloring; Harary [11] listed all 2-colorings of $K_6$ which result in exactly two monochromatic triangles. Thus, any graph containing $K_6$ satisfies such a property.

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From this fact, it is natural to ask for a structure of graphs with such a property that *does not* contain $K_6$, which was posed by Erdős and Hajnal [5]. Here, we say that a graph $G = (V, E)$ satisfies property $\mathtt{Z}_1$ if for any 2-coloring to $E$, there is (at least) one monochromatic triangle in $G$. In what follows, we consider graphs that do not contain $K_6$, called $K_6$*-free* graphs. Answering the question posed by Erdős and Hajnal [5], Graham [8] presented a $K_6$-free graph satisfying $\mathtt{Z}_1$, called here the *Graham graph*, which is on eight vertices, depicted in Figure 3 in the next section. In fact, it is unique of all $K_6$-free graphs satisfying $\mathtt{Z}_1$ on (at most) eight vertices.

More generally, a graph is *minimal* (with respect to 2-colorings to edges and monochromatic triangles) if the graph does not properly contain any graph satisfying $\mathtt{Z}_1$. Thus, $K_6$ as well as the Graham graph are both minimal. Following the Graham graph, Nenov [16] presented a minimal graph on nine vertices, called here the *Nenov graph* (see Figure 1). Moreover, it is unique of all minimal graphs on nine vertices, and all minimal graphs on at most thirteen vertices are known in [1].

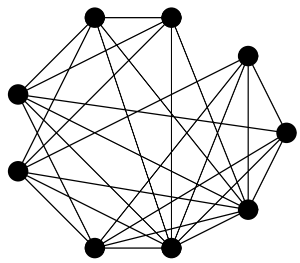


Figure 1: The Nenov graph

In this paper, we generalize the study on minimal graphs towards another way, in which we care for the number of monochromatic triangles. This is motivated by the fact that there are at least *two* monochromatic triangles in $K_6$ itself for any 2-coloring. That is, we further ask for a structure of $K_6$-free graphs such that there are at least *two* triangles for any edge 2-coloring. In fact, the Graham graph does not satisfy this property. (See the colorings presented in Figure 4.) Furthermore, neither does the Nenov graph [16], and all minimal graphs on at most thirteen vertices in [1] except $K_6$ do not have this property. Thus, the lower bound on the minimum number of vertices of graphs with such a property is nine. It is easy to see that the upper bound is eleven, which is guaranteed by the graph obtained by combining two Graham graphs via sharing the two cycles of size five. Note that the graph does not share any triangle in the two Graham graphs. Thus, given any 2-coloring, at least one monochromatic triangle comes from each of the two Graham graphs, giving (at least) two monochromatic triangles in total.

We show that the minimum number of vertices is ten, giving a concrete graph $G_0$ on ten vertices with such a property (see Figure 2)[1]. In fact, the graph $G_0$ contains four Graham graphs in an elaborate way. Thus, it takes more benefit from the Graham graph than the Nenov graph. In fact, the graph on ten vertices does not

[1]The graph $G_0$ is a join of $K_2$ and the maximal graph of order 8 showing the Ramsey number $R(4, 3) \geq 9$.

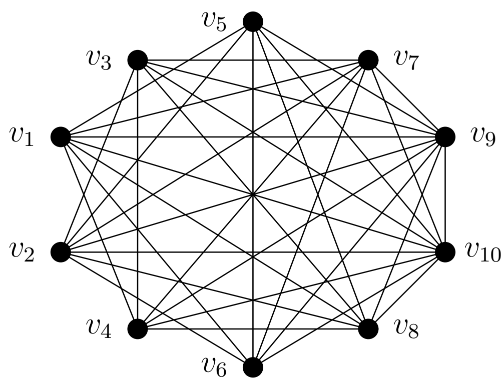



Figure 2: The graph $G_0$

contain the Nenov graph. Moreover, we show the uniqueness of $G_0$ for all $K_6$-free graphs on (at most) ten vertices.

**Theorem 1.1.** *The minimum number of vertices of $K_6$-free graphs containing (at least) two monochromatic triangles for any edge 2-coloring is ten. Moreover, the graph $G_0$ is unique of all $K_6$-free graphs with such property on (at most) ten vertices.*

### Related results

Generalizing the stream of research raised by Erdős and Hajnal [5], and Graham [8], Folkman [6] introduced a new concept similar to the Ramsey number, called the *Folkman number*. Let $\mathcal{F}(r,k,l)$ with $k < l$ be the set of $K_l$-free graphs $G$ such that every edge $r$-coloring of $G$ produces a monochromatic $K_k$. The *Folkman number* (or *edge Folkman number*) $f(r,k,l)$ is the minimum order of $G \in \mathcal{F}(r,k,l)$, i.e., $f(r,k,l) = \min_{G\in\mathcal{F}(r,k,l)} |V(G)|$, where $V(G)$ denotes the vertex set of $G$. It is shown in [6] that $\mathcal{F}(2,k,l) \neq \emptyset$, and more generally, $\mathcal{F}(r,k,l) \neq \emptyset$ for any $r \geq 2$ holds [18]. Observe that $f(2,3,l) = 6$ for any $l > 6$ by the fact on $K_6$. Thus the most interesting and important study on the Folkman number is to determine $f(2,3,l)$ for $4 \leq l \leq 6$. For $l = 6$, we see $f(2,3,6) = 8$ from the Graham graph. For $l = 5$, the upper bound by Nenov [17] and the lower bound by Piwakowski, Radziszowski, and Urbański [19] determined $f(2,3,5) = 15$. For $l = 4$, the lower and upper bounds of $f(2,3,4)$ are summarized in Table 1.

As is mentioned above, it is known that every edge 2-coloring of $K_6$ produces at least two monochromatic triangles. On the other hand, the Graham graph admits an edge 2-coloring with exactly one monochromatic triangle. Thus, it is natural to extend the Folkman number $f(r,k,l)$ to $f_s(r,k,l)$ along with the number $s$ of monochromatic $K_k$. In this notation, $f(r,k,l) = f_1(r,k,l)$, and Theorem 1.1 states $f_2(2,3,6) = 10$.

| year | authors | lower | upper |
|---|---|---|---|
| 1970 | Frankl and Rödl [7] | | $8 \cdot 10^{11}$ |
| 1988 | Spencer [23] | | $3 \cdot 10^{9}$ |
| 2008 | Lu [14] | | 9697 |
| 2008 | Dudek and Rödl [4] | | 941 |
| 2014 | Lange, Radziszowski, and Xu [15] | | 786 |
| 2017 | Bikov and Nenov [2] | 20 | |
| 2020 | Bikov and Nenov [3] | 21 | |

Table 1: The summary of the upper and lower bounds for $f(2,3,4)$

**Organization**

In Section 2, we depict the Graham graph, denoted by GH, as well as two edge 2-colorings of GH, which are used in the sequel. In Section 3, we enumerate all maximal $K_6$-free graphs on ten vertices, and show that there is an edge 2-coloring for all of them, except for $G_0$, such that at most one monochromatic triangle exists. In Section 4, we present the exceptional graph $G_0$ as well as its minimality and we prove Theorem 1.1.

## 2 Preliminaries

In this paper, we mostly follow the standard notation and concepts of graph theory. For example, $P_n, C_n$, and $K_n$ are a path graph, a cycle graph, and a complete graph on $n$ vertices, respectively. For a graph $G = (V, E)$, a path $v_1, \ldots, v_k \in V$ (respectively a cycle $u_1, \ldots, u_\ell, u_1 \in V$) in $G$ is denoted by $P_k = (v_1, \ldots, v_k)$ (respectively $C_\ell = (u_1, \ldots, u_\ell)$). Thus, an edge $e \in E$ is denoted by $e = (u, v)$ for the end vertices $u$ and $v$. We call $K_3$ a *triangle*. The complement graph of $G$ is denoted by $\overline{G}$. Thus, $\overline{K_n}$ is the empty graph, which is the graph on $n$ vertices that does not have any edges. For a subset $V' \subseteq V$, the induced subgraph of $G$ by $V'$ is denoted by $G[V']$.

In what follows, for a graph $G = (V, E)$, the set of vertices of $G$ is denoted by $V(G)$, and the set of edges of $G$ by $E(G)$. (That is, $V = V(G)$ and $E = E(G)$.) We say that a graph $G = (V, E)$ *contains* a graph $G' = (V', E')$ if $V' \subseteq V$ and $E' \subseteq E$, which is (crudely) denoted by $G' \subseteq G$. Furthermore, for a subset $E' \subseteq E$, we (crudely) denote the graph $G' = (V, E \setminus E')$ by $G' = G \setminus E'$. An *isolated vertex* of a graph $G$ is a vertex which any edge of $G$ does not have as an endpoint. For a graph $G = (V, E)$, a set $V' \subseteq V$ is called an *independent set* if the induced subgraph $G[V']$ by $V'$ is an empty graph. The *independence number* of $G$, denoted by $\alpha(G)$, is the maximum size of independent sets of $G$. A graph $G$ is *$\alpha$-critical* if for any edge $e \in E(G)$, $\alpha(G \setminus \{e\}) = \alpha(G) + 1$.

In this paper, we focus on the simplest case of graph theoretic Ramsey theory, that is, edge 2-colorings and monochromatic triangles. In what follows, we use blue and red as two colors. (In fact, we use blue and red for coloring edges of graphs depicted below in figures.)

**Definition 2.1.** For a natural number $r$, let $R(r)$ be the minimum number $n$ of vertices such that there is a monochromatic $K_r$ in $K_n$ for any 2-coloring to $E(K_n)$. (Note that $R(r)$ is the same as the Ramsey number $R(r, r)$.)

**Fact 1.** $R(3) = 6$.

In fact, if we differently color the inside $C_5$ and the outside $C_5$ of $E(K_5)$, there is no monochromatic triangle in $K_5$. Moreover, it is easy to see (by a counting argument) that there are at least two monochromatic triangles in $K_6$ for any 2-coloring. From this fact, there is one (further at least two) monochromatic triangle in any graph containing $K_6$. Thus, people search for a structure of graphs that have one monochromatic triangle for any 2-coloring but do not contain $K_6$.

**Definition 2.2** ([8])**.** We call the graph depicted in the left in Figure 3 the *Graham graph*, denoted by GH. The graph in the right is the complement of the Graham graph, denoted by $\overline{\texttt{GH}}$.

