The Mathematical Morpho-Logical View on Reasoning about Space

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Abstract

Qualitative reasoning about mereotopological relations has been extensively investigated, while more recently geometrical and spatio-temporal reasoning are gaining increasing attention. We propose to consider mathematical morphology operators as the inspiration for a new language and inference mechanism to reason about space. Interestingly, the proposed morpho-logic captures not only traditional mereotopological relations, but also notions of relative size and morphology. The proposed representational framework is a hybrid arrow logic theory for which we define a resolution calculus which is, to the best of our knowledge, the first such calculus for arrow logics.

1 Introduction

Intelligent agents sensing or exchanging information about the physical world need ways to represent and reason about space. For instance, humans navigating new areas may build 'cognitive maps' of the environment [Redish, 1999], robots engaged in a soccer game may use Monte-Carlo-Localization based on probabilistic representations from the raw sensory data [Thrun *et al.*, 2001], and crawlers analyze web pages using relative position and size, such as in the Lixto system [Gottlob and Koch, 2004], to index the pages. In fact, the way space is represented and reasoned upon varies tremendously depending on the nature of the agents and on their tasks.

A paramount criterion for representing space is that of having an agile language able to capture basic qualitative aspects of space and providing simple reasoning mechanisms for making diagrammatic deductions. In this line is the study of part-whole and topological relationships for describing qualitative aspects of region connections. This study, known as mereotopology, has been widely investigated by philosophers, logicians and, recently, computer scientists. The best known proposal in the AI literature is the region connection calculus (RCC) [Randell *et al.*, 1992]: a calculus of regions based on 8 basic qualitative mereotopological relations. With RCC one may express that a region is contained into another one, that it overlaps with another one or that it is disconnected. Reasoning is left to composition tables for deter-

mining the spatial relations among regions given the binary relations.

In general, one wants more from spatial reasoning. A way of achieving more is that of considering the logic of some theory of space and then perform deduction using the logic's inference mechanisms. [Bennett, 1995] showed that a decidable fragment of RCC can be embedded into a multimodal logic. In general, one is interested in logical theories of space: Tarski's axiomatization of elementary geometry is the most successful example and is the motivation for the recent effort of considering logical theories of space (see [Aiello *et al.*, 2007] for a wide set of spatial and spatio-temporal logics).

Our goal in the present treatment is to consider a logical approach to qualitative spatial reasoning that, without giving up on the mereotopological expressiveness, is also able to capture prominent morphological and geometrical properties of space. To fulfill this goal we turn our attention to a nifty mathematical theory.

Mathematical morphology (MM), developed in the 60s by Matheron and Serra for creating methods for the estimation of ore deposits [Matheron, 1967; Serra, 1982], underlies modern image processing, where it has a wide variety of applications. Compared with classical signal processing approaches it is more efficient in image preprocessing, enhancing object structure, and segmenting objects from the background. The idea behind MM is that one can find objects with different properties by probing an image with so called 'structuring elements'. Although Serra and Materhorn developed their theory for binary images, morphological operators exist for both grey scale and color images as well.

The connections between mathematical morphology and logic were recently highlighted by Bloch et al. for both spatial reasoning [Bloch, 2000] and agent coordination [Bloch and Lang, 2002]; while in [Aiello and van Benthem, 2002], we spelled out the connections with arrow and linear logics.

We dig into the connections hinted in these works. We show how mathematical morphology sheds a new light on spatial reasoning by considering a novel encoding into hybrid arrow logic and by providing a complete reasoning method for the introduced logic. Our proposal is not 'yet another modal RCC encoding' as the language we introduce is more expressive than RCC. For instance, we show that our morpho-logic is able to express the concept of relative size which is not expressible in RCC and which introduces a new notion

of granularity. Similarly to Aiello and van Benthem and differently from Bloch, in our representation worlds are points in space, rather than abstract worlds, allowing to reason on actual images. In [Aiello and van Benthem, 2002] we presented only the initial connections among MM and modal logic with no further study, here we take it to the next step and turn Mathematical Morpho-logic into a powerful spatial reasoning tool. Finally, we remark that the proposed logic increases the expressive power in the direction of morphology and vector spaces, and not towards incidence or projective geometries as in, e.g., [Balbiani *et al.*, 1997].

The remainder of the paper is organized as follows. In Section 2, after recalling basic MM definitions, we introduce the morpho-logic and show how to take advantage of its expressive power. In Section 3 we introduce a resolution calculus for the morpho-logic which we implemented in Haskell. Final considerations are presented in Section 4.

