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An elementary geometric characterization of the integrable motions of a curve

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Abstract

We show that the following elementary geometric properties of the motion of a curve select hierarchies of integrable dynamics: i) the curve moves in an N-dimensional sphere of radius R; ii) the motion is nonstretching; iii) the dynamics is independent of the radius of the sphere. For $N = 2$ we obtain the modified Korteweg-de Vries hierarchy, for $N = 3$ the nonlinear Schrödinger hierarchy and for $N > 3$ we obtain integrable multicomponent generalizations of the above hierarchies.

1993 PACS: 02.30.Jr, 02.40.Hw.

*This work was supported by the 1993 agreement between Rome and Warsaw Universities; by the grant 2-0168-91-01 KBN and by the INFN.

1 A historical introduction

One of the classical problems of the XIX-century geometers was the study of the connection between differential geometry of submanifolds and nonlinear PDE's. For instance, Liouville found the general solution of the equation (known now as the Liouville equation) which describes minimal surfaces in E^3 [1]. Bianchi solved the general Goursat problem for the sine-Gordon (SG) equation [2], which encodes the whole geometry of the pseudospherical surfaces. Moreover the method of construction of a new pseudospherical surface from a given one, proposed by Bianchi [3], gives rise to the so-called Bianchi-Bäcklund transformation for the SG equation [4].

In 1967 Gardner, Green, Kruskal and Miura [5] discovered a method (now called the inverse-scattering (or spectral) transform (IST) method) that can be used to solve the Cauchy problem, for rapidly decreasing data, of several PDE's, like the Korteweg-de Vries (KdV), the modified Korteweg-de Vries (mKdV), the nonlinear Schrödinger (NLS) equations, and of the SG equation as well [6].

The IST method, as well as the finite gap method which generalizes it in the context of periodic boundary conditions [7], is based on the association of the given PDE with a spectral problem, which plays a central role in the theory. Actually, to each spectral problem one can associate a whole hierarchy of integrable PDE's generated by a recursion operator, as it was shown by Ablowitz, Kaup, Newell and Segur [8].

After these achievements, the connection between geometry and integrable PDE's became even deeper and reappeared many times. It was Hasimoto [9] who found the transformation between the equations governing the curvature and torsion of an isolated nonstretching thin vortex filament moving in an incompressible inviscid fluid and the NLS equation. Lamb [10] used the Hasimoto transformation to connect other motions of curves to integrable equations like the mKdV and SG equations. Lund and Regge [11] observed that the Gauss-Weingarten equations of the surface created by a particular motion of a relativistic string can be viewed as a pair of spectral problems whose compatibility gives the so-called Lund and Regge system.

Lakshmanan et al. studied the dynamics of a flexible string of constant length in E^3, deriving the Zakharov-Shabat (ZS)-Ablowitz-Kaup-Newell-Segur (AKNS) spectral problem [12]: an integrable NLS equation with x-dependent coefficients was also derived from the motion of a suitable nonlinear string [13].

Sasaki [14] gave a geometric interpretation to the ZS-AKNS spectral problem in terms of pseudospherical surfaces. Chern and Tenenblat [15] characterized the mKdV hierarchy as relations between local invariants of certain foliations on a surface of constant non-zero Gauss curvature. In the papers [11]-[15] the spectral parameter, which plays a crucial role in the identification of the hierarchies of integrable PDE's, was introduced either identifying it with the (constant) curvature or torsion of the string, or using gauge and/or space-time invariances of the resulting PDE's.

Almost at that time Sym introduced the "soliton surfaces approach" [16], in which the powerful tools of the IST method are used to construct explicit formulas for the immersions of one-parameter families (labeled by the spectral parameter) of surfaces corresponding to given solutions of integrable PDE's [17]; see also the recent developments of Bobenko [18].

Barbashov, Nesterenko and Chervyakov studied the world surface generated by the motion of a relativistic string, obtaining and solving an integrable generalization of the Liouville equation [19].

More recently Langer and Perline [20] showed that the dynamics of a nonstretching vortex filament in R^3 gives rise, through the Hasimoto transformation, to the recursion operator of the NLS hierarchy. Similarly, Goldstein and Petrich [21] showed that the dynamics of a nonstretching string on the plane produces the recursion operator of the mKdV hierarchy, and Nakayama, Segur and Wadati [22] explained this result observing that the Frenet equations for the curve are equivalent to the ZS-AKNS spectral problem with zero spectral parameter.

