Coalitions in Action Logic

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Abstract

If modal logics for coalitions need to be applied, one must know how to translate coalition power into agents' actions. To these days, the connection between coalition power and the actions of the agents has not been studied in the literature. This paper fills the gap by presenting a semantic translation from Pauly's Coalition Logic to a (fragment of an) action logic. The interpretation of the empty coalition in Pauly's system and the representation of the notion of ability are discussed.

Keywords: modal logic, coalitions, coordination and cooperation, multiagent systems, concurrency

1 Introduction

In the last decade, there has been increasing interest in logics suited for reasoning about groups of agents [1; 9; 4; 11; 10] and some systems emerged as paradigmatic in this area. Among these, Pauly's *Coalition Logic* [8; 9] seems to identify the core properties of coalitions and it is practically taken as a parametric system to verifying the adequacy of other logics in this area.

Generally, these modal logics for coalitions address the relationship between models and relative expressivity of the systems without tackling the relationship between coalition power and each agent's actions. They essentially capture *implicit* coalition power as in "coalition C can enforce proposition φ", which here we contrast with logics of *explicit* coalition power, "coalition C, doing such and such, can enforce proposition φ". The relationship between these two types of systems must be studied if we want to be able to use the general view provided by coalition logics in frameworks where actions and their consequences can be evaluated (cf. [10]). The study of such a relationship is the goal of this work.

The first issue is the choice of the target language that can relate coalition power to agents' actions. One way to represent coalition power is to introduce a unary modality with two indices, say $[C, a]$, where C is a coalition and a is an action; informally, given a proposition p, formula $[C, a]p$ stands for "coalition C can enforce p by performing action a". If a coalition C is capable of enforcing p, we need a coalition action to express this fact in the language. If a model has n possible worlds, there are 2^n distinct subsets of worlds and (roughly) the same number of coalition actions are needed to effectively capture C's power in this model.

A way out is to refer to actions without actually naming them. This seems to match the general idea of Coalition Logic since, informally, what it captures is not "action such and such performed by C enforces p" but "C has a way to enforce p", i.e., "there exists a *coalition action* for C to enforce p". This latter approach guarantees a concise language by introducing quantifiers. We can implement it in two ways: on the one hand using variables for coalition actions and, on the other hand, using variables for agent actions. At a closer analysis, the first option gives us $[C, \exists x]$ which does not differ in expressive power from Pauly's coalition modality $[C]$.

The other option is what we study in this paper and requires a logic with new features. Such a logic has already been studied with other motivations in [3]. Here we isolate a fragment of that system and show how to translate coalition logic in it.

Overview: sections 2 and 3 present the coalition logic ($\mathsf{C}\mathcal{L}$) and the action logic ($\mathsf{A}\mathcal{L}$). Section 4 discusses the assumptions of the translation and defines the coalition action logic $\mathsf{CA}\mathcal{L}$. Sections 5 and 6 give the model correspondence. Section 7 discusses the approach and the notion of ability.

2 The Coalition Logic $\mathsf{C}\mathcal{L}$

The coalition logic $\mathsf{C}\mathcal{L}$ [9] is a modal system for reasoning about groups of agents. The logic is developed to formally prove what a coalition of agents can bring about in a multi-agent system. The semantics associates an (extensive) game of perfect information to each state in the system. The central notion is *effectiveness*: a coalition of agents is effective for a set of states in a game if the coalition can guarantee that the system will necessarily move to one state in the set [7].

The language of $\mathsf{C}\mathcal{L}$ is a type of propositional multi-modal logic. It is built out of a countable set of proposition identifiers *PropId*, with $\perp \in PropId$, and contains $2^{|N|}$ modal operators for some fixed finite set N. The modality markers are the subsets of N: $[C]$ is a modality in $\mathsf{C}\mathcal{L}$ if and only if $C \subseteq N$. *Formulas* in $\mathsf{C}\mathcal{L}$ are defined inductively:

a) all elements of $PropId$ are formulas (*atomic formulas*)

b) $\varphi \vee \psi$ and $\neg\varphi$ are formulas if φ and ψ are formulas

c) $[C]\varphi$ is a formula if $C \subseteq N$ and φ is a formula

The intuition behind the formalism is that the modal operator $[C]$ expresses the power of the group C of agents. The formula $[C]\varphi$ states that the agents in C, acting as a coalition, can enforce φ no matter what the remaining agents do.