2 Mathematical Morpho-languages

Honoring the mathematical in the name, Mathematical Morphology has an algebraic base [Heijmans and Ronse, 1990]. Its two basic operators, the dilation and the erosion, work on a complete lattice. For example, the $\mathcal{P}(\mathbb{R}^2)$ together with the subset relation constitutes a complete lattice. Dilation is an operator that distributes over the supremum, while erosion distributes over the infimum. Given a group of automorphisms (translations) in a complete lattice $(\mathcal{L}, \leq)$ and a sup-generating subset $l \subseteq \mathcal{L}$ one can create a group structure on l, $G := (l, +, -, e)$. Using this group, the translation invariant dilation and erosion can be written in the following manner

$$A \oplus B = \bigvee \{a + b | a \in A, b \in B\} \quad \textbf{dilation} \quad (1)$$

$$A \ominus B = \bigvee \{z \in l | A_z \leq B\} \quad \textbf{erosion} \quad (2)$$

An example of a dilation is shown in Figure 1 where a binary image containing a region denoting India is dilated by a small disk region. Applying the dilation and erosion successively, one can create the so called opening and closing. An opening is an erosion followed by a dilation with the same structuring element and is used to remove connections between regions. A closing is a dilation followed by an erosion with the same structuring element and is used to fill holes of a certain size in a region. If one then considers the group G for binary images, one is actually dealing with the vector space $\mathbb{R}^2$. In this way, one sees how mathematical morphology is also a language for vector spaces.

2.1 Morpho-logic

Arrow logic is a form of modal logic where the objects, rather than being possible worlds, are transitions structured by various relations [Venema, 1996], in particular there is a binary modality for composition of arrows and a unary modality for the inverse of an arrow. An arrow usually has the following structure $w = (a, b)$ with a the beginning of the arrow and b the end. Such a language naturally models a vector space which, in turn, is the most intuitive underlying model of mathematical morphology. The connections between arrow logics and mathematical morphology were first highlighted in

[Aiello and van Benthem, 2002]. Let us first recall the basic arrow logic with its truth definition.

Definition 2.1 (arrow logic) *Let* PROP *be a set of proposition letters, then the well-formed formulas F of the* arrow logic *are:*

$$F := p|e|\neg\varphi|\varphi \vee \psi|\hat{\otimes}\varphi|\varphi\hat{\oplus}\psi,$$

where $p \in$ PROP, e is a distinguished identity element, and $\varphi, \psi \in F$

A model consists of a set of arrows, a ternary, a binary and unary relations, and a valuation function.

Definition 2.2 (arrow logic semantics) *An* arrow model $\mathcal{M}$ *is a tuple (W, C, R, I, ν) in which W is a set of arrows, $C \subseteq W \times W \times W$, $R \subseteq W \times W$ and $I \subseteq W$. Furthermore, ν is a valuation function such that $\nu :$ PROP $\mapsto \mathcal{P}(W)$. The tuple (W, C, R, I) is called* frame. *The truth of formulas is defined in a model at a given arrow w in the following way (omitting the usual base case and boolean connectives).*

$\mathcal{M}, w \Vdash e$	*iff*	$(w) \in I$	
$\mathcal{M}, w \Vdash \hat{\otimes}\varphi$	*iff*	*there exists a $v \in W$ such that* $(w, v) \in R$ and $\mathcal{M}, v \Vdash \varphi$	
$\mathcal{M}, w \Vdash \varphi\hat{\oplus}\psi$	*iff*	*there exists $v, v' \in W$ such that* $(w, v, v') \in C$, $\mathcal{M}, v \Vdash \varphi$ and $\mathcal{M}, v' \Vdash \psi$	

Thus, a formula of the form $\hat{\otimes}\varphi$ is true in a model $\mathcal{M}$ on a world w if and only if there exists a world v that one can 'reach' from w using the relation R such that v satisfies φ. In the same manner, a formula of the form $\varphi \oplus \psi$ is true in a model $\mathcal{M}$ on a world w if and only if there exist two worlds v and v' such that w, v and v' are connected via the relation C, v satisfies φ and v' satisfies ψ. For a concrete example of arrow models, showing the link between vector spaces and mathematical morphology, consider the meaning of addition for the $\hat{\oplus}$ operator, of vector negation for the $\hat{\otimes}$ and of identity vector to e, i.e.,