The absence of the spectral parameter in the papers [20][21][22] does not allow one to select integrable dynamics of a curve from a purely geometric point of view, without prior knowledge of the theory of integrable systems.

One of the goals of our paper is to show that there exists a very elementary geometric characterization of the motion of a curve, which selects only integrable dynamics.

2 An elementary geometric characterization of integrable dynamics

The mathematical objects of our investigation are curves; the main result of our investigation consists in the identification of the following **three elementary geometric properties** of the motion of a curve which select, among all possible dynamics, **integrable dynamics.**

Property 1. The motion of the curve takes place in the N-dimensional sphere of radius R, denoted by $S^N(R)$, $N > 1$.

Property 2. The curve does not stretch during the motion.

Property 3. The dynamics of the curve does not depend explicitly on the radius R.

We remark that the set of dynamics satisfying *Properties 1-3* is not empty: for instance it obviously contains the rigid motion of the curve on the sphere, described by the trivial evolutions

$$\kappa_{j,t} = 0 \,, \quad j = 1, \ldots, N-1 \,, \tag{1}$$

and the motion of the curve along itself, described by the linear wave equations

$$\kappa_{j,t} = c\kappa_{j,s} \,, \quad j = 1, \ldots, N-1 \tag{2}$$

for the geodesic curvatures κ_j of the curve, where s is the arc-length parameter.

We shall see in the following sections that the rigid motion and the motion along itself are just two of the infinitely many integrable dynamics of the curve, selected by *Properties 1-3*. These dynamics are described by integrable commuting nonlinear evolution equations in 1+1 (one spatial and one temporal) dimensions for the geodesic curvatures of the curve. More precisely, for $N = 2$ we obtain the mKdV hierarchy for the curvature κ. For $N = 3$ we obtain a hierarchy of evolution equations for the geodesic curvature $\kappa = \kappa_1$ and torsion $\tau = \kappa_2$, which can be transformed, through the Hasimoto transformation, into the NLS hierarchy. For $N > 3$ we obtain integrable multicomponent generalizations of the above hierarchies.

It is very important to remark that *Properties 1-3* not only select integrable PDE's, but also provide their integrability scheme; in other words, in the process of deriving the dynamics selected by *Properties 1-3* one discovers "for free" the integrable nature of such dynamics! In particular, the spectral problem is given by the Frenet equations of the curve and is a consequence of *Property 1*, and the spectral parameter is given by the inverse of the radius of the sphere.

We expect that *Properties 1-3*, when applied to the motion of n-dimensional submanifolds of S^N, will select integrable PDE's in $n+1$ variables: this problem is presently under investigation by the authors. An example of system of integrable PDE's in an arbitrary number of variables connected with geometry was found by Tenenblat and Terng [23] and solved by Ablowitz, Beals and Tenenblat [24].

The Reader could wonder why spheres should be so important for integrability. The embedding of a curve in a space of constant non-zero curvature, instead of a flat Euclidean space, has the great advantage of introducing in a natural geometric way the spectral parameter, without having to use scaling properties of the resulting PDE's. Furthermore spheres are not only simple objects, but they are also the universal covering spaces for manifolds of positive constant sectional curvature [25]: therefore they are generic for local purposes.

We hope that our work, motivated by the results of the papers [15][21], will give a contribution to a deeper understanding of the elementary geometric nature of an integrable dynamics.

The paper is organized as follows. In Section 3 we show in detail how *Properties 1-3*, for $N = 2$, select the mKdV hierarchy; moreover we stress the basic role played by the rigid motion and we discuss the geometric meaning of the mKdV and SG equations. In Section 4 we show how *Properties 1-3*, for $N = 3$, select the NLS hierarchy and we give a geometrical meaning to the Hasimoto transformation. In Section 5 we extend the above results to higher N's.

3 The curve on $S^2(R)$ and the mKdV hierarchy

Property 1. If the curve lies on $S^2(R)$, the unit tangent vector $\hat{\mathbf{t}}$ to the string is related to the geodesic curvature κ of the string and to the unit vector $\hat{\mathbf{n}}$, normal to the string and tangent to the sphere, by the Frenet equations [26]

$$\begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\mathbf{t}} \\ \hat{\mathbf{n}} \end{bmatrix}_{,s} = \begin{bmatrix} 0 & \lambda & 0 \\ -\lambda & 0 & \kappa \\ 0 & -\kappa & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\mathbf{t}} \\ \hat{\mathbf{n}} \end{bmatrix} \tag{3}$$

where $\lambda = \frac{1}{R}$ and $\hat{\mathbf{r}} = \lambda\vec{\mathbf{r}}$ is the unit vector in the radial direction.