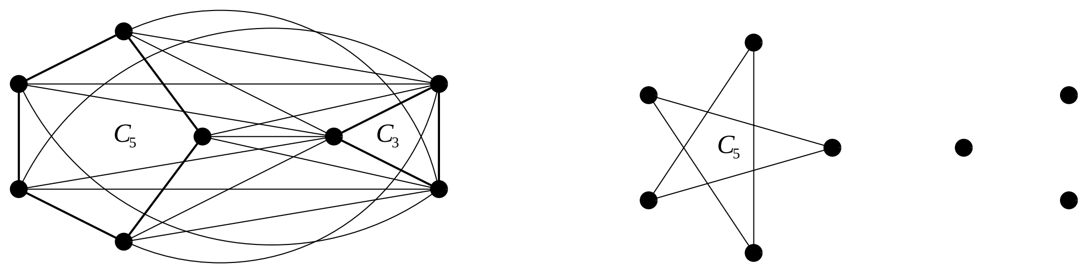


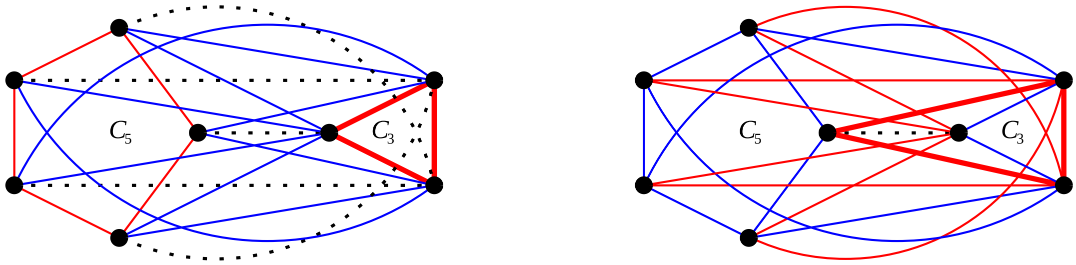

Figure 3: The Graham graph

Note that GH consists of $C_5$, $C_3$, and edges between $C_5$ and $C_3$. Thus, emphasizing the two cycles, we sometimes denote it by GH$(C_5, C_3)$, and depict it omitting all the edges between $C_5$ and $C_3$. Note further that the complement of $C_5$ is again $C_5$.

**Fact 2** ([8])**.** *The Graham graph does not contain $K_6$, and there is at least one monochromatic triangle in GH for any 2-coloring to $E($GH$)$. Furthermore, the structure of GH is unique of all graphs with such a property on at most eight vertices.*

Figure 4 shows the 2-colorings of GH such that there is exactly one monochromatic triangle. In fact, there are several 2-colorings of GH such that there is exactly one monochromatic triangle in GH. The colorings in Figure 4 are simple and symmetrical, where dashed lines and arcs can be colored in either way. We call the coloring in (A) *of type A* and in (B) *of type B*, respectively[2]. In the following sections, we will make

[2]Here, we explain the coloring of type B in more detail. Let $(a, b)$ be the backbone for $a \in V(C_5)$ and $b \in V(C_3)$. Consider a vertex of $C_3$ which is not incident to the backbone $(a, b)$, the upper vertex $u$, for example. The edge $(u, a)$ is colored red. Starting $x$ with $x = a$, walking on $C_5$ in the anticlockwise way, $(u, x)$ is alternately colored red and blue. On the other hand, for the lower vertex $v$, starting $x$ with $x = a$, walking on $C_5$ in the clockwise way, $(v, x)$ is alternately colored red and blue.




(a) The monochromatic triangle in $C_3$

(b) The monochromatic triangle between $C_5$ and $C_3$

Figure 4: Colorings of the Graham graph

use of the two 2-colorings, where dashed lines and arcs are colored properly. We call the dashed line in the middle in (A) and (B), respectively, the *backbone.*

In Ramsey theory, people focus on the existence of a monochromatic clique in a large complete graph, that is, they do not care for the number of monochromatic cliques. Here, we focus on the property that there are at least *two* monochromatic triangles.

**Definition 2.3.** Given a graph $G = (V, E)$ that does not contain $K_6$, we say that $G$ satisfies property $\mathtt{Z}_2$ if for any edge 2-coloring, there are at least two monochromatic triangles in $G$.

Note that the Graham graph GH indeed satisfies $\mathtt{Z}_1$ (defined in the introduction), but does not satisfy $\mathtt{Z}_2$, as shown in the 2-colorings in Figure 4. As is mentioned in the introduction, there is a graph on eleven vertices that satisfies $\mathtt{Z}_2$, which is the graph consisting of two Graham graphs $\mathtt{GH}(C_5, C_3)$ and $\mathtt{GH}(C_5, C_3')$ via sharing $C_5$. In Section 4, we will see that there is a graph on ten vertices that satisfies $\mathtt{Z}_2$. In the next section, we will see that all $K_6$-free graphs on ten vertices, except for that graph, do not satisfy $\mathtt{Z}_2$.

## 3 The enumeration of maximal $K_6$-free graphs on ten vertices

Consider the lattice over all the graphs on ten vertices, where the top is $K_{10}$ and the bottom is the empty graph. We enumerate all *maximal* graphs that do not contain $K_6$, and show that all of them, except for one graph, do not satisfy $\mathtt{Z}_2$. We here denote the exceptional graph by $G_0$ (see Figure 2), the complement graph of which is enumerated in Lemma 3.2, and in fact presented in Figure 19 in the next section. For our enumeration, we divide graphs into two classes: graphs that contain the Graham graph GH or not. In what follows, we consider complement graphs so that we enumerate *minimal* (complement) graphs[3] that do not contain $\overline{K_6}$, and hence it

[3]Here, we use the term of minimal as the one in the usual way, i.e., a graph is minimal if $G \setminus \{e\}$ for any edge $e$ in $E(G)$ does contain $\bar{K_6}$ as an induced subgraph.

suffices to consider graphs with independence number *exactly* 5. We do it in terms of the number of isolated vertices.

For a graph $G = (V, E)$ (on ten vertices) that does not contain $K_6$, consider the complement graph of $G$, denoted by $H$. It is easy to see that the number of isolated vertices of $H$ is at most four since otherwise there exists at least one $K_6$ in $G$. It is also easy to see that it is not four since otherwise $H$ must be $\overline{K_4} \cup K_6$ so that $G$ does not contain $K_6$, and hence $G$ does not satisfy $\mathtt{Z}_2$. (In fact, we can color it without monochromatic triangles; see Theorem 3.2) Therefore, it suffices to consider complement graphs with at most three isolated vertices.

For guaranteeing the correctness of our enumeration, we make use of the following fact on $\alpha$-critical graphs of small order, which can be seen in Table 1 in [20]. In Figure 5, we present all the $\alpha$-critical connected graphs of order at most 7 with independence number at least 2.

**Fact 3.** *Any $\alpha$-critical connected graphs of order at most 7 is isomorphic to a complete graph or one of the graphs presented in Figure 5.*

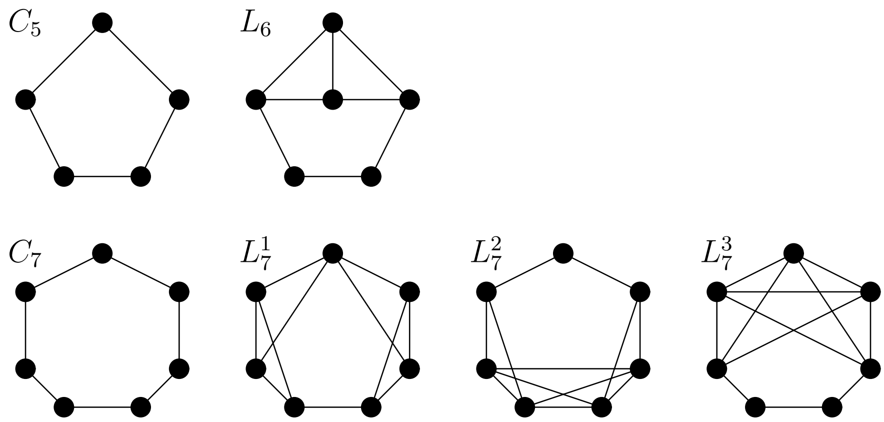



Figure 5: All the $\alpha$-critical connected graphs of order at most 7 with independence number at least 2

We further make use of the following fact on $\alpha$-critical graphs of order almost equal to $2\alpha$. (In fact, the order of any $\alpha$-critical graph is at least $2\alpha$. See, for example, [21] for details.)

**Fact 4.** *Let $G$ be an $\alpha$-critical connected graph of order $n$. Then the following holds:*

- *If $n$ is even and $\alpha(G) = n/2$, then $G = K_2$.*
- *If $n$ is odd and $\alpha(G) = (n-1)/2$, then $G = C_n$.*

Let $\mathcal{M}$ be the set of graphs on ten vertices whose complements are maximal $K_6$-free graphs. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be subsets of $\mathcal{M}$ in which any graph in $\mathcal{M}_1$ (respectively $\mathcal{M}_2$) whose complement contains (respectively does not contain) $\mathtt{GH}$. In the following, we consider graphs in $\mathcal{M}_1$ in Subsection 3.1 and ones in $\mathcal{M}_2$ in Subsection 3.2. As a consequence of results in those subsections, we have the following theorem.

**Theorem 3.1.** *The set $\mathcal{M}$ contains exactly eighteen graphs; $\overline{G_0}$ (Figure 19), $H_1^{GH}$, $H_2^{GH}$, $H_3^{GH}$ (Figure 6), $H_4^{GH}$, $H_5^{GH}$, $H_6^{GH}$ (Figure 7), $H_7^{GH}$ (Figure 10), $H_1^{nGH}$, $H_2^{nGH}$ (Figure 12), $H_3^{nGH}$, $H_4^{nGH}$, $H_5^{nGH}$ (Figure 13), $H_6^{nGH}$, $H_7^{nGH}$, $H_8^{nGH}$ (Figure 16), $\overline{K_4} \cup K_6$, $5K_2$. Equivalently, all 18 maximal $K_6$-free graphs on ten vertices are completely enumerated.*

## 3.1 Graham graph

We first present all maximal graphs that contain GH, but do not contain $K_6$. As is mentioned above, we consider complement graphs $H$ of those graphs, that is, $H \in \mathcal{M}_1$. Fix five vertices that constitute $C_5$ of GH, denoted by $S \subseteq V$. Note that $H[S]$ itself must be isomorphic to $C_5$. (Remember $\overline{\text{GH}}$ depicted in Figure 3.) Thus, we can not have the other edges (i.e., chords) within $S$.

Consider first that there are exactly three isolated vertices in $H$, the set of which is denoted by $S_1$.