- $(x, y, z) \in C$ if $x = y + z$
- $(x, y) \in R$ if $x = -y$
- $(x) \in I$ if $x = e$

In this way the definition of the interpretation of $\varphi\hat{\oplus}\psi$ goes from

$$\{w | \exists v, v' \in W \text{ s.t. } (w, v, v') \in C, \ \mathcal{M}, v \Vdash \varphi, \mathcal{M}, v' \Vdash \psi\}$$

to being

$$\{w | \exists v, v' \in W \text{ s.t. } w = v + v', \ v \in \nu(\varphi) v' \in \nu(\psi)\}$$
$$=$$
$$\{v + v' | v \in \nu(\varphi), v' \in \nu(\psi)\}$$

where we have lifted the valuation function to the formulas. Considering the set defined by the dilation definition (Equation 1), one appreciates the similarities with the interpretation of $\varphi\hat{\oplus}\psi$ using vectors. In axiomatizing the language such that the relations behave as described above, the axiom $x + (-x) = e$ poses a problem because it is not valid for arbitrary subsets of the universe. Only if the subsets are singletons, the axiom is true. To avoid this problem and finally

arrive to usual vector spaces, we introduce a set of nominals which provide the expressive power to differentiate among worlds, that is, among arrows. We thus have the power to force a singleton set. We are entering the realm of hybrid logics.

Where in modal logic there is no explicit reference to the world at the language level, in hybrid logic instead one can refer to specific worlds in a model. The nominals are labels for the elements of W. If syntactically, the nominals have the same function as propositions; semantically, nominals have the restriction that the valuation function maps a nominal to a singleton set. Furthermore, the language contains the satisfaction operator $@_i\varphi$, with the intuitive meaning of bringing the valuation of the formula φ to the world labeled i, i.e., a world that satisfies i.

In addition, in a group the operation $+$ must be defined for every pair of elements, i.e., $\forall x, y \exists z$ s.t. $z = x + y$. Thus we need a way to quantify over the set of worlds. This can be done by introducing the universal modality E. Where in normal modal logic a formula is inherently local, modal logic extended with the universal modality has the power to talk about the entire model. For example, a formula of the form $E\varphi$ is true in a model if φ is true somewhere in the model. Note that the validity of a formula does not neccissarily depend on where in the model a formula is evaluated. The dual, U means that a formula must be true everywhere in the model.

We are now in the position to define the Morpho-logic. The morpho-logic is a theory in the hybrid arrow logic whose axioms are shown in Table 1. We present it as an extension of the arrow logic of Definition 2.1 in the following way.

Definition 2.3 (morpho-logic) *Let* ATOM $=$ PROP$\cup$NOM *be a set of proposition letters and names, then the well-formed formulas F of the morpho-logic are:*

$$F := a \,|\, e \,|\, \neg\varphi \,|\, \varphi \vee \psi \,|\, \hat{\otimes}\varphi \,|\, \varphi\hat{\oplus}\psi \,|\, @_i\varphi \,|\, E\varphi$$

where $a \in$ ATOM, $i \in$ NOM and $\varphi, \psi \in F$.

Definition 2.4 (morpho-model) *A model $\mathcal{M}$ is a morpho-model if $\mathcal{M}$ is a model according to definition 2.2, with ν extended to the nominals in such a way that $\nu(i)$ is a singleton set for every $i \in$ NOM.*

The semantics defined in Definition 2.2 is straightforwardly extended by introducing truth definitions for the at @ operator and for the nominals as follows:

$\mathcal{M}, w \Vdash a$	iff	$w \in \nu(a)$ with $a \in$ ATOM
$\mathcal{M}, w \Vdash @_i\varphi$	iff	there exists a $v \in W$ such that $\mathcal{M}, v \Vdash i$ and $\mathcal{M}, v \Vdash \varphi$
$\mathcal{M}, w \Vdash E\varphi$	iff	there exists a $v \in W$ such that $\mathcal{M}, v \Vdash \varphi$

We note that the @ operator becomes superfluous because one can define it using E, however, for readability we shall use the @ operator.

Remark 2.5 (Notation) We overload the terms dilation and erosion from mathematical morphology to the morpho-logic and define them as $\varphi\hat{\oplus}\psi$ and $\varphi \ominus \psi = \neg(\neg\varphi\hat{\oplus}\hat{\otimes}\psi)$, respectively.