The Frenet equations (3) are nothing but the matrix spectral problem of the theory of integrable systems. More precisely:
i) the Frenet frame (i.e. the orthonormal basis $\{\hat{\mathbf{r}}, \hat{\mathbf{t}}, \hat{\mathbf{n}}\}$ defined along the curve) corresponds to a 3×3 matrix eigenfunction;
ii) the inverse of the radius of the sphere is the spectral parameter λ;
iii) the geodesic curvature κ is the "direct datum" or "potential".
We remark that this spectral problem is equivalent to the 2×2 ZS-AKNS spectral problem

$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}_{,s} = \frac{i}{2} \begin{bmatrix} \lambda & \kappa \\ \kappa & -\lambda \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \tag{4}$$

using the well-known isomorphism between $su(2)$ and $so(3)$ Lie algebras.

Let us consider the velocity field $\vec{\mathbf{v}}$ along the curve which gives us the kinematics. Then *Property 1* implies that $\vec{\mathbf{v}}$ has no component in the radial direction:

$$\vec{\mathbf{r}}_{,t} = \vec{\mathbf{v}} = V\vec{\mathbf{t}} + U\vec{\mathbf{n}} \ . \tag{5}$$

Property 2. If the curve does not stretch during the motion, the distance between any two points of the curve (measured along the curve) does not vary in time, then s and t can serve as local coordinates on the sphere, and

$$\partial_s\partial_t = \partial_t\partial_s \ . \tag{6}$$

Since the Frenet frame is orthonormal, its time evolution is governed by an antisymmetric matrix:

$$\begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\mathbf{t}} \\ \hat{\mathbf{n}} \end{bmatrix}_{,t} = \begin{bmatrix} 0 & \lambda V & \lambda U \\ -\lambda V & 0 & A \\ -\lambda U & -A & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\mathbf{t}} \\ \hat{\mathbf{n}} \end{bmatrix} \ . \tag{7}$$

Futhermore the commutativity condition (6), applied to equations (3) and (7), specifies the coefficient A

$$A = U_{,s} + \kappa V \tag{8}$$

and implies the time evolution of the geodesic curvature

$$\kappa_{,t} = (U_{,s} + \kappa V)_{,s} + \lambda^2 U \qquad (9)$$

with the condition

$$V_{,s} = \kappa U \ . \qquad (10)$$

Equations (9) and (10) can be written in the following way

$$\kappa_{,t} = \mathcal{R}U + \lambda^2 U \qquad (11)$$

in terms of the recursion operator

$$\mathcal{R} = \partial_s^2 + \kappa^2 + \kappa_{,s}\partial_s^{-1}\kappa \qquad (12)$$

of the mKdV hierarchy [8][27][28]. For $\lambda = 0$, the case of the curve on the plane, we recover the result obtained in [21].

Property 3. Equations (9) and (10) show that the dynamics of the curve, subjected to *Properties 1-2*, depends explicitly on the curvature λ^2 of the sphere.

A drastic way to satisfy *Property 3* is to choose the normal velocity U as the real eigenfunction of the recursion operator $\mathcal{R}$ for the eigenvalue $-\lambda^2$:

$$\mathcal{R}U = -\lambda^2 U \ . \qquad (13)$$

This velocity field generates the rigid motion of the curve on the sphere, described by the equation

$$\kappa_{,t} = 0 \ . \qquad (14)$$

Therefore:

the rigid motion of a curve on the sphere of curvature λ^2 is generated by a normal velocity being the real eigenfunction of the recursion operator of the mKdV hierarchy, corresponding to the eigenvalue $-\lambda^2$.

We remark that equation (13) plays a central role in the theory of integrable systems and it is satisfied by the so called "squared eigenfunctions" of the spectral problem (4) [8]. From simple geometric considerations one can show that the solution U of equation (13) is given by any linear combination of the Cartesian components of the unit tangent vector $\hat{\mathbf{t}}$, which can be expressed in terms of ϕ_1 and ϕ_2 in the following way:

$$\hat{\mathbf{t}} = \left[Im(\bar{\phi}_1^2 - \phi_2^2) \ , \ Re(\phi_2^2 + \bar{\phi}_1^2) \ , \ 2Im\bar{\phi}_1\phi_2 \right] \ . \qquad (15)$$

This representation is a direct consequence of the already mentioned isomorphism between $so(3)$ and $su(2)$.