**Lemma 3.1.** *The graphs in $\mathcal{M}_1$ with three isolated vertices are the graphs $H_1^{GH}$, $H_2^{GH}$ and $H_3^{GH}$, presented in Figure 6. Moreover, all the complement graphs of these graphs do not satisfy $Z_2$.*

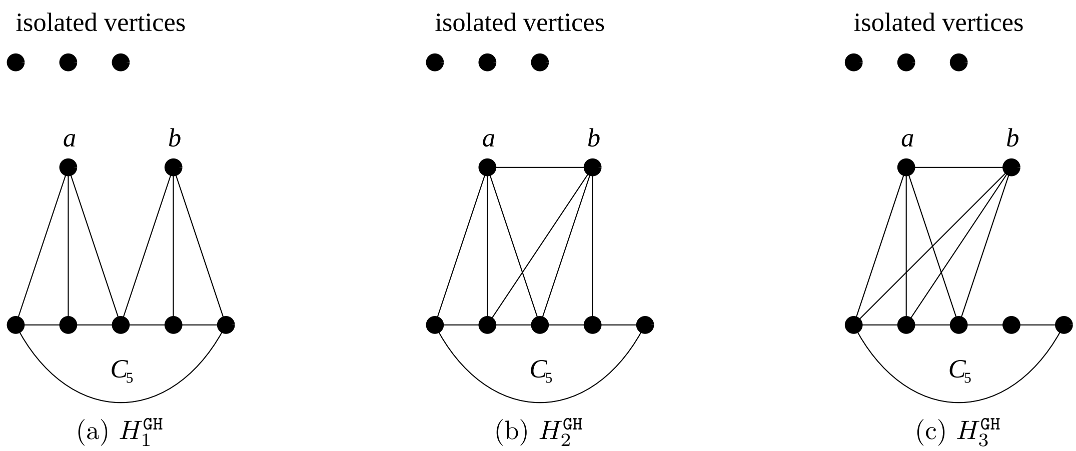


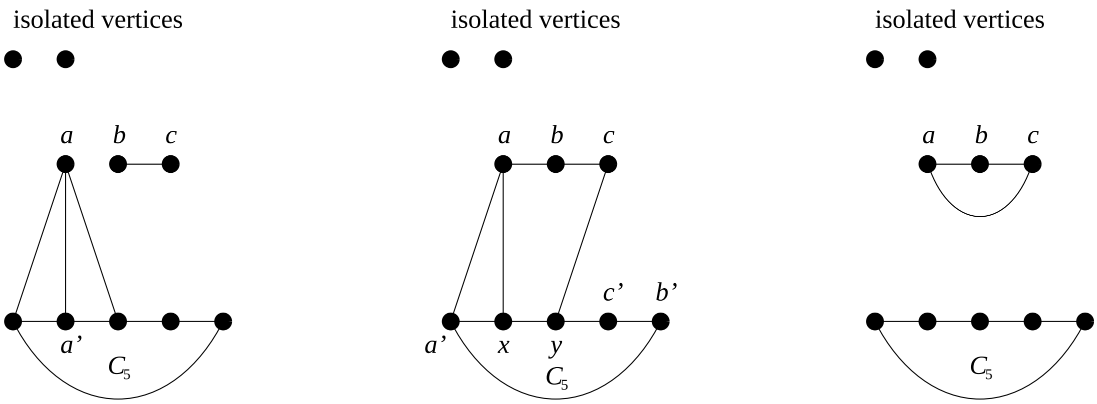

Figure 6: The minimal graphs in $\mathcal{M}_1$ with three isolated vertices

*Proof.* Let $H$ be a graph in $\mathcal{M}_1$ with three isolated vertices. Note first that $G[S \cup S_1]$ is isomorphic to GH. Let $a, b$ be the two vertices other than $S \cup S_1$. We consider $E(H[S \cup \{a, b\}])$. Note that $\alpha(H[S \cup \{a, b\}]) = 2$, and that any vertex $u \in \{a, b\}$ is adjacent to at least a vertex in $S$ since otherwise $G$ contains $K_6$. Thus, $H[S \cup \{a, b\}]$ is an $\alpha$-critical connected graph of order 7. Since $\alpha(H[S \cup \{a, b\}]) = 2$, by Fact 3, we have $H_1^{\text{GH}}$, $H_2^{\text{GH}}$, and $H_3^{\text{GH}}$ as $H$ if $H[S \cup \{a, b\}]$ is isomorphic to $L_7^1$, $L_7^2$, and $L_7^3$ in Figure 5, respectively.

We show that all the complement graphs of these graphs in Figure 6, denoted by $G_1^{\mathtt{GH}}, G_2^{\mathtt{GH}}, G_3^{\mathtt{GH}}$, do not satisfy $\mathtt{Z}_2$. For this, we present a concrete coloring to each graph of $G_1^{\mathtt{GH}}, G_2^{\mathtt{GH}}, G_3^{\mathtt{GH}}$ that produces at most one monochromatic triangle. In fact, we show it only for $G_1^{\mathtt{GH}}$ since it is almost same for the other two graphs. Note first that $G_1^{\mathtt{GH}}$ contains exactly one GH. (So do the other two graphs). The coloring of GH in $G_1^{\mathtt{GH}}$ is of type A, that is, the cycles of $C_3$ and $C_5$ are colored red, and the all the edges between $C_3$ and $C_5$ blue. In this case, the triangle of $C_3$ is monochromatic. We will see that this is the only one monochromatic triangle in $G_1^{\mathtt{GH}}$. The edges between $C_3$ and $\{a, b\}$ are colored blue, and the edges between $C_5$ and $\{a, b\}$ are colored red. Finally, the edge $(a, b)$ is colored red. (Note that there is no edge between $a$ and $b$ in the other two graphs.) Note that there is no triangle in $G[V(C_5) \cup \{a, b\}]$. It is easy to check that this coloring gives only one monochromatic triangle, that is, the one consisting of the three isolated vertices in $H$. □

Next, suppose that there are exactly two isolated vertices in $H$, the set of which is denoted by $S_2$.

**Lemma 3.2.** *The graphs in $\mathcal{M}_1$ with three isolated vertices are the graphs $H_4^{GH}$, $H_5^{GH}$, $H_6^{GH}$ presented in Figure 7 and $\overline{G_0}$ presented in Figure 19. Moreover, the complement graphs of $H_4^{GH}$, $H_5^{GH}$ and $H_6^{GH}$ do not satisfy $\mathtt{Z}_2$.*




(a) $H_4^{\mathtt{GH}}$ (b) $H_5^{\mathtt{GH}}$ (c) $H_6^{\mathtt{GH}}$

Figure 7: The minimal graphs in $\mathcal{M}_1$ with two isolated vertices

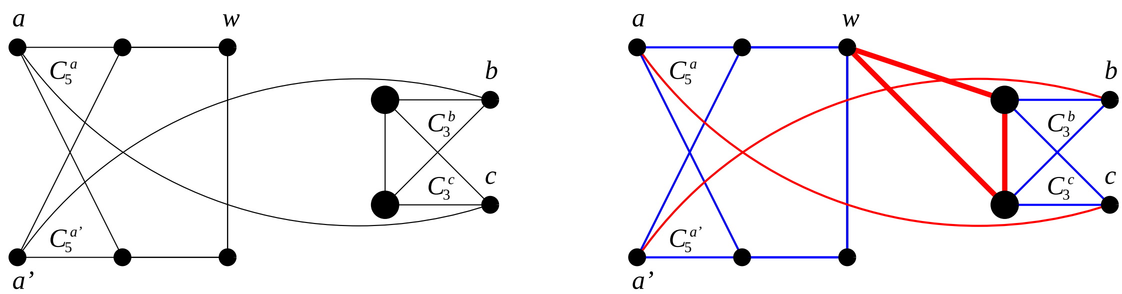
*Proof.* Let $H$ be a graph in $\mathcal{M}_1$ with two isolated vertices. Let $a, b, c$ be the three vertices other than $S \cup S_2$. Designate $b$ so that $G[S \cup S_2 \cup \{b\}]$ is isomorphic to GH. We consider $E(H[S \cup \{a, b, c\}])$. Note that $\alpha(H[S \cup \{a, b, c\}]) = 3$, and that $b$ is not adjacent to any vertex of $S$, but is adjacent to at least one vertex of $\{a, c\}$, say $c$.

If $H[S \cup \{a, b, c\}]$ is disconnected, then the graph has exactly two connected components $D_1$ and $D_2$, where $S \subseteq V(D_1)$ and $\{b, c\} \subseteq V(D_2)$. If $D_1$ contains $a$, then $H$ is isomorphic to $H_4^{\mathtt{GH}}$ since $\alpha(D_1) = 2$, and hence $D_1$ must be $L_6$ in Figure 5. Otherwise, $H$ is isomorphic to $H_6^{\mathtt{GH}}$ since $D_2$ must be $K_3$.

Suppose that $H[S \cup \{a, b, c\}]$ is connected. It is known in [13, Corollary 12.1.8] that every $\alpha$-critical graph has no cut vertex. By this fact, $b$ is adjacent to both $a$ and $c$. Moreover, there must be three consecutive vertices in $S$, say $a', x, y$, which are neighbors of $a$ or $c$ (since otherwise $H[S \cup \{a, b, c\}] \geq 4$). Thus, depending on neighbors of $a$ and $c$, we have the following three cases:

- If $a$ is adjacent to all of $a', x, y$, then $H$ is a super-graph of $H_4^{\mathtt{GH}}$.
- If $a$ is adjacent to both of $a', x$, and if $c$ is adjacent to $y$, then $H$ is isomorphic to $H_5^{\mathtt{GH}}$.
- If $a$ is adjacent to both of $a', y$, and if $c$ is adjacent to $x$, then $H$ is isomorphic to $\overline{G_0}$.

As before, we show that all the complement graphs of these graphs in Figure 7, denoted by $G_4^{\mathtt{GH}}, G_5^{\mathtt{GH}}, G_6^{\mathtt{GH}}$, do not satisfy $\mathtt{Z}_2$. However, the colorings of $G_4^{\mathtt{GH}}$ and $G_5^{\mathtt{GH}}$ are not so simple as before. First, we show the coloring of $G_4^{\mathtt{GH}}$. For this, we explain the structure of $G_4^{\mathtt{GH}}$. (See the left in Figure 8.) The graph $G_4^{\mathtt{GH}}$ consists of the sum of two