(Ass1)	$(i\hat{\oplus}j)\hat{\oplus}k \rightarrow i\hat{\oplus}(j\hat{\oplus}k)$
(Ass2)	$i\hat{\oplus}(j\hat{\oplus}k) \rightarrow (i\hat{\oplus}j)\hat{\oplus}k$
(comm)	$i\hat{\oplus}j \rightarrow j\hat{\oplus}i$
(rev1)	$\neg\hat{\otimes}(i) \rightarrow \hat{\otimes}(\neg j)$
(rev2)	$\hat{\otimes}(\neg j) \rightarrow \neg(\hat{\otimes}i)$
(rev3$_1$)	$i\hat{\oplus}\hat{\otimes}i \rightarrow e$
(rev3$_2$)	$e \rightarrow i\hat{\oplus}\hat{\otimes}i$
(id1)	$i\hat{\oplus}e \rightarrow i$
(id2)	$i \rightarrow i\hat{\oplus}e$
(total)	$Ei \wedge Ej \rightarrow Ei\hat{\oplus}j$
(unique)	$@_ij_1\hat{\oplus}j_2 \wedge @_kj_1\hat{\oplus}j_2 \rightarrow @_ik$

Table 1: The morphological axioms.

Having defined the morpho-logic and its axioms, the next natural question to ask is what are the laws which the operators of the morpho-logic obey. The answer comes from looking at their mathematical morphological counterparts. We do not report here the full axiomatization of the hybrid arrow language as it can be found in [de Freitas *et al.*, 2002], but rather report the new axioms with morphological significance (Table 1).

Remark 2.6 (Notation) As a point of notation, we use K_{HAL} for the axiomatization of the hybrid arrow-logic.

Of all the axioms presented in Table 1, the axiom (rev3$_1$) and (rev3$_2$) are the most notable ones. The purpose of the axioms is to give the relations C, R and I a group semantics. One of the group axioms is $x + (-x) = e$. The algebraic counterpart of $\hat{\oplus}$ is $\langle+\rangle$, the $+$ operator lifted to the complex group of G. $\langle+\rangle$ operates on sets, and $a\langle+\rangle - a = e$ only holds if a is a singleton. In axiom (rev3$_1$) and (rev3$_2$), the nominals represent the singleton sets. This is best seen by looking at the atoms as sets of worlds. Since the set belonging to a nominal must be a singleton set we have met the precondition of the group axiom.

We are now ready to state the completeness result and sketch its proof. Here, by completeness we mean that given a set of frames $\mathcal{F}$ a set of axioms Λ is complete with respect to $\mathcal{F}$ if for each formula φ it is the case that $\mathcal{F} \models \varphi$ implies $\Lambda \vdash \varphi$.

Theorem 2.7 (completeness) *The axioms presented in Table 1, together with K_{HAL} and the extended set of derivation rules, are complete with respect to the set of frames defined by the axioms in Table 1.*

Proof. First, we note that the axioms in Table 1 are pure formulas. Then, we remark that every pure formula is di-persistent. Generalizing Theorem 5.3.16 in [Cate, 2005] to the hybrid arrow logic, we have that the axioms are complete for the family of frames they define. QED

2.2 Expressive power and QSR

The morpho-logic combines the power of talking about relational structures from a local perspective, typical of modal languages, with expressing global properties using the nominals to 'jump' globally from one point to another in the models, typical of hybrid ones.
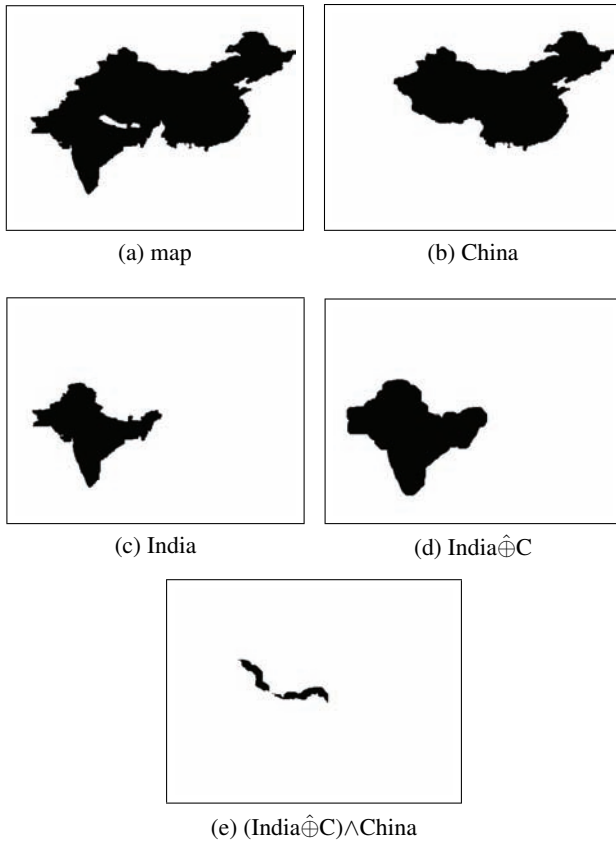
(a) map

