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ON ODD-SUMMING NUMBERS

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Abstract. In this paper we investigate two theorems dealing with those natural numbers which can be written as the sum of two or more consecutive odd numbers.

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1. Introduction

Olson [3] proved that a natural number n is the sum of two or more consecutive natural numbers if and only if n is not a power of 2.

C. Ray and S. Harris [3] proved the following:

The natural number n can be written as the sum of consecutive odd natural numbers $2r+1, 2r+3, \dots, 2s-1$ if and only if

$$n = s^2 - r^2 = (s-r)(s+r).$$

The natural number n is odd-summing if and only if either n is the product of two odd numbers, each greather than 1, or n is the product of two even numbers.

Suppose that $n = p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}$, where $p_1, p_2, \dots, p_t$ are distinct primes, $p_1 < p_2 < \dots < p_t$, and each $k_i > 0$. In [3] the following statements have been proved:

(i) If n is odd and is not a square then

$$\frac{(k_1+1)(k_2+1)\cdots(k_t+1)-2}{2}$$

representation of n exist.

(ii) If n is odd square then

$$\frac{(k_1+1)(k_2+1)\cdots(k_t+1)-1}{2}$$

representation of n exists.

(iii) If $p_1 = 2$ and n is not a square then

$$\frac{(k_1 - 1)(k_2 + 1) \cdots (k_t + 1)}{2}$$

representation of n exists.

(iv) If $p_1 = 2$ and n is a square then

$$\frac{(k_1 - 1)(k_2 + 1) \cdots (k_t + 1) + 1}{2}$$

representation of n exists.

The natural number n has a unique representation as the sum of consecutive odd numbers if and only if n is the square of a prime number, if n is the cube of a prime number, if n is four times a prime number, or if n is the product of two different odd prime numbers.

The author proved in [2] that no set of four consecutive natural numbers exists that are all odd-summing or that are all not odd-summing.

The purpose of this paper is to form some new results of the properties of the odd-summing numbers. First we define by [2] and [3] the concept of these special numbers, then we give our theorems and proofs.

2. Results and proofs

Definition. All natural numbers that are the sum of two or more consecutive odd numbers are called odd-summing numbers.

Remark. It is clear that all square numbers are odd-summing numbers but keep in mind that not all odd-summing numbers are square numbers, take 8 as a counterexample: $8 = 3 + 5$. In this paper we denote the set of the odd-summing numbers by N_o.

Theorem 1. *If $n \geq 2$ and $k \geq 2$ are integers then n^k can be written as the sum of n consecutive odd-numbers, ($n^k \in N_o$, or n^k is an odd-summing number).*

Proof. Write n^k as the sum of equal terms.

$$n^k = nn^{k-1} = n^{k-1} + n^{k-1} + \cdots + n^{k-1}. \tag{1}$$

Next we show, that the sum (1) can be written as the sum of consecutive odd numbers. Form pairs of the first and last terms, the second and the one but last terms, and so on. We separate the proof into two parts according to the parity of n.

1.1. If n is an even number then the terms are also even numbers, because $k-1 \geq 1$.

Subtract 1 from the first term of the middle pair and add 1 to the second term of the middle pair. Thus we get $n^{k-1}-1$ and $n^{k-1}+1$; are consecutive odd numbers.

Similarly dencrease the $\left(\frac{n}{2}-1\right)$st and increase the $\left(\frac{n+2}{2}+1\right)$st terms by the next odd number, 3 or $2\cdot 1+1$; the $\left(\frac{n}{2}-2\right)$th and $\frac{n+2}{2}+2$th by 5, or $2\cdot 2+1$, and so on, at the end the $\frac{n}{2}-(\frac{n}{2}-1)=1$st and the $\frac{n+2}{2}+(\frac{n}{2}-1)=n$th terms by $2(\frac{n}{2}-1)+1=n-1$.