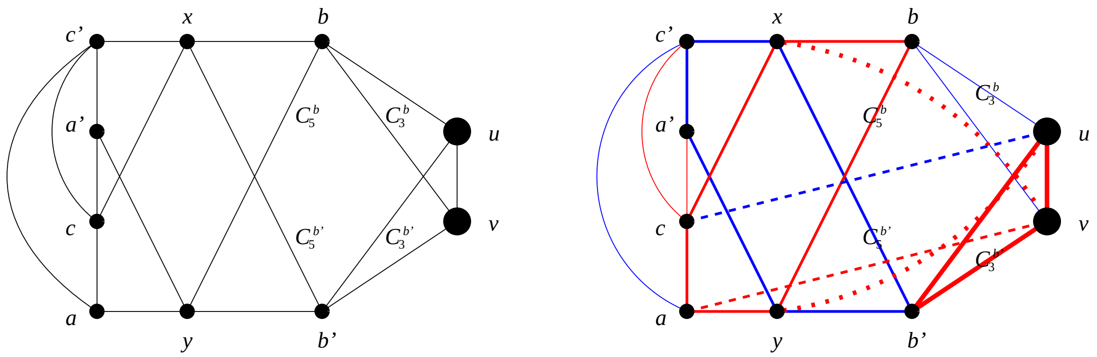



(a) $G_4^{\mathtt{GH}}$ (b) The coloring of $G_4^{\mathtt{GH}}$

Figure 8: The graph $G_4^{\mathtt{GH}}$ and the coloring of $G_4^{\mathtt{GH}}$

GHs, say, $\mathtt{GH}(C_5^a, C_3^b)$ where $C_5^a$ (respectively $C_3^b$) is the cycle of size five (respectively three) containing $a$ (respectively $b$) and $\mathtt{GH}(C_5^{a'}, C_3^c)$ where $C_5^{a'}$ (respectively $C_3^c$) is the cycle of size five (respectively three) containing $a'$ (respectively $c$), as well as the two additional edges $(a, c)$ and $(a', b)$. Note that all the edges between $C_5^a$ and $C_3^b$ and between $C_5^{a'}$ and $C_3^c$ are omitted in the figure. The two vertices corresponding to the isolated vertices in $H_4^{\mathtt{GH}}$ are depicted rather largely in the figure of $G_4^{\mathtt{GH}}$. The cycles $C_5^a$ and $C_5^{a'}$ share the four vertices except for $a$ and $a'$, and the cycles $C_3^b$ and $C_3^c$ share the two vertices (corresponding to the isolated vertices) except for $b$ and $c$. Thus, the coloring of $G_4^{\mathtt{GH}}$ is the one coupling the two colorings of $\mathtt{GH}(C_5^a, C_3^b)$ and $\mathtt{GH}(C_5^{a'}, C_3^c)$ both of type B that share the unique monochromatic triangle. See the right in Figure 8, where the monochromatic triangle is colored red so that the two backbones are $(b, w)$ and $(c, w)$, which are omitted in the figure and colored in either way. Note that coloring red to the two additional edges $(a, c)$ and $(a', b)$ does not yield any monochromatic triangle. This comes from the following observation: consider a triangle $(a, c, v)$ containing the edge $(a, c)$, for example. Then, $v$ must be a neighbor to $a$ and $c$, and hence $v$ must be a vertex on $C_5^a$ or $C_3^c$. Thus, either $(a, v)$ or $(c, v)$ must be colored blue, and hence the triangle $(a, c, v)$ can not be monochromatic.

Next, we show the coloring of $G_5^{\mathtt{GH}}$. For this, we explain the structure of $G_5^{\mathtt{GH}}$. (See the left in Figure 9.) The graph $G_5^{\mathtt{GH}}$ consists of the sum of two GHs, say, $\mathtt{GH}(C_5^b, C_3^{b'})$




(a) $G_5^{\mathtt{GH}}$ (b) The coloring of $G_5^{\mathtt{GH}}$

Figure 9: The graph $G_5^{\mathtt{GH}}$ and the coloring of $G_5^{\mathtt{GH}}$

and $\mathtt{GH}(C_5^{b'}, C_3^b)$, where $u$ and $v$ depicted largely in the figure of $G_5^{\mathtt{GH}}$ correspond to the isolated vertices in $H_5^{\mathtt{GH}}$, and

$$\begin{aligned} C_5^b &= (a, y, b, x, c), & C_3^{b'} &= (b', u, v), \\ C_5^{b'} &= (a', y, b', x, c'), & C_3^b &= (b, u, v), \end{aligned}$$

as well as the three additional edges $(a, c')$, $(a', c)$, and $(c, c')$. Note that all the edges between $C_5^b$ and $C_3^{b'}$ and between $C_5^{b'}$ and $C_3^b$ (except for those overlapping with the cycles of size five and three) are omitted in the figure. The cycles $C_5^b$ and $C_5^{b'}$ share the two vertices $x$ and $y$, and the cycles $C_3^b$ and $C_3^{b'}$ share $u$ and $v$ (corresponding to the isolated vertices). Thus, the coloring of $G_5^{\mathtt{GH}}$ is the one coupling the coloring of $\mathtt{GH}(C_5^b, C_3^{b'})$ of type A and the coloring of $\mathtt{GH}(C_5^{b'}, C_3^b)$ of type B that share the unique monochromatic triangle. See the right in Figure 9, where the monochromatic triangle $C_3^{b'}$ is colored red so that the two backbones of type A and B are commonly $(b, b')$, which are omitted in the figure and colored in either way.

Here, we explain the coloring of $G_5^{\mathtt{GH}}$ in more detail. For the coloring of type A depicted in the left in Figure 4, letting the backbone be $(b, b')$, $C_5$ colored red corresponds to $C_5^b = (a, y, b, x, c)$ and $C_3$ corresponds to $C_3^{b'} = (b', u, v)$. On the other hand, for the coloring of type B depicted in the right in Figure 4, letting the backbone be $(b', b)$, $C_5$ colored blue corresponds to $C_5^{b'} = (a', y, b', x, c')$ and $C_3$ corresponds to $C_3^b = (b, u, v)$. Thus, $(u, x)$ (respectively $(v, y)$) is colored blue and $(u, c')$ (respectively $(v, a')$) is colored red, and so on anticlockwise (respectively clockwise) on $C_5^{b'}$.

Observe that $(u, y)$ and $(v, x)$, which correspond to the two dashed arcs in Figure 4, should be colored red in the coloring of $\mathtt{GH}(C_5^b, C_3^{b'})$ of type A so that it is coincident with the coloring of $\mathtt{GH}(C_5^{b'}, C_3^b)$ of type B. In fact, for this case of coupling the two colorings of type A and B, we have been making the coloring of type A in Figure 4 rather flexible, that is, the dashed lines and arcs can be colored in either way. It is easy to see that if we ignore the three additional edges $(a, c')$, $(a', c)$, and $(c, c')$, there is no monochromatic triangle other than $C_3^{b'}$. We claim that for the three additional edges $(a, c')$, $(a', c)$, and $(c, c')$, coloring $(a', c)$, $(c, c')$ (respectively $(a, c')$) red (respectively blue) does not yield any monochromatic triangle. This is done by coloring $(u, c)$ and $(v, a)$ blue and red, respectively, which correspond to the two dashed lines in Figure 4. Consider a triangle $(a, c', w)$ containing the edge $(a, c')$, for example. Then, $w$ must be a neighbor of $a$ and $c'$, and hence $w \in \{u, v, c\}$. Thus, since the edges $(u, c'), (v, a), (c, c')$ are all colored red, the triangle $(a, c', w)$ can not be monochromatic. (It is similarly shown for a triangle $(a', c, w)$ containing the edge $(a', c)$, via the fact that the edges $(u, c), (v, c), (c', a')$ are all colored blue.) Similarly, consider a triangle $(c, c', w)$ containing the edge $(c, c')$. Then, $w$ must be a neighbor of $c$ and $c'$, and hence $w \in \{u, v, a, a', x\}$. Thus, since the edges $(u, c), (v, c), (a, c'), (a', c'), (x, c')$ are all colored blue, the triangle $(c, c', w)$ can not be monochromatic.

Finally, the coloring of $G_6^{\mathtt{GH}}$ is settled by appealing to those for GH. Note that $\{a, b, c\}$ are isolated from all the other vertices in $H_6^{\mathtt{GH}}$, and hence the graph obtaining from $G_6^{\mathtt{GH}}$ by identifying $\{a, b, c\}$ to one vertex $v$ is isomorphic to GH. Since $\{a, b, c\}$ are independent in $G_6^{\mathtt{GH}}$, we make use of any coloring of GH where the unique monochromatic triangle avoids the vertex $v$, say, the coloring of type B. ☐

Next, suppose that there is exactly one isolated vertex in $H$, the set of which is denoted by $S_3$.

**Lemma 3.3.** *There exists a unique graph in $\mathcal{M}_1$ with one isolated vertex, namely $H_7^{\mathtt{GH}}$, presented in Figure 10. Moreover, the complement graph of the graph does not satisfy $Z_2$.*

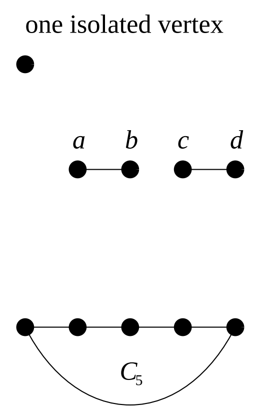



Figure 10: The minimal graph in $\mathcal{M}_1$ with one isolated vertex

*Proof.* Let $H$ be a graph in $\mathcal{M}_1$ with one isolated vertex. Let $a, b, c, d$ be the four vertices other than $S \cup S_3$. Designate $b$ and $c$ so that $G[S \cup S_3 \cup \{b, c\}]$ is isomorphic to GH. We consider $E(H[S \cup \{a,b,c,d\}])$. Note that $\alpha(H[S \cup \{a,b,c,d\}]) = 4$, and that $(b,c) \notin E(H[S \cup \{a,b,c,d\}])$, and both of $b$ and $c$ are not adjacent to any vertex of $S$, but is adjacent to at least one vertex of $\{a,d\}$.

Consider first the case that the two vertices adjacent to $b$ and $c$ are different, say, $(a,b),(c,d) \in E(H)$. Then, $H$ is isomorphic to $H_7^{\mathtt{GH}}$. Consider next the case that the two vertices adjacent to $b$ and $c$ are same, say, $(a,b),(a,c) \in E(H)$. In this case, we may assume that $(b,d),(c,d) \notin E(H)$ since otherwise it gives a super-graph of $H_7^{\mathtt{GH}}$. Then, as before, there are three consecutive vertices in $S$ adjacent to $d$, which gives a super-graph of $H_4^{\mathtt{GH}}$.

As before, we show that the complement graph of the graph in Figure 10, denoted by $G_7^{\mathtt{GH}}$, do not satisfy $\mathtt{Z}_2$. The coloring of $G_7^{\mathtt{GH}}$ is again settled by appealing to those for GH, as in the previous lemma. Note that $\{a,b\}$ and $\{c,d\}$ respectively are isolated from all the other vertices in $H_7^{\mathtt{GH}}$, and hence the graph obtaining from $G_7^{\mathtt{GH}}$ by identifying $\{a,b\}$ and $\{c,d\}$ to one vertex $u$ and $v$ respectively is isomorphic to GH. Since $\{a,b\}$ and $\{c,d\}$ are independent in $G_7^{\mathtt{GH}}$, we make use of any coloring of GH where the unique monochromatic triangle avoids the two vertices $u$ and $v$. Such a coloring is neither of type A nor type B, which is, for example, shown in Figure 11, where the other vertex in $C_3$ is denoted by $w$.