(b) China

(c) India

(d) India$\hat{\oplus}$C

(e) (India$\hat{\oplus}$C)$\wedge$China

Figure 1: Finding the EC* relation between China and India.

$DC^*(x,y)$	$U\neg(x \wedge y)$
$EC^*(x,y)$	$E(((x\hat{\oplus}C) \wedge y)) \wedge U\neg(x \wedge y)$
$PO^*(x,y)$	$E(x \wedge y) \wedge \neg U(x \to y) \wedge$ $\neg U(y \to x)$
$x =^* y$	$U(x \leftrightarrow y)$
$TPP^*(x,y)$	$U(x \to y) \wedge \neg U(y \to x) \wedge$ $\neg U((x\hat{\oplus}C) \to y)$
$NTPP^*(x,y)$	$U(x \to y) \wedge \neg U(y \to x) \wedge$ $U((x\hat{\oplus}C) \to y)$
$TP^{*-1}(x,y)$	$U(y \to x) \wedge \neg U(x \to y) \wedge$ $\neg U((y\hat{\oplus}C) \to x)$
$NTPP^{*-1}(x,y)$	$U(y \to x) \wedge \neg U(x \to y) \wedge$ $U((y\hat{\oplus}C) \to x)$

Table 2: RCC-8 relations in the morpho-logic.

(Figure 1.d).[1] Finally, considering the region denoted by (India$\hat{\oplus}$C)$\wedge$China (Figure 1.e), we see that it is not empty, thus the image (the model) verifies the EC* relation and we safely conclude that indeed India and China are indeed neighboring countries.

Interestingly, one can use the geometric expressive power in the morpho-logic to go beyond mere topological relations and thus being more expressive than RCC. In fact, taking advantage of the dilation operation, one is able to define a notion of relative size. We say that x is smaller than y, and write $St(x,y)$ if

$$St(x,y) = Ei \wedge (TPP^*(x\hat{\oplus}i, y) \vee NTPP^*(x\hat{\oplus}i, y))$$

where and TPP^* and $NTTP^*$ are the RCC tangential proper part and non-tangential proper part, respectively. This definition of smaller than takes advantage of the fact that a dilation with a singleton set is equivalent to a translation. In plain words, the region x is smaller than region y, if there exists a translation such that x is a proper part of y.

3 Reasoning via Resolution

The morpho-logic is an expressive formalism to represent spatial properties of points and regions of space capturing topological and morphological content. The next natural question to ask is how one can use it to reason about space. We introduce a resolution calculus for the morpho-logic that is also, to the best of our knowledge, the first resolution procedure for an arrow language.

3.1 Resolution for reasoning in the morpho-logic

Resolution is a refutation theorem-proving technique [Bachmair and Ganzinger, 2001]. If in model checking one works with a specific model and verifies whether a formula is true, in theorem proving one is concerned with verifying whether there is a model for a formula. In a refutation theorem prover the goal is to show that there is no model for its negation. Before introducing the resolution calculus for the morpho-logic in the next section, we consider how theorem proving sheds

Let us now consider the expressive power in terms of qualitative spatial reasoning (QSR). The morpho-logic is able to express topological and morphological properties of points and regions. First, we turn our attention to the best-known example of topological calculus, the region connection calculus [Randell *et al.*, 1992]. The RCC language uses the primitive $C(x,y)$ holding among two regions x, y if x and y are connected and then derives a number of relations indicating the overlapping, the being part or being disconnected of regions. In the morpho-logic, the concept of connectedness is encoded using dilations: two regions A and B are connected if $A\hat{\oplus}C$ overlaps with B. C denotes the notion of connectivity that is being used. In discrete binary images for example this could be 4- or 8-connectivity. One can define RCC-like relations (denoted with a terminating *) using the morpho-logic as shown in Table 2 where E is the existential modality defined as dual of the universal one: $E\varphi := \neg U\neg\varphi$.