It is known that one can find the explicit expression of the solution U of equation (13) in terms of κ only for special curves, like multisoliton and finite gap solution type of direct data. For $\lambda = 0$ (the case of the curve on the plane) such integration gives rise to the explicit formula

$$U = Ker\mathcal{R} = a\sin(\theta - \theta_0) \qquad (16)$$

where θ is the incomplete total curvature

$$\theta = \partial_s^{-1}\kappa \ . \qquad (17)$$

If the coefficients a and θ_0 are time independent, the corresponding velocity field

$$\vec{\mathbf{v}} = a\left(\sin(\theta - \theta_0)\hat{\mathbf{n}} - \cos(\theta - \theta_0)\hat{\mathbf{t}} \right) \qquad (18)$$

generates a rigid translation on the plane.

The linear dependence on λ^2 of equation (11) suggests that *Property 3* can be satisfied also looking for velocity fields given by suitable Laurent expansions in λ^2

$$U = \sum_{j=-m_1}^{m_2} \lambda^{2j} U_j \ , \ m_1 \, , \, m_2 \geq 0 \ . \qquad (19)$$

Substituting equation (19) into (11) and requiring independence of λ, one obtains a class of integrable dynamics in the form

$$g(\mathcal{R})\kappa_{,t} = h(\mathcal{R})\kappa_{,s} \ . \qquad (20)$$

where g and h are arbitrary entire functions of their argument. The simplest three examples are the following.

i) If $g = h = 1$ we have

$$\vec{\mathbf{v}} = \hat{\mathbf{t}} \qquad \kappa_{,t} = \kappa_{,s} \ ; \qquad (21)$$

therefore:

the motion of the curve along itself is described by the linear wave equation.

ii) If $g = 1$ and $h(x) = x$ we have

$$\vec{\mathbf{v}} = \kappa_{,s}\hat{\mathbf{n}} + (-\lambda^2 + \frac{1}{2}\kappa^2)\hat{\mathbf{t}} \tag{22}$$

$$\kappa_{,t} = \mathcal{R}\kappa_{,s} = \kappa_{,sss} + \frac{3}{2}\kappa^2\kappa_{,s} \ ,$$

therefore:
the motion of the nonstreching curve under a normal velocity proportional to the s-variation of the curvature is described by the mKdV equation.
iii) If $g(x) = x$ and $h = 0$ we have

$$\vec{\mathbf{v}} = \lambda^{-2}\left(\sin\theta\,\hat{\mathbf{n}} - \cos\theta\,\hat{\mathbf{t}}\right) \tag{23}$$

$$\mathcal{R}\kappa_{,t} = 0 \quad \Leftrightarrow \quad \kappa_t = \sin\theta \quad \Leftrightarrow \quad \theta_{,st} = \sin\theta \ .$$

We remark that the s-derivative of this velocity field has no components in the tangent and normal directions; then the corresponding dynamics gives rise to the so called "Tchebyshev net". Therefore (see also [29]):
the λ-independent dynamics of a curve which gives rise to a Tchebyshev net on the sphere is described by the SG equation.

It is important to remark that, although the integrable dynamics (20) we derived from *Property 3* do not depend explicitly on λ, the corresponding motions on different spheres are not in general equivalent, in the sense that, if two curves on concentric spheres form initially the same cone, this property will be lost during the evolution. However if we consider **only** a single element of the hierarchy, say the equation

$$\kappa_{,t} = \mathcal{R}^n\kappa_{,s} \tag{24}$$

its scale invariance with respect to λ:

$$s \rightarrow \lambda^{-1}s \ , \quad \kappa \rightarrow \lambda\kappa \ , \quad t \rightarrow \lambda^{-(2n+1)}t \tag{25}$$

implies that the corresponding motions on different spheres are equivalent, provided that the time used on the sphere of radius λ^{-1} is $\lambda^{-(2n+1)}$ times the one used on the sphere of radius 1.