Here, we explain the coloring in more detail. Let $(a,b)$ be the backbone for $a \in V(C_5)$ and $b \in V(C_3)$, as in Figure 4. Consider a vertex of $C_3$ which is not incident to the backbone $(a,b)$, the upper vertex $w$, for example. The edge $(w,a)$ is colored red. Starting $x$ with $x = a$, walking on $C_5$ in the clockwise way, $(w,x)$ is alternately colored red and blue, resulting in the fact that the first and the end edges are both colored red, that makes the monochromatic triangle. On the other hand, for the lower vertex $c$, the edge $(c,a)$ is colored blue. Starting $x$ with $x = a$, walking on $C_5$ in the clockwise way, $(c,x)$ is alternately colored blue and red. In this case, the fact that the first and the end edges are both colored blue does not make any monochromatic triangle since all the edges of $C_5$ are colored red. In this notation, the two vertices $\{u,v\}$ in the proof correspond to $\{b,c\}$ in this footnote. □

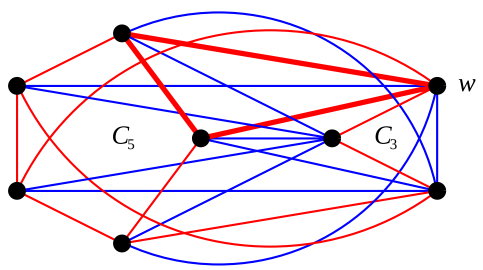



Figure 11: A coloring of GH used for $H_7^{\mathtt{GH}}$

Finally, suppose that there is no isolated vertex in $H$.

**Lemma 3.4.** *No graph $H$ in $\mathcal{M}_1$ without an isolated vertex exists.*

*Proof.* Let $a, b, c, d, e$ be the five vertices other than $S$. Designate $a, b, c$ so that $G[S \cup \{a, b, c\}]$ is isomorphic to $\mathtt{GH}$. Since $H$ has no isolated vertex (and there is no edge among $\{a, b, c\}$), each vertex of $\{a, b, c\}$ is adjacent to $d$ or $e$. If the set of neighbors of $\{a, b, c\}$ is $\{d, e\}$, then $H$ is a super-graph of $H_7^{\mathtt{GH}}$. Thus, we may assume that the neighbor of $\{a, b, c\}$ is, say, $d$. Then, as before, whether $(d, e) \in E(H)$ or not, there are three consecutive vertices in $S$ adjacent to $e$, which gives a super-graph of $H_4^{\mathtt{GH}}$. □

### 3.2 Non-Graham graph

We next present all maximal graphs that *do not* contain $\mathtt{GH}$. As is mentioned before, we consider complement graphs $H$ of those graphs so that we enumerate minimal (complement) graphs that do not contain $\overline{K_6}$, that is, $H \in \mathcal{M}_2$. Recall that it suffices to consider graphs $H$ with independence number *exactly* five, and the number of isolated vertices in $H$ is at most three.

Before starting the proof, we introduce the following theorem which is a complementary version (with some small change) of the result in [12] that every vertex 5-colorable graph $G$ has an edge 2-coloring of $G$ without a monochromatic triangle[4].

**Theorem 3.2.** *Let $G$ be a graph. If each component of $\overline{G}$ is a complete graph and the number of components in $\overline{G}$ is at most 5, then there exists an edge 2-coloring of $G$ without a monochromatic triangle.*

Consider first that there are exactly three isolated vertices in $H$, the set of which is denoted by $S_1$.

**Lemma 3.5.** *The graphs in $\mathcal{M}_2$ with three isolated vertices are the graphs $H_1^{nGH}$ and $H_2^{nGH}$, presented in Figure 12. Moreover, all the complement graphs of these graphs do not satisfy $\mathtt{Z}_2$.*

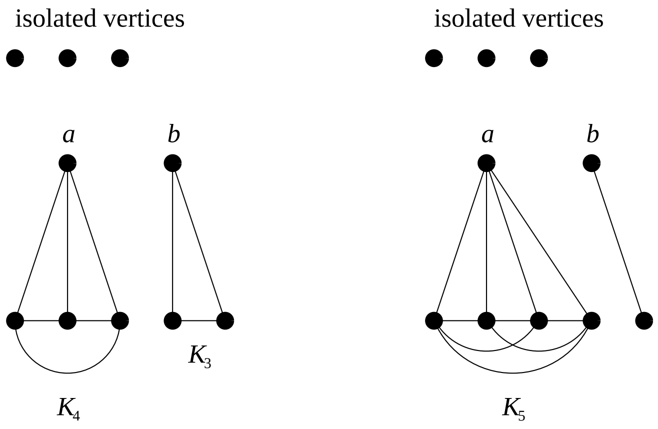
*Proof.* Let $H$ be a graph in $\mathcal{M}_2$ with three isolated vertices. Since $\alpha(H[V \setminus S_1]) = 2$, fix two independent vertices arbitrarily, denoted by $a, b$. We first suppose that $H[V \setminus S_1]$ is disconnected. Since both of $a$ and $b$ have degree at least one, and the two components must be complete, we immediately have that $H$ is isomorphic to $H_1^{\mathtt{nGH}}$ or $H_2^{\mathtt{nGH}}$.

We next suppose that $H[V \setminus S_1]$ is connected. Then, $H[V \setminus S_1]$ is an $\alpha$-critical connected graph of order 7. As before, since $\alpha(H[V \setminus S_1]) = 2$, by Fact 3 we have $H[V \setminus S_1]$ is isomorphic to one of $L_7^1$, $L_7^2$, and $L_7^3$, and hence it is isomorphic to $H_1^{\mathtt{GH}}$, $H_2^{\mathtt{GH}}$ or $H_3^{\mathtt{GH}}$.

By Theorem 3.2, the complement graphs of $G_1^{\mathtt{nGH}}$ and $G_2^{\mathtt{nGH}}$ do not satisfy $\mathtt{Z}_2$. □

Next, suppose that there are exactly two isolated vertices in $H$, the set of which is denoted by $S_2$.

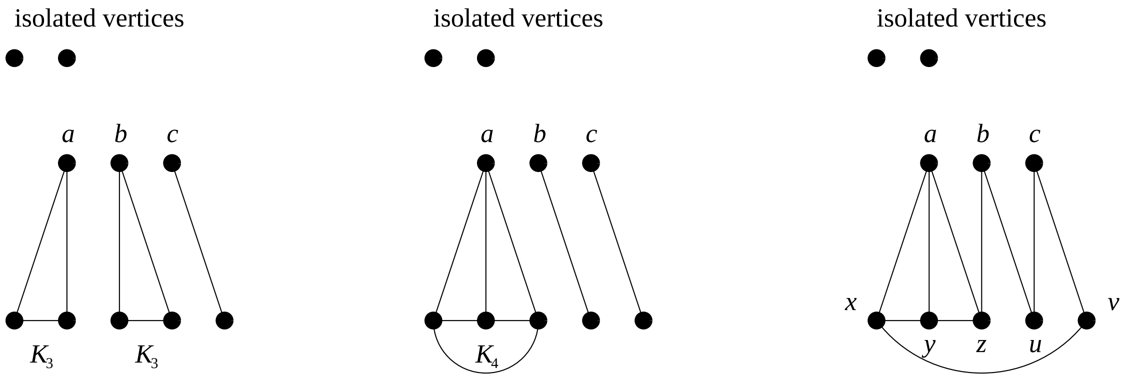
[4]More precisely, Theorem 3.2 is a complement version of Lin's result for a maximal vertex 5-colorable graph $G$, that is, $G$ is isomorphic to a complete five partite graph $K_{a_1,a_2,a_3,a_4,a_5}$.




(a) $H_1^{\mathtt{nGH}}$ (b) $H_2^{\mathtt{nGH}}$

Figure 12: The minimal graphs in $\mathcal{M}_2$ with three isolated vertices

**Lemma 3.6.** *The graphs in $\mathcal{M}_2$ with two isolated vertices are the graphs $H_3^{nGH}$, $H_4^{nGH}$ and $H_5^{nGH}$, presented in Figure 13. Moreover, all the complement graphs of these graphs do not satisfy $\mathcal{Z}_2$.*




(a) $H_3^{\mathtt{nGH}}$ (b) $H_4^{\mathtt{nGH}}$ (c) $H_5^{\mathtt{nGH}}$

Figure 13: The minimal graphs in $\mathcal{M}_2$ with two isolated vertices

*Proof.* Let $H$ be a graph in $\mathcal{M}_2$ with two isolated vertices. Since $\alpha(H[V \setminus S_2]) = 3$, fix three independent vertices arbitrarily, denoted by $a, b, c$. We first suppose that $H[V \setminus S_2]$ is disconnected, and has three connected components. Similar to the first case in the previous lemma, we have that $H$ is isomorphic to $H_3^{\mathtt{nGH}}$ or $H_4^{\mathtt{nGH}}$.

We next suppose that $H[V \setminus S_2]$ has two connected components $D_1, D_2$, in decreasing order of $\alpha(D_i)$. We may assume that $\alpha(D_1) = 2$ and $\alpha(D_2) = 1$, and hence $D_1$ has at least five vertices by Fact 3. Since $D_2$ has at least two vertices, $D_1$ has at most six vertices. If $D_2 = C_5$, then $D_1 = K_3$, and hence $H = H_6^{\mathtt{GH}}$. If $D_2 = L_6$, then $H = H_4^{\mathtt{GH}}$.

Finally, we suppose that $H[V \setminus S_2]$ is connected. Recall that every $\alpha$-critical graph has no cut vertex [13, Corollary 12.1.8]. Moreover, it is known in [10] that the degree of any vertex in an $\alpha$-critical graph of order $n$ is at most $n - 2\alpha + 1$. Thus, $H[V \setminus S_2]$ is 3-regular or it has a vertex of degree 2. In the former case, it is known in [13, Exercise 12.1.16] that such a graph is of order at most 4. In the latter case, it is known in [13, Lemma 12.1.4] that if an $\alpha$-critical graph of order $n$ has a vertex of degree 2, then the graph can be obtained from some $(\alpha - 1)$-critical graph of order $n - 2$ by splitting some vertex into two vertices $y$ and $z$, and by creating a new vertex $x$ so that $x$ is adjacent to $y$ and $z$. Thus, by applying the operation to a vertex of $L_6$, we can obtain all $\alpha$-critical connected graphs of order 8 with $\alpha = 3$ which have a vertex of degree 2. See Figure 14 for all those graphs. Therefore, if

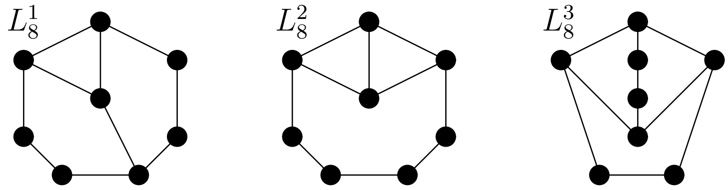
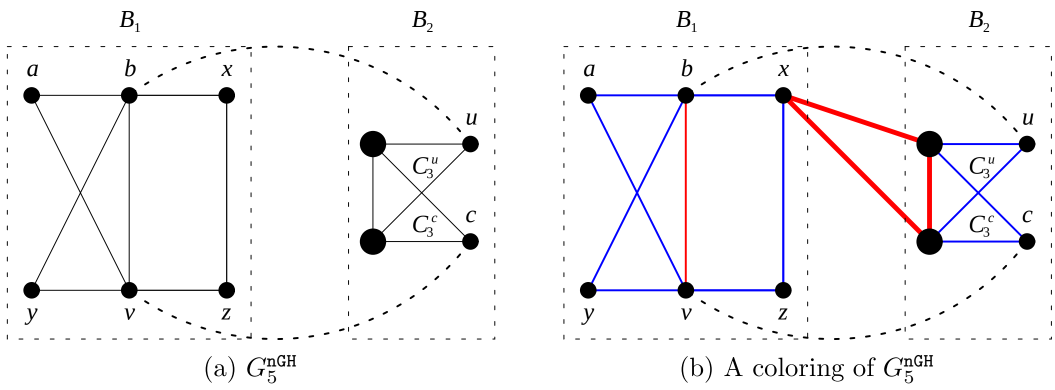



Figure 14: Three $\alpha$-critical connected graphs of order 8 with $\alpha = 3$ having a vertex of degree 2

$H[V \setminus S_2]$ is isomorphic to $L_8^1$, $L_8^2$, and $L_8^3$, then $H$ is isomorphic to $H_5^{\mathtt{GH}}$, $H_5^{\mathtt{nGH}}$, and $G_0$, respectively.