Let us now consider an example in the domain of binary images using the RCC relations as defined via the morpho-logic. For instance, we want to check whether India and China are neighboring countries (i.e., in the EC* relation) in the map shown in Figure 1.a. Suppose that China and India are denoted by the propositions 'China' and 'India' (Figure 1.b and Figure 1.c, respectively). Then, we can consider the dilation of the India region by a relatively small disc C

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[1]Note that the region in Figure 1.d is slightly bigger than the one in Figure 1.c due to the dilation India$\hat{\oplus}$C.

light on mathematical morphology logical view on reasoning about space and benefits mathematical morphology in return.

In traditional spatial reasoning calculi (such as RCC or Allen's one dimensional interval calculus [Cohn and Hazarika, 2001]) a paramount task is that of defining composition tables for the calculi relations. A composition table is a compact representation for assessing which relation holds among two locations based on knowledge of other relations. For example, suppose that there is a relation R_1 between location a and b, and a relation R_2 between b and c. The entry in the composition table tells us which relations are possible between a and c. Composition tables are created resorting to human reasoning, an ad-hoc program performing exhaustive search, or a theorem prover. A successful example of the latter is Bennett's use of resolution for reasoning with RCC relations encoded in intuitionistic logic [Bennett, 1994]. By having resolution for the morpho-logic it is also possible to create composition tables for morphological relations. Not only, it is also possible to check dynamically the validity of formulas and the composition of any two given relations expressible in the morpho-language.

Conversely, a resolution based theorem prover is also a powerful tool in the hands of the mathematical morphology expert. In mathematical morphology one of the typical tasks is that of identifying filters, verify their formulation, and then test experimentally their effectiveness on collections of images. For example, the salt-and-pepper filter, used to filter out noise from an image, is known to be idempotent. The task of design and verification of the filter is based on the expertise of the mathematical morphology scientist. With a resolution based theorem prover the verification of filter properties can be automated. Furthermore, if one couples the theorem prover with a formula generator, one has a way of identifying new and potentially useful filters.

3.2 The resolution calculus

The resolution calculus for the morpho-logic builds on the fact that nominals are available, making it possible to perform resolution recursively inside the modal operators. Resolution rules can be applied to clauses. A clause is a set of formulae and it is true if one of the formula in the clause is true. We extend the resolution calculus for the basic hybrid logic presented in [Areces *et al.*, 2001] by means of the additional rules shown in Tables 3, 5, 6. This calculus uses the assumption that all the formulas present in the clauses are given in negated normal form, i.e. only atoms can occur in the scope of a negation. The first set of rules (Table 3) deals with the binary $\oplus$ and unary $\hat{\otimes}$ modalities, where the symbol $\hat{\underline{\oplus}}$ is the dual of $\hat{\oplus}$ and is defined as $\varphi \hat{\underline{\oplus}} \psi = \neg(\neg\varphi \hat{\oplus} \neg\psi)$. Equivalently, $\hat{\underline{\otimes}}$ is the dual of $\hat{\otimes}$ and is defined as $\hat{\underline{\otimes}} = \neg \hat{\otimes} \neg$. The second set of rules (Table 4) takes care of the universal modalities E and U. Finally, the third set of rules (Tables 5, 6) deals with the axioms of the morpho-logic (Definition 1).

In the case of morpho resolution, we have that a clause is a set of formulas of the form $@_i\varphi$. As usual, the semantics of a clause is the disjunction of its elements. The resolution calculus then works on the clauses by applying the morpho resolution rules. Given a set of formulas $S = \varphi_1, ..., \varphi_n$, this

$$(\hat{\underline{\otimes}}) \quad \frac{Cl_1 \cup \{@_i\hat{\underline{\otimes}}\varphi\} \quad Cl_2 \cup \{@_i\hat{\otimes}j\}}{Cl_1 \cup Cl_2 \cup \{@_j\varphi\}}$$

$$(\hat{\otimes}) \quad \frac{Cl \cup \{@_i\hat{\otimes}\varphi\}}{\substack{Cl \cup \{@_i\hat{\otimes}j\} \\ Cl \cup \{@_jnf(\neg\varphi)\}}} \quad \text{where j is new}$$