4 The curve in $S^3(R)$ and the NLS hierarchy

In the case of the curve in S^3 the Frenet equations read

$$\begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\mathbf{t}} \\ \hat{\mathbf{n}} \\ \hat{\mathbf{b}} \end{bmatrix}_{,s} = \begin{bmatrix} 0 & \lambda & 0 & 0 \\ -\lambda & 0 & \kappa & 0 \\ 0 & -\kappa & 0 & \tau \\ 0 & 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\mathbf{t}} \\ \hat{\mathbf{n}} \\ \hat{\mathbf{b}} \end{bmatrix} \tag{26}$$

where the new ingredients are the torsion τ of the curve and the binormal vector $\hat{\mathbf{b}}$. The kinematics is now described by

$$\vec{\mathbf{r}}_{,t} = \vec{\mathbf{v}} = V\hat{\mathbf{t}} + U\hat{\mathbf{n}} + W\hat{\mathbf{b}} \ . \tag{27}$$

As before, *Property 2* implies equation (6); moreover the time evolution of the Frenet frame is given in terms of an antisymmetric matrix:

$$\begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\mathbf{t}} \\ \hat{\mathbf{n}} \\ \hat{\mathbf{b}} \end{bmatrix}_{,t} = \begin{bmatrix} 0 & \lambda V & \lambda U & \lambda W \\ -\lambda V & 0 & A & B \\ -\lambda U & -A & 0 & C \\ -\lambda W & -B & -C & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\mathbf{t}} \\ \hat{\mathbf{n}} \\ \hat{\mathbf{b}} \end{bmatrix} \ . \tag{28}$$

The commutativity condition (6) allows to express the coefficients A, B, C in terms of the velocity field and of the data:

$$\begin{aligned} A &= U_{,s} + \kappa V - \tau W \\ B &= W_{,s} + \tau U \\ C &= \frac{1}{\kappa}\left(\tau A + B_{,s} + \lambda^2 W\right) \end{aligned} \tag{29}$$

and it also implies the time evolution of the data:

$$\begin{bmatrix} \kappa \\ \tau \end{bmatrix}_{,t} = \Phi_2 \begin{bmatrix} U \\ W \end{bmatrix} + \lambda^2\Phi_1 \begin{bmatrix} U \\ W \end{bmatrix} \ . \tag{30}$$

$$V_{,s} = \kappa U \ .$$

where

$$\Phi_2 = \begin{bmatrix} \partial_s^2 + \partial_s\kappa\partial_s^{-1}\kappa - \tau^2 & -\partial_s\tau - \tau\partial_s \\ \partial_s\frac{1}{\kappa}\left(\tau(\partial_s + \kappa\partial_s^{-1}\kappa) + \partial_s\tau\right) + \kappa\tau & \partial_s\frac{1}{\kappa}(\partial_s^2 - \tau^2) + \kappa\partial_s \end{bmatrix} \ . \tag{31}$$

$$\Phi_1 = \begin{bmatrix} 1 & 0 \\ 0 & \partial_s \frac{1}{\kappa} \end{bmatrix} \ .$$

As in the previous Section *Property 3* will be satisfied by the rigid motion, obtained solving the eigenvalue problem

$$\Phi_2 \begin{bmatrix} U \\ W \end{bmatrix} = -\lambda^2 \Phi_1 \begin{bmatrix} U \\ W \end{bmatrix} , \tag{32}$$

and also by suitable Laurent expansions of U and W in terms of λ^2. In order to obtain recursively the coefficients of such expansions one has to rewrite equation (30) in the following form

$$\Phi_1^{-1} \begin{bmatrix} \kappa \\ \tau \end{bmatrix}_{,t} = \begin{bmatrix} \kappa_{,t} \\ \kappa \partial_s^{-1} \tau_{,t} \end{bmatrix} = \mathcal{R} \begin{bmatrix} U \\ W \end{bmatrix} + \lambda^2 \begin{bmatrix} U \\ W \end{bmatrix} \quad , \ \mathcal{R} = \Phi_1^{-1} \Phi_2 \ . \tag{33}$$

For $\lambda = 0$, the case of the curve in E^3, we recover the formulas of [20]. The final result is the following hierarchy of integrable dynamics

$$g(\mathcal{R}) \begin{bmatrix} \kappa_{,t} \\ \kappa \partial_s^{-1} \tau_{,t} \end{bmatrix} = h(\mathcal{R}) \left(c_1 \begin{bmatrix} \kappa_{,s} \\ \kappa \tau \end{bmatrix} + c_2 \begin{bmatrix} \kappa \tau_{,s} + 2\tau \kappa_{,s} \\ -\kappa_{,ss} - \kappa(\frac{\kappa^2}{2} - \tau^2) \end{bmatrix} \right) \ . \tag{34}$$