As before, we show that all the complement graphs of these graphs in Figure 13, denoted by $G_3^{\mathtt{nGH}}, G_4^{\mathtt{nGH}}, G_5^{\mathtt{nGH}}$, do not satisfy $\mathsf{Z}_2$. By Theorem 3.2, the complements of $G_3^{\mathtt{nGH}}$ and $G_4^{\mathtt{nGH}}$ do not satisfy $\mathsf{Z}_2$. On the other hand, the coloring of $G_5^{\mathtt{nGH}}$ is not so simple as the two graphs. We first explain the structure of $G_5^{\mathtt{nGH}}$, which indeed is close to that of $G_4^{\mathtt{GH}}$. (See the left in Figure 15, comparing to the left in Figure 8.) The




(a) $G_5^{\mathtt{nGH}}$ (b) A coloring of $G_5^{\mathtt{nGH}}$

Figure 15: The graph $G_5^{\mathtt{nGH}}$ and the coloring of $G_5^{\mathtt{nGH}}$

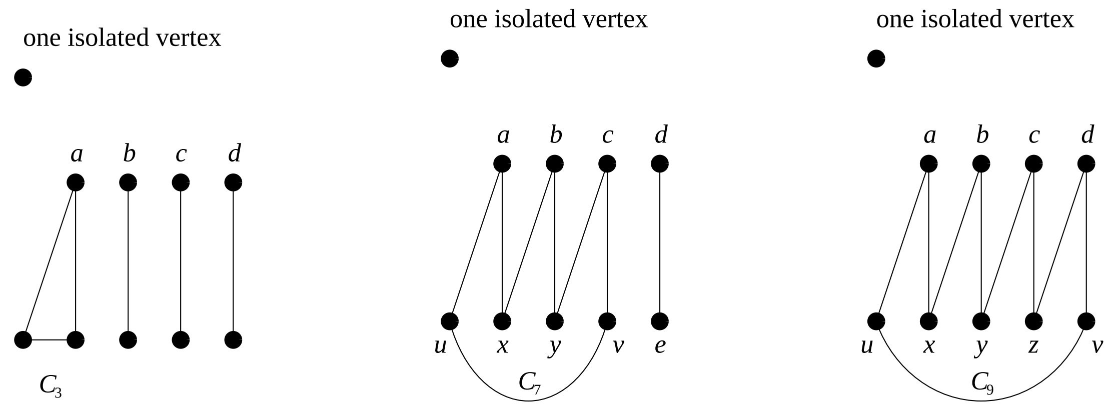
graph $G_5^{\mathtt{nGH}}$ consists of the two parts $B_1$ and $B_2$. The part $B_1$ is over $a, b, x, y, z, v$ and the part $B_2$ is over $c, u$ and the two isolated vertices (depicted rather largely), which correspond to vertices of $H_5^{\mathtt{nGH}}$ in Figure 13. Observe that there is one edge on every pair between vertices of $B_1$ and $B_2$, except for the two pairs $(b,u)$ and $(c,v)$, where the formers are omitted and the latters are depicted in dashed lines. We next present a coloring of $G_5^{\mathtt{nGH}}$ so that there is only one monochromatic triangle [5], which is on $x$ and the two isolated vertices. For this, we focus on the differences between $G_4^{\mathtt{GH}}$ and $G_5^{\mathtt{nGH}}$ by the following equation:

$$G_5^{\mathtt{nGH}} = (G_4^{\mathtt{GH}} \cup \{(b,v)\}) \setminus \{(b,u),(c,v)\},$$

where we employ the labels of vertices in Figure 15, discarding those in Figure 8. Then, the coloring of $G_5^{\mathtt{nGH}}$ is almost same as that of $G_4^{\mathtt{GH}}$ shown in the right in Figure 8. Recall that we have made use of the coloring of type B so that we share the unique monochromatic triangle between the two GHs. It is easy to see that there is only one monochromatic triangle if we apply the coloring of $G_4^{\mathtt{GH}}$ to edges of $E(G_4^{\mathtt{GH}}) \cap E(G_5^{\mathtt{nGH}})$, and do not care for the color of the edge $(b,v)$. We claim that the additional edge $(b,v)$ colored red does not produce any monochromatic triangle. This is because (1) there is no edge on $(b,u)$ and $(c,v)$ in $G_5^{\mathtt{nGH}}$, and (2) $(b,w)$ and $(v,w)$ must be differently colored for any isolated vertex $w$. □

Next, suppose that there is exactly one isolated vertex in $H$, the set of which is denoted by $S_3$.

**Lemma 3.7.** *The graphs in $\mathcal{M}_2$ with one isolated vertex are the graphs $H_6^{nGH}$, $H_7^{nGH}$ and $H_8^{nGH}$, presented in Figure 16. Moreover, all the complement graphs of these graphs do not satisfy $Z_2$.*




(a) $H_6^{\mathtt{nGH}}$ (b) $H_7^{\mathtt{nGH}}$ (c) $H_8^{\mathtt{nGH}}$

Figure 16: The minimal graphs in $\mathcal{M}_2$ with one isolated vertex

*Proof.* Let $H$ be a graph in $\mathcal{M}_2$ with one isolated vertex. Since $\alpha(H[V \setminus S_3]) = 4$, fix four independent vertices arbitrarily, denoted by $a,b,c,d$. We first suppose that $H[V \setminus S_3]$ is disconnected, and has four connected components. Similar to the proofs of the previous lemmas, we have that $H$ is isomorphic to $H_6^{\mathtt{nGH}}$,

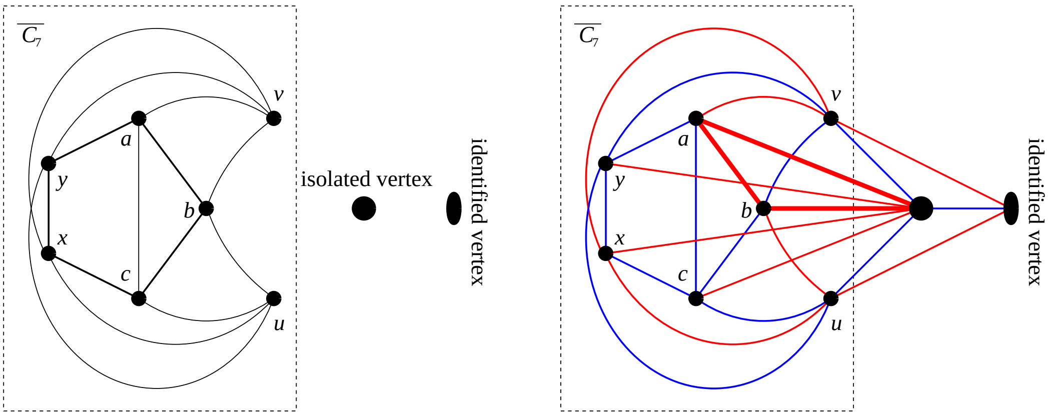
[5]There might exist a coloring of $G_5^{\mathtt{nGH}}$ so that that there is no monochromatic triangle.

Suppose that $H[V \setminus S_3]$ has three connected components $D_1, D_2, D_3$, in decreasing order of $\alpha(D_i)$. We may assume that $\alpha(D_1) = 2$ and $\alpha(D_2) = \alpha(D_3) = 1$. Since each of $D_2$ and $D_3$ has at least two vertices, $D_1$ has at most five vertices. This means that $D_1 = C_5$ by Fact 3, and hence $H = H_7^{\mathtt{GH}}$. Suppose that $H[V \setminus S_3]$ has two connected components $D_1, D_2$, in decreasing order of $\alpha(D_i)$. If $\alpha(D_1) = 3$, then $D_1 = C_7$ by Fact 4 since $|V(D_1)| \leq 7$, and hence $H = H_7^{\mathtt{nGH}}$. Otherwise, i.e., $\alpha(D_1) = \alpha(D_2) = 2$, at least one of $D_1$ and $D_2$ does not exist by Fact 3 since $\min\{|V(D_1)|, |V(D_2)|\} \leq 4$.

Finally, suppose that $H[V \setminus S_3]$ is connected. Then, $H[V \setminus S_3] = C_9$ by Fact 4 since $|V \setminus S_3| = 9$ and $\alpha(H[V \setminus S_3]) = 4$, and hence $H = H_8^{\mathtt{nGH}}$.

As before, we show that all the complement graphs of these graphs in Figure 16, denoted by $G_6^{\mathtt{nGH}}, G_7^{\mathtt{nGH}}, G_8^{\mathtt{nGH}}$, do not satisfy $\mathsf{Z}_2$. By Theorem 3.2, the complement of $G_6^{\mathtt{nGH}}$ does not satisfy $\mathsf{Z}_2$. On the other hand, the colorings of $G_7^{\mathtt{nGH}}$ and $G_8^{\mathtt{nGH}}$ are not so simple, which are presented individually in the following two claims.