$$(\oplus) \quad \frac{\substack{Cl_1 \cup \{@_i\varphi\oplus\psi\} \\ Cl_2 \cup \{@_i\neg(\neg j_1\oplus\neg j_2)\}}}{Cl_1 \cup Cl_2 \cup \{@_{j_1}\varphi, @_{j_2}\psi\}}$$

$$(\oplus) \quad \frac{Cl \cup \{@_i(\varphi \oplus \psi)\}}{\substack{Cl \cup \{@_i j_1\oplus j_2)\} \\ Cl \cup \{@_{j_1}\varphi\} \\ Cl \cup \{@_{j_2}\psi\}}} \quad \substack{\text{where } j_1 \text{ and } j_2 \\ \text{are new}}$$

Table 3: Morpho resolution rules for the morpho modalities.

$$(@E) \quad \frac{Cl \cup \{@_iE\varphi\}}{Cl \cup \{E\varphi\}}$$

$$(@U) \quad \frac{Cl \cup \{@_iU\varphi\}}{Cl \cup \{U\varphi\}}$$

$$(E) \quad \frac{Cl \cup \{E\varphi\}}{Cl \cup \{@_i\varphi\}} \quad \text{for some new } i \in \text{NOM}$$

$$(U) \quad \frac{Cl_1 \cup \{U\varphi\} \quad Cl_2 \cup \{@_i\psi\}}{Cl \cup \{@_i\varphi\}}$$

Table 4: Resolution rules for the universal modality

is satisfiable if and only if the set of clauses $@_i\varphi_1, ...@_i\varphi_n$ is satisfiable. Formally, we have the following definition of morpho resolution.

Definition 3.1 (morpho resolution) *Given φ, a refutation by morpho resolution of φ is a sequence of morpho clauses $C_1, ..., C_n$ such that for all $i \in \{1..n\}$ either*

  1. C_i is in φ, or

  2. C_i is a resolvent of C_j, C_k according to a morpho rule

where C_n is the empty clause and a morpho rule is one of the rules in Tables 3, 5, 6, or in [Areces et al.*, 2001].*

If there exists no such refutation, φ is *satisfiable*. We say that φ is *valid* if there is a refutation of $\neg\varphi$ (i.e., $\neg\varphi$ is unsatisfiable). We write $\vdash \varphi$ for a valid φ.

Finally, we want to be sure that reasoning with the resolution calculus is correct and complete. The following theorem does the job. For the full proof we refer the reader to [Ottens, 2006] and provide only a proof sketch here.

Theorem 3.2 (completeness) *Given a set of clauses Σ of the morpho-logic, Σ is unsatisfiable if and only if there exists a refutation of Σ using the morpho resolution of Definition 3.1.*

(Rev$_1$)	$\dfrac{Cl_1 \cup \{@_i\underline{\hat{\otimes}}\varphi\}}{Cl_1 \cup \{@_i\overline{\hat{\otimes}}\varphi\}}$
(Rev$_2$)	$\dfrac{Cl_1 \cup \{@_i\overline{\hat{\otimes}}\varphi\}}{Cl_1 \cup \{@_i\underline{\hat{\otimes}}\varphi\}}$
(Rev$_{3_1}$)	$\dfrac{Cl_1 \cup \{@_i j_1\hat{\oplus}j_2\} \quad Cl_2 \cup \{@_{j_2}\hat{\otimes}j_1\} \quad Cl_3 \cup \{@_k e\}}{Cl_1 \cup Cl_2 \cup Cl_3 \cup \{@_i k\}}$
(Rev$_{3_2}$)	$\dfrac{Cl_1 \cup \{@_i e\} \quad CL_2 \cup \{@_j \varphi\}}{Cl_1 \cup Cl_2 \cup \{@_i j\hat{\oplus}(\hat{\otimes}j)\}}$
(Id$_1$)	$\dfrac{Cl_1 \cup \{@_{j_1} e\} \quad Cl_2 \cup \{@_i j_1\hat{\oplus}j_2\}}{Cl_1 \cup Cl_2 \cup \{@_i j_2\}}$
(Id$_2$)	$\dfrac{Cl_1 \cup \{@_i \varphi\} \quad Cl_2 \cup \{@_j e\}}{Cl_1 \cup \{@_i(j\hat{\oplus}i)\}}$

Table 5: Morpho resolution rules for the morpho axioms I.