As it was shown in [20] this hierarchy can be transformed into the NLS hierarchy using the Hasimoto transformation

$$\psi = \kappa e^{i\sigma} \ , \ \sigma = \partial_s^{-1} \tau \ . \tag{35}$$

Here we explain the simple geometric meaning of it. Motivated by the case of the planar curve, one can partially integrate the Frenet equations (26) for the curve in S^3 introducing the following new basis of the normal plane

$$\hat{\mathbf{n}}_1 = \cos\sigma \, \hat{\mathbf{n}} - \sin\sigma \, \hat{\mathbf{b}} \ , \tag{36}$$

$$\hat{\mathbf{n}}_i = \sin\sigma \, \hat{\mathbf{n}} + \cos\sigma \, \hat{\mathbf{b}} \ .$$

This allows us to represent any vector of the normal plane as a complex number:

$$\vec{\phi} = Re\,\phi \, \hat{\mathbf{n}}_1 + Im\,\phi \, \hat{\mathbf{n}}_i \ \Leftrightarrow \ \phi = Re\,\phi + i \, Im\,\phi \tag{37}$$

In terms of the new orthonormal frame $\{\hat{\mathbf{r}}, \hat{\mathbf{t}}, \hat{\mathbf{n}}_1, \hat{\mathbf{n}}_i\}$ we have the equations

$$\begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\mathbf{t}} \\ \hat{\mathbf{n}}_1 \\ \hat{\mathbf{n}}_i \end{bmatrix}_{,s} = \begin{bmatrix} 0 & \lambda & 0 & 0 \\ -\lambda & 0 & Re\,\psi & Im\,\psi \\ 0 & -Re\,\psi & 0 & 0 \\ 0 & -Im\,\psi & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\mathbf{t}} \\ \hat{\mathbf{n}}_1 \\ \hat{\mathbf{n}}_i \end{bmatrix} \tag{38}$$

$$\begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\mathbf{t}} \\ \hat{\mathbf{n}}_1 \\ \hat{\mathbf{n}}_i \end{bmatrix}_{,t} = \begin{bmatrix} 0 & \lambda V & \lambda Re\,\phi & \lambda Im\,\phi \\ -\lambda V & 0 & Im\,\dot{\phi} & -Re\,\dot{\phi} \\ -\lambda Re\,\phi & -Im\,\dot{\phi} & 0 & \tilde{V} \\ -\lambda Im\,\phi & Re\,\dot{\phi} & -\tilde{V} & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\mathbf{t}} \\ \hat{\mathbf{n}}_1 \\ \hat{\mathbf{n}}_i \end{bmatrix} \ . \tag{39}$$

where

$$\vec{\phi} = U\hat{\mathbf{n}} + W\hat{\mathbf{b}} \ \Leftrightarrow \ \phi = (U + iW)e^{i\sigma} \ . \tag{40}$$

$$V_{,s} = Re(\phi\bar{\psi}) \ , \quad \dot{\phi} = i(\phi_{,s} + V\psi) \ , \quad \tilde{V}_{,s} = Re(\dot{\phi}\bar{\psi}) \tag{41}$$

and

$$\psi_{,t} = \lambda^2 \phi - \mathcal{Q}^2 \phi \ . \tag{42}$$

where

$$\mathcal{Q}\phi = i(\phi_{,s} + \psi\partial_s^{-1} Re(\bar{\psi}\phi)) \tag{43}$$

is the recursion operator of the NLS hierarchy [8][27]. By using *Property 3* we obtain the following class of integrable equations:

$$g(\mathcal{Q}^2)\psi_{,t} = h(\mathcal{Q}^2)\left(c_1\psi_{,s} + c_2\left(\psi_{,ss} + \frac{1}{2}|\psi|^2\psi\right)\right) \ . \tag{44}$$