We get from (1)

$$\begin{aligned} n^k &= (n^{k-1}-n+1)+(n^{k-1}-n+3)+\ldots+(n^{k-1}-3)+(n^{k-1}-1)+ \\ &(n^{k-1}+1)+(n^{k-1}+3)+\cdots+(n^{k-1}+n-3)+(n^{k-1}+n-1). \end{aligned} \tag{2}$$

The terms of (2) are odd, the difference of two consecutive terms is 2, the number of terms is

$$1+\frac{(n^k+n-1)-(n^k-n+1)}{2}=n. \tag{3}$$

1.2. If n is an odd number then the middle term of (1) is alone, the number of pairs is $\frac{n-1}{2}$.

The middle term is the $\frac{n-1}{2}+1=\frac{n+1}{2}$th one, the adjacent elements are $\frac{n-1}{2}$ and $\frac{n+3}{2}$.

In this case the terms are odd numbers. So starting from the middle term we change the terms of pairs by $2, 4, \ldots, 2\frac{n-1}{2}=n-1$ so from (1) we get

$$\begin{aligned} n^k &= (n^{k-1}-n+1)+(n^{k-1}-n+3)+\cdots+(n^{k-1}-4)+(n^{k-1}-2)+ \\ &n^{k-1}+(n^{k-1}+2)+(n^{k-1}+4)+\cdots+(n^{k-1}+n-3)+(n^{k-1}+n-1). \end{aligned} \tag{4}$$

The number of terms is n, all terms are odd numbers, and the difference of adjacent terms is 2. Theorem 1 is proved.

Note. Theorem 1 can be proved by a simpler method as well. Adding the n numbers $-n+1, -n+3, \ldots, n-1$ to the numbers of the sum we get:

$$(n^{k-1}-n+1)+(n^{k-1}-n+3)+\cdots+(n^{k-1}+n-3)+(n^{k-1}+n-1).$$

The difference of the consecutive numbers in the sum is 2 and each of the numbers added are odd since $k \geq 2$.

Theorem 2. *If $n \geq 1$ then the $n(n+1)(n+2)(n+3)+1$ is an odd-summing number.*

Proof. The proof follows immediately from the fact that $n(n+1)(n+2)(n+3)+1 = k^2$ for all natural numbers $k \geq 1$.

If we add 1 to the product of four consecutive natural number then

$$\begin{aligned} n(n+1)(n+2)(n+3)+1 =&(n^2+n)(n^2+5n+6)+1 \\ =&n^4+n^3+5n^3+5n^2+6n^2+6n+1 \\ =&n^4+6n^3+11n^2+6n+1 \end{aligned}$$

holds. This can be written in the form

$$\begin{aligned} [(n^2+3n)+1]^2 =&(n^2+3n)^2+2(n^2+3n)+1 \\ =&n^4+6n^3+9n^2+2n^2+6n+1 = n^4+6n^3+11n^2+6n+1 \\ =&n(n+1)(n+2)+(n+3)+1 = k^2. \end{aligned}$$

It is well known that a perfect square is an odd-summing number. Thus Theorem 2 is proved.

The converse of Theorem 2 does not hold.

Remarks

1. The proof of Theorem 1 furnishes an algorithm to find all terms of consecutive odd numbers that adds to n^k.
2. Theorem 1 and Theorem 2 can be proved by the results of C. Ray and S. Harris in [3].
3. If n is a natural number then $n(n+1)(n+2)(n+3)$ and $n(n+1)(n+2)(n+3)+1$ are consecutive odd-summing numbers. Theorem 2 amplifies and clarifies this fact.

   Examples:

   If $n=1$ then $1 \cdot 2 \cdot 3 \cdot 4 + 1 = 25$, $25 = 1+3+5+7+9$ and $24 = 11+13$.

   If $n=2$ then $2 \cdot 3 \cdot 4 \cdot 5 + 1 = 121$ and $120 = 59+61$ are odd-summing numbers.

References

[1] Olson, M., Sequentially so. *Mathematics Magazin* (1991), 297–298.

[2] Orosz, Gyuláné, Partíciók páratlan számokkal, *Acta. Acad. Paed. Agriensis, Sect. Math.* **29** (2002), 107–114.

[3] Ray, C., Harris, S., An odd sum. *Mathematics Teacher* **95**, Number 3 (2002).

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