**Claim 1.** *The graph $\widetilde{G_7^{nGH}}$ obtained from $G_7^{nGH}$ by identifying the two vertices $d$ and $e$ in $H_7^{nGH}$ is depicted in the left in Figure 17. All the edges among $V(\overline{C_7})$ and the two vertices, the isolated vertex and identified vertex, are omitted in the figure. Then, there is only one monochromatic triangle in the coloring of $\widetilde{G_7^{nGH}}$ depicted in the right in Figure 17, where the following colorings are omitted: let $C_5 = (a, y, x, c, b)$ be the cycle of size five. The coloring of edges between the identified vertex $w$ and $V(C_5)$ is as follows: starting $p$ with $p = a$, walking on $C_5$ in the anticlockwise way, the edge $(w, p)$ is alternately colored blue and red so that $(w, a)$ and $(w, b)$ are both colored blue.*




(a) $\widetilde{G_7^{\mathtt{nGH}}}$ (b) The coloring of $\widetilde{G_7^{\mathtt{nGH}}}$

Figure 17: The graph $\widetilde{G_7^{\mathtt{nGH}}}$ and the coloring of $\widetilde{G_7^{\mathtt{nGH}}}$

*Proof.* We explain the structure of $\widetilde{G_7^{\mathtt{nGH}}}$, in particular, the complement of $C_7 = (a, x, b, y, c, v, u)$ in $H_7^{\mathtt{nGH}}$. We extract $C_5$ with the chord $(a, c)$ from $\overline{C_7}$, and put aside the other two vertices $u$ and $v$. Note that $u$ (respectively $v$) is neither adjacent to $v$ nor $a$ (respectively $u$ nor $c$) in $\widetilde{G_7^{\mathtt{nGH}}}$. Moreover, triangles within $\overline{C_7}$ other than $(a, b, c)$ are incident to either $u$ or $v$.

We confirm that there is only one monochromatic triangle in the coloring of $\widetilde{G_7^{\mathtt{nGH}}}$. Here, we make sure of important features in the coloring. Let $q$ be the isolated vertex and $w$ be the identified vertex. Then,

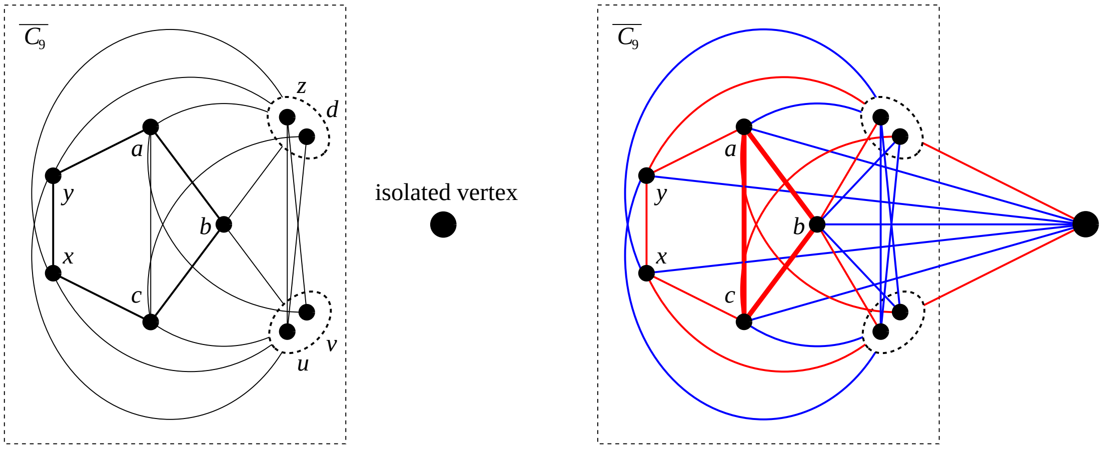
1. All the edges except for $(a, b)$ on $C_5$ as well as the chord $(a, c)$ are colored blue.
2. The edges between $u$ (respectively $v$) and $\{b, c, x, y\}$ (respectively $\{b, a, y, x\}$) are alternately colored.
3. All the edges between $V(C_5)$ and $q$ are colored red.
4. All the edges between $V(C_5)$ and $w$ are alternately colored as in the claim.

Firstly, it is easy to see that there is no monochromatic triangle within $\overline{C_7}$ because of the feature 1 and 2 above. Next, consider triangles containing the isolated vertex $q$ within $\widetilde{G_7^{\mathtt{nGH}}}[V(C_7) \cup \{q\}]$. It is easy to see that there is only one monochromatic triangle between $C_5$ and $q$ because of the feature 1 and 3 above. It is also easy to see that there is no monochromatic triangle containing $(q, u)$ and $(q, v)$ since they are colored blue and because of the feature 3 above. Finally, consider triangles containing the identified vertex $w$ in $\widetilde{G_7^{\mathtt{nGH}}}$. Firstly, it is obvious that there is no monochromatic triangle within $\widetilde{G_7^{\mathtt{nGH}}}[\{w, q, u, v\}]$. It is easy to see that there is no monochromatic triangle between $C_5$ and $w$ because of the feature 1 and 4 above. It is also easy to see that there is no monochromatic triangle containing $(w, u)$ and $(w, v)$ since these are colored red and because of the feature 2 and 4 above. Note here about the feature 2 that the edges $(x, u), (x, v)$ (respectively $(y, u), (y, v)$) are colored red (respectively blue) while $(x, w)$ (respectively $(y, w)$) is colored blue (respectively red). It is also easy to see that there is no monochromatic triangle containing $(w, q)$ since it is colored blue and because of the feature 3 above. □

From this claim, it is easy to see that $G_7^{\mathtt{nGH}}$ does not satisfy $\mathtt{Z}_2$ since the monochromatic triangle in the claim avoids the identified vertex which corresponds to $d$ and $e$ in $G_7^{\mathtt{nGH}}$.

**Claim 2.** *The graph $G_8^{nGH}$ is depicted in the left in Figure 18. All the edges between $V(\overline{C_9})$ and the isolated vertex are omitted in the figure. Moreover, adjacent vertices $z, d$ (respectively $u, v$) on $C_9$ in $H_8^{nGH}$ are bundled up (but not identified) in one dotted circle in the figure, where one common edge is depicted for each vertex adjacent to $z$ and $d$ (respectively $u$ and $v$) in $G_8^{nGH}[V(C_9)]$. Then, there is only one monochromatic triangle in the coloring of $G_8^{nGH}$ depicted in the right in Figure 18.*

*Proof.* We explain the structure of $G_8^{\mathtt{nGH}}$, in particular, the complement of $C_9 = (a, x, b, y, c, z, d, v, u)$ in $H_8^{\mathtt{nGH}}$. As the previous claim, we extract $C_5 = (a, y, x, c, b)$ with the chord $(a, c)$ from $\overline{C_9}$, and put aside the other four vertices $z, d, u, v$, bundling adjacent vertices $z$ and $d$ (respectively $u$ and $v$) as one vertex. Remark that we do not *identify* the two vertices. Note that $z$ (respectively $d$) is neither adjacent to $d$ nor $c$ (respectively $z$ nor $v$) in $G_8^{\mathtt{nGH}}$. It is symmetrically similar for $u$ and $v$.




(a) $G_8^{\mathtt{nGH}}$ (b) The coloring of $G_8^{\mathtt{nGH}}$

Figure 18: The graph $G_8^{\mathtt{nGH}}$ and the colorings of $G_8^{\mathtt{nGH}}$

We confirm that there is only one monochromatic triangle in the coloring of $G_8^{\mathtt{nGH}}$. Here, we make sure of important features in the coloring. Let $q$ be the isolated vertex. Then,

1. All the edges are colored red on $C_5$.

2. The common edges between the bundle $\{z, d\}$ (respectively $\{u, v\}$) and $\{a, y, x\}$ (respectively $\{c, x, y\}$) are alternately colored, but the two edges corresponding to the commonly depicted edge incident to $b$ is exclusively colored in $\{z, d\}$ (respectively $\{u, v\}$).

3. All the edges between the two bundles are colored blue.

4. All the edges between $V(C_5)$ and $q$ are colored blue.

5. All the edges between the two bundles and $q$ are colored red.

Firstly, it is easy to see that there is only one monochromatic triangle within $G_8^{\mathtt{nGH}}[V(C_5)]$. Next, consider triangles containing $z$ within $\overline{C_9}$. It is easy to see that there is no monochromatic triangle between $C_5$ and $z$ because of the feature 2 above. It is also easy to see that there is no monochromatic triangle containing $(z, u)$ because of the feature 3 and the following observation on the feature 2: the edge between the bundle $\{z, d\}$ and $x$ (respectively $y$) is colored blue (respectively red) while the edge between the bundle $\{u, v\}$ and $x$ (respectively $y$) is colored red (respectively blue). This fact on $(z, u)$ is similar for $(z, v)$ except for the existence of a triangle $(z, v, a)$. These facts on $z$ are similar for $d$ except for the existence of a triangle $(b, c, d)$. Moreover, those facts about $z$ and $d$ are symmetrically same for $u$ and $v$. Finally, consider an arbitrary triangle $(q, s, t)$ in $G_8^{\mathtt{nGH}}$ for some $s, t \in V(C_9)$. Firstly, for the case of $s, t \in V(C_5)$, it is obvious that $(q, s, t)$ is not monochromatic because of the feature 1 and 4. For the case of $s, t \in V(C_9) \setminus V(C_5)$, it is obvious that $(q, s, t)$ is not monochromatic because of the feature 3 and 5. For the case of $s \in V(C_5)$ and $t \in V(C_9) \setminus V(C_5)$, it is obvious that $(q, s, t)$ is not monochromatic because of the feature 4 and 5. □

From this claim, we see that $G_8^{\mathtt{nGH}}$ does not satisfy $\mathtt{Z}_2$. □

Finally, suppose that there is no isolated vertex in $H$.

**Lemma 3.8.** *There exists a unique graph in $\mathcal{M}_2$ without an isolated vertex, namely the bipartite graph $5K_2$. Moreover, $K_{2,2,2,2,2} = \overline{5K_2}$ does not satisfy $\mathtt{Z}_2$.*

*Proof.* Let $H$ be a graph in $\mathcal{M}_2$ without an isolated vertex. Since $\alpha(H) = 5$, we may suppose that $H$ is disconnected (since otherwise $H = K_2$ by Fact 4). If $H$ has five connected components, then each component is $K_2$, and hence, $H = 5K_2$.

Suppose that $H$ has four connected components $D_1, D_2, D_3, D_4$, in decreasing order of $\alpha(D_i)$. Since $\alpha(D_1) = 2$ and $|V(D_1)| \leq 4$, such a connected graph $D_1$ does not exist by Fact 3.

Next, suppose that $H$ has three connected components $D_1, D_2, D_3$, in decreasing order of $\alpha(D_i)$. If $\alpha(D_1) = 3$, such a connected graph $D_1$ does not exist by Fact 3 since $|V(D_1)| \leq 6$. Otherwise, $\alpha(D_1) = \alpha(D_2) = 2$, and hence at least one of $D_1$ and $D_2$ does not exist by Fact 3 since $\min\{|V(D_1)|, |V(D_2)|\} \leq 4$.

Finally, suppose that $H$ has two connected components $D_1, D_2$, in decreasing order of $\alpha(D_i)$. There are two cases; $(\alpha(D_1), \alpha(D_2))$ is equal to $(4, 1)$ or $(3, 2)$. In either case, similar to the above cases, no such graph exists by Fact 3 and Fact 4.