In particular:
i) if $g = h = 1$, $c_1 = 0$ and $c_2 = 1$ we obtain

$$\vec{\mathbf{v}} = \kappa \hat{\mathbf{b}} \tag{45}$$

$$i\psi_{,t} = \psi_{,ss} + \frac{1}{2}|\psi|^2\psi \ \Leftrightarrow \ \begin{cases} \kappa_{,t} & = \quad 2\tau\kappa_{,s} + \kappa\tau_{,s} \\ \kappa\partial_s^{-1}\tau_{,t} & = \quad -(\kappa_{,ss} + \frac{1}{3}\kappa^3 - \kappa\tau^2) \end{cases} \ ;$$

therefore:
the motion of the nonstretching curve subjected to a binormal velocity field proportional to the curvature is described by the NLS equation.

ii) If $g = 1$, $h(x) = x$, $c_1 = 1$ and $c_2 = 0$ we obtain

$$\vec{\mathbf{v}} = \left(\frac{1}{2}\kappa^2 - \lambda^2\right)\hat{\mathbf{t}} + \kappa_{,s}\hat{\mathbf{n}} + \kappa\tau\hat{\mathbf{b}} \qquad (46)$$

$$v_{,t} = v_{,sss} + \frac{3}{2}|v|^2 v_{,s} \Leftrightarrow \left\{ \begin{array}{rcl} \kappa_{,t} & = & \kappa_{,sss} + \frac{3}{2}\kappa^2\kappa_{,s} - 3\tau^2\kappa_{,s} - 3\kappa\tau\tau_{,s} \\ \kappa\partial_s^{-1}\tau_{,t} & = & 3\tau\kappa_{,ss} + 3\kappa_{,s}\tau_{,s} + \kappa\tau_{,ss} - \kappa\tau^3 + \frac{3}{2}\kappa^3\tau \end{array} \right.$$

therefore:
the nonstretching motion of a curve subjected to the velocity field (46) is described by the complex mKdV equation.

5 The curve in $S^N(R)$

In the case of the curve in S^N the Frenet equation reads

$$\begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\mathbf{t}} \\ \hat{\mathbf{n}}_1 \\ \vdots \\ \hat{\mathbf{n}}_{N-1} \end{bmatrix}_{,s} = \begin{bmatrix} 0 & \lambda & 0 & & & \\ -\lambda & 0 & \kappa_1 & \ddots & & \\ 0 & -\kappa_1 & \ddots & \ddots & 0 & \\ & \ddots & \ddots & 0 & \kappa_{N-1} \\ & & 0 & -\kappa_{N-1} & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\mathbf{t}} \\ \hat{\mathbf{n}}_1 \\ \vdots \\ \hat{\mathbf{n}}_{N-1} \end{bmatrix} \qquad (47)$$

and the kinematics is described by

$$\vec{\mathbf{r}}_{,t} = \vec{\mathbf{v}} = V\hat{\mathbf{t}} + \sum_{j=1}^{N-1} U^j \hat{\mathbf{n}}_j \ . \qquad (48)$$

The time evolution of the Frenet frame is described by an $(N+1)\times(N+1)$ antisymmetric matrix M: the commutativity (6) allows one to express the coefficients of this matrix in terms of the velocity field and of the curvatures:

$$M_{1,j} = \lambda U^{j-2} \ , \quad j > 1 \qquad (49)$$

$$M_{i+1,j} = \frac{1}{\kappa_{i-1}}\left(M_{i,j,s} - \kappa_{j-1}M_{i,j+1} + \kappa_{j-2}M_{i,j-1} + \kappa_{i-2}M_{i-1,j}\right) \ , \quad j > i+1$$

$$M_{i,j} = -M_{j,i} \ .$$

where $\kappa_j = U^j = 0$ for $j < 0$ and $\kappa_0 = \lambda$, $U^0 = V$. Equation (6) also implies the following time evolution of the curvatures

$$\kappa_{j,t} = (M_{j+1,j+2})_{,s} + \kappa_{j-1}M_{j,j+2} - \kappa_{j+1}M_{j+1,j+3} \ , \quad j = 0,\ldots,N-1 \ . \qquad (50)$$

Like in the previous Section, these equations are in the form

$$\vec{\kappa}_{,t} = \Phi_2\vec{U} + \lambda^2\Phi_1\vec{U} \ . \qquad (51)$$

where the vectors $\vec{\kappa}$ and $\vec{U}$ have respectively components κ_j and U^j for $j = 1,\ldots,N-1$. The procedure to obtain the hierarchy of integrable dynamics from *Property 3* is the same as in the previous Sections.

Acknowledgements

We acknowledge useful discussions with A. S. Fokas and A. Sym.

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