Proof. The proof of refutational completeness works by showing that if there is no refutation, a model of Σ exists on which all the axioms are valid. QED

4 Concluding Remarks

Driven by the Mathematical Morphology view of space, we introduced a language based on hybrid arrow logic to reason about space. The logic, which is a theory in the hybrid arrow logic defined by the axioms of Table 1, is a powerful language to express morphological as well as mereotopological properties of space. To reason in the morpho-language we introduced a resolution calculus. To the best of our knowledge, this is the first resolution calculus for arrow logics. We proved completeness of our language and calculus, while leaving open for future research issues of complexity [Renz and Nebel, 1999].

We implemented the morpho resolution in Haskell (http://www.haskell.org) as an extension of the HyLoRes theorem prover [Areces and Heguiabehere, 2001]. We used the theorem prover for the correctness of a number of theorems (including theorems based on the 'smaller than' definition of Section 2). We leave for future research the enhancement and evaluation of the implementation. The theorem prover is of particular usefulness when showing that the morpho-logic is not only interesting in AI for performing spatial reasoning, but is also a tool for the computer vision expert that needs to check, or perhaps even generate, new morphological filters.

Ackowledgements

(Ass$_1$)	$\dfrac{\begin{array}{c}Cl_1 \cup \{@_i\neg(\neg\, j_1\oplus\neg j_2)\} \\ Cl_2 \cup \{@_{j_2}\neg(\neg\, s_1\oplus\neg s_2)\} \\ Cl_3 \cup \{@_{j_1}\varphi\} \quad Cl_4 \cup \{@_{s_1}\psi\} \\ Cl_5 \cup \{@_{s_2}\xi\}\end{array}}{\begin{array}{c}Cl_1 \cup Cl_2 \cup Cl_3 \cup Cl_4\cup \\ Cl_5 \cup \{@_{j_1}\neg(\neg\, k_1\oplus\neg k_2)\} \\ Cl_1 \cup Cl_2 \cup Cl_3 \cup Cl_4 \cup Cl_5 \cup \{@_{k_1}\varphi\} \\ Cl_1 \cup Cl_2 \cup Cl_3 \cup Cl_4 \cup Cl_5 \cup \{@_{k_2}\psi\} \\ Cl_1 \cup Cl_2 \cup Cl_3 \cup Cl_4 \cup Cl_5 \cup \{@_{j_2}\xi\}\end{array}}$
(Ass$_2$)	$\dfrac{\begin{array}{c}Cl_1 \cup \{@_i\neg(\neg\, j_1\oplus\neg j_2)\} \\ Cl_2 \cup \{@_{j_1}\neg(\neg\, s_1\oplus\neg s_2)\} \quad Cl_3 \cup \{@_{j_2}\varphi\} \\ Cl_4 \cup \{@_{s_1}\psi\} \quad Cl_5 \cup \{@_{s_2}\xi\}\end{array}}{\begin{array}{c}Cl_1 \cup Cl_2 \cup Cl_3 \cup Cl_4\cup \\ Cl_5 \cup \{@_{j_2}\neg(\neg\, k_1\oplus\neg k_2)\} \\ Cl_1 \cup Cl_2 \cup Cl_3 \cup Cl_4 \cup Cl_5 \cup \{@_{k_1}\psi\} \\ Cl_1 \cup Cl_2 \cup Cl_3 \cup Cl_4 \cup Cl_5 \cup \{@_{k_2}\xi\} \\ Cl_1 \cup Cl_2 \cup Cl_3 \cup Cl_4 \cup Cl_5 \cup \{@_{j_1}\varphi\}\end{array}}$
(Comm)	$\dfrac{Cl_1 \cup \{@_i j_1 \oplus j_2\}}{Cl_1 \cup \{@_i j_2 \oplus j_1\}}$
(total)	$\dfrac{Cl_1 \cup \{\varphi\} \quad Cl_2 \cup \{\psi\}}{Cl_1 \cup Cl_2 \cup \{Ei\hat{\oplus}j\}}$, for i occurs in φ and j occurs in ψ
(unique)	$\dfrac{Cl_1 \cup \{@_i j_1\hat{\oplus}j_2\} \quad Cl_2 \cup \{@_k j_1\hat{\oplus}j_2\}}{Cl_1 \cup Cl_2 \cup \{@_i k\}}$

Table 6: Morpho resolution rules for the morpho axioms II.

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