By Theorem 3.2, $K_{2,2,2,2,2} = \overline{5K_2}$ has an edge 2-coloring without a monochromatic triangle. □

## 4 Proof of Theorem 1.1

We present the maximal $K_6$-free graph on ten vertices satisfying $\mathtt{Z}_2$, denoted by $G_0 = (V, E_0)$ (see Figure 2) where $V = \{v_1, \ldots, v_{10}\}$. We depict the complement graph of $G_0$ in Figure 19. Before we show that the graph $G_0$ satisfies $\mathtt{Z}_2$, we explain

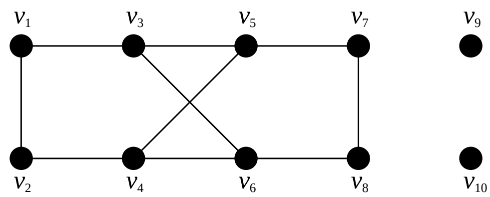



Figure 19: The graph $\overline{G_0}$

the structure of the complement graph $\overline{G_0}$. The following induced sub-graphs of $G_0$

are all isomorphic to $\mathtt{GH}$.

$$\begin{aligned}
G_1 &\stackrel{\text{def}}{=} G_0[\{v_1, v_2, v_3, v_4, v_5\} \cup \{v_8, v_9, v_{10}\}],\\
G_2 &\stackrel{\text{def}}{=} G_0[\{v_1, v_2, v_3, v_4, v_6\} \cup \{v_7, v_9, v_{10}\}],\\
G_3 &\stackrel{\text{def}}{=} G_0[\{v_3, v_5, v_6, v_7, v_8\} \cup \{v_2, v_9, v_{10}\}],\\
G_4 &\stackrel{\text{def}}{=} G_0[\{v_4, v_5, v_6, v_7, v_8\} \cup \{v_1, v_9, v_{10}\}]
\end{aligned}$$

Thus, emphasizing the two cycles in $\mathtt{GH}$, for each $1 \leq i \leq 4$, we denote $G_i$ by $\mathtt{GH}(C_5^i, C_3^i)$. It is easy to see that $G_0$ is $K_6$-free. This comes from the following observation. It suffices to show that there is no independent set of size four in $\overline{G_0}[\{v_1, \ldots, v_8\}]$. Let $I \subseteq \{v_1, \ldots, v_8\}$ be an arbitrary independent set in $\overline{G_0}[\{v_1, \ldots, v_8\}]$. Consider the case of $I \cap \{v_3, v_4, v_5, v_6\} = \emptyset$. Then, it is easy to see $|I| \leq 2$. Otherwise, suppose w.l.o.g. that $v_3 \in I$. In this case, $v_1, v_5, v_6 \notin I$. Then, it is easy to see $|I| \leq 3$. It is also easy to see the maximality of $G_0$. This comes from the fact that there is an independent set of size four in $\overline{G_0}[\{v_1, \ldots, v_8\}] \setminus \{e\}$ for any $e \in E(\overline{G_0})$.

**Lemma 4.1.** *The graph $G_0 = (V, E_0)$ satisfies $\mathtt{Z}_2$.*

*Proof.* Fix an edge 2-coloring of $G_0$ arbitrarily. Note that there is at least one monochromatic triangle in $G_0$ since the graph $G_0$ contains $\mathtt{GH}$ and because of Fact 2. Let $\{a, b, c\} \subseteq V$ be the set of the vertices of a monochromatic triangle in $G_0$. We show that there exists another monochromatic triangle in $G_0$.

**Claim 3.** *For any $v \in \{v_1, \ldots, v_8\}$, there is one $G_i$ such that $v \notin V(G_i)$.*

*Proof.* Consider $v = v_1$, for example. Then, $v_1 \notin V(G_3)$. It is similarly proven for any $v \in \{v_2, \ldots, v_8\}$. □

At least one from $\{a, b, c\}$ must be from $\{v_1, \ldots, v_8\}$. Thus, from this claim, there is one $G_i$ which does not contain the triangle $(a, b, c)$. Thus, there is another monochromatic triangle in the graph $G_i$, which is guaranteed by Fact 2. □

**Lemma 4.2.** *The graph $G_0$ is minimal with respect to the property $\mathtt{Z}_2$, that is, for any edge $e \in E(G_0)$, there is a coloring for $G_0 \setminus \{e\}$ such that at most one monochromatic triangle exists.*

*Proof.* For proving the lemma, we deal with the complement graph of $G_0$, denoted by $H_0$, so that we consider to add an edge $e$ to $H_0$. We will see that any graph $H_0 \cup \{e\}$ except for some graph, denoted by $H_0'$, is a super-graph of some graph presented in the previous section, resulting in the contradiction to $\mathtt{Z}_2$.

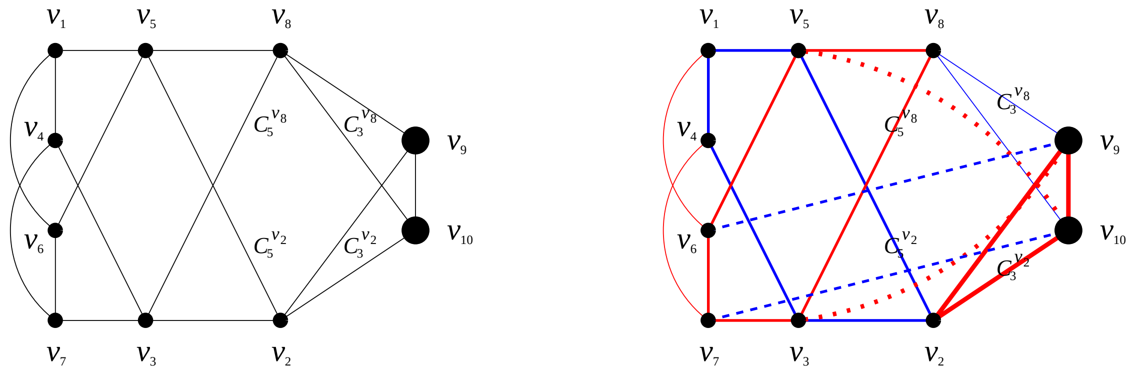
We first consider an edge $e$ incident to $v_{10}$. (By symmetry, we similarly consider an edge $e$ incident to $v_9$.) For any $v \in \{v_1, \ldots, v_9\}$, it is easy to check that the graph $H_0 \cup \{e\}$ for $e = (v, v_{10})$ is a super-graph of $H_7^{\mathtt{GH}}$.

We next consider an edge $e$ incident to $v_1$. (By symmetry, we similarly consider an edge $e$ incident to $v_2, v_7, v_8$.) Observe that for any $v \in \{v_4, \ldots, v_8\}$, the graph

$H_0 \cup \{(v_1, v)\}$ with $v = v_5$ (respectively $v_7$) is isomorphic to the graph with $v = v_6$ (respectively $v_8$). The exceptional graph $H'_0$ is the graph $H_0 \cup \{(v_1, v)\}$ with $v = v_7$, which is not a super-graph of any graph presented in the previous section. It is easy to check that the graph $H_0 \cup \{e\}$ for $e = (v_1, v_5)$ is a super-graph of $H_5^{\mathtt{GH}}$. It is also easy to check that the graph $H_0 \cup \{e\}$ for $e = (v_1, v_4)$ is a super-graph of $H_6^{\mathtt{GH}}$.

We next consider an edge $e$ incident to $v_3$. (By symmetry, we similarly consider an edge $e$ incident to $v_4, v_5, v_6$.) It suffices to consider the case of $e = (v_3, v_4)$. Then, it is easy to check that the graph $H_0 \cup \{e\}$ for $e = (v_3, v_4)$ is a super-graph of $H_4^{\mathtt{GH}}$.

We finally show the coloring of the complement graph of $H'_0$, denoted by $G'_0$. For this, we explain the structure of $G'_0$. (See the left in Figure 20.) The graph




(a) $G'_0$

(b) The coloring of $G'_0$

Figure 20: The graph $G'_0$ and the coloring of $G'_0$

as well as the coloring are very close to those of $G_5^{\mathtt{GH}}$ depicted in Figure 9. The difference of the two graphs is about additional edges, that is, the three additional edges $(a, c'), (a', c), (c, c')$ in $G_5^{\mathtt{GH}}$ are replaced with the two edges $(v_1, v_6)$ and $(v_4, v_7)$ in $G'_0$. The coloring of $G'_0$ is almost same as that of $G_5^{\mathtt{GH}}$ so that there is the unique monochromatic triangle $(v_2, v_9, v_{10})$, denoted by $C_3^{v_2}$. (See the right in Figure 20.) As same as that case, it is easy to see that if we ignore the two additional edges $(v_1, v_6)$ and $(v_4, v_7)$, there is no monochromatic triangle other than $C_3^{v_2}$. We claim that coloring the two additional edges $(v_1, v_6)$ and $(v_4, v_7)$ red does not yield any monochromatic triangle. This is done by coloring $(v_6, v_9)$ and $(v_7, v_{10})$ blue, which correspond to the two dashed lines in Figure 4. Consider a triangle $(v_1, v_6, w)$ containing the edge $(v_1, v_6)$. Then, $w$ must be a neighbor of $v_1$ and $v_6$, and hence $w \in \{v_5, v_9, v_{10}\}$. Thus, since the edges $(v_5, v_1), (v_9, v_6), (v_{10}, v_1)$ are all colored blue, the triangle $(v_1, v_6, w)$ can not be monochromatic. Similarly, consider a triangle $(v_4, v_7, w)$ containing the edge $(v_4, v_7)$. Then, $w$ must be a neighbor of $v_4$ and $v_7$, and hence $w \in \{v_3, v_9, v_{10}\}$. Thus, since the edges $(v_3, v_4), (v_9, v_4), (v_{10}, v_7)$ are all colored blue, the triangle $(v_4, v_7, w)$ can not be monochromatic. □

Finally, we give a proof of Theorem 1.1.

*Proof of Theorem* 1.1. By Lemmas 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7 and 3.8, any maximal $K_6$-free graph except $G_0$ does not have property $\mathtt{Z}_2$. Moreover, by Lemmas 4.1 and 4.2, $G_0$ satisfies $\mathtt{Z}_2$ and it is minimal with respect to the property $\mathtt{Z}_2$. Therefore, the theorem holds. □

## 5 Conclusions

We have shown that the minimum number of vertices of $K_6$-free graphs containing (at least) two monochromatic triangles for any edge 2-coloring is ten, giving a concrete (minimal) graph on ten vertices with such a property. Moreover, we show the uniqueness of the graph of all $K_6$-free graphs on (at most) ten vertices.

Recall that $f_1(2,3,6) = f(2,3,6) = 8$ and that our result implies $f_2(2,3,6) = 10$, where $f_s(r,k,l)$ is a generalized concept of the Folkman number defined in the introduction. A straightforward future work is to generalize the construction of the graph $G_0$, which gives an upper bound on the value of $f_s(2,3,6)$ for any $s \geq 1$.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