The semantics is based on *game frames* [9] that here we dub *standard coalition frames*. Note that standard coalition frames are not Kripke frames because modalities in $\mathsf{C}\mathcal{L}$ do not satisfy the *normality* condition, i.e, they do not distribute over implication.

Definition 2.1 (Coalition Frame)
A Coalition Frame for a set N *is a pair* $\mathcal{F}_N = \langle W, R\rangle$ *where* W *is a non-empty set (the set of states) and* R *is a subset of* $W \times \mathcal{P}(N) \times \mathcal{P}(W)$. *Furthermore, for all* $s \in W$, $C \subseteq N$, *and* $X, X' \subseteq W$, R *satisfies the following:*

I) $(s, C, \emptyset) \notin R$

II) $(s, C, W) \in R$

III) If $(s, \emptyset, X) \notin R$, *then* $(s, N, W \setminus X) \in R$

IV) If $X \subseteq X'$ *and* $(s, C, X) \in R$, *then* $(s, C, X') \in R$

V) If $C_1 \cap C_2 = \emptyset, (s, C_1, X_1) \in R$, *and* $(s, C_2, X_2) \in R$, *then* $(s, C_1 \cup C_2, X_1 \cap X_2) \in R$

Pauly [9] introduces R as a function $R : W \to (\mathcal{P}(N) \to \mathcal{P}(\mathcal{P}(W)))$. The two definitions are equivalent.

Proposition 2.1 (Coalition monotonicity) *For all* $s \in W$ *and* $C \subseteq C' \subseteq N$, $(s, C, X) \in R \to (s, C', X) \in R$.

Definition 2.2 (Coalition Structure)
A Coalition Structure for a set N *is a triple* $\mathcal{M}_{\mathcal{F}} = \langle W, R, [\![\cdot]\!]\rangle$ *where:*

- $\langle W, R\rangle$ *is a coalition frame* $\mathcal{F}_N$;
- $[\![\cdot]\!]$ *is a valuation function PropId* $\mapsto \mathcal{P}(W)$ *s.t.* $[\![\bot]\!] = \emptyset$.

Since N is fixed in advance, we drop it in $\mathcal{F}_N$. Also, we write $R(C)$ for the set of pairs (s, V) such that $(s, C, V) \in R$ and $R(s)(C)$ for the set of sets V such that $(s, C, V) \in R$.

Fix a coalition structure $\mathcal{M}_{\mathcal{F}}$ and a state s. We write $\mathcal{M}_{\mathcal{F}}, s \models \varphi$ to mean that the $\mathsf{C}\mathcal{L}$-formula φ is *true* (equivalently, *satisfied*) at state *s* of structure $\mathcal{M}_{\mathcal{F}}$.

1) $\mathcal{M}_{\mathcal{F}}, s \models p$ if $s \in [\![p]\!]$ (for p atomic)
2) $\mathcal{M}_{\mathcal{F}}, s \models \neg\varphi$ if $\mathcal{M}_{\mathcal{F}}, s \not\models \varphi$
3) $\mathcal{M}_{\mathcal{F}}, s \models \varphi \vee \psi$ if $\mathcal{M}_{\mathcal{F}}, s \models \varphi$ or $\mathcal{M}_{\mathcal{F}}, s \models \psi$
4) $\mathcal{M}_{\mathcal{F}}, s \models [C]\varphi$ if $(s, C, \{s' \in W \mid \mathcal{M}_{\mathcal{F}}, s' \models \varphi\}) \in R$

We write $\mathcal{M}_{\mathcal{F}} \models \varphi$ to mean that formula φ is *valid in* $\mathcal{M}_{\mathcal{F}}$, that is, it is true at each state of $\mathcal{M}_{\mathcal{F}}$. A *coalition model* for a set of formulas Σ in $\mathsf{C}\mathcal{L}$ is a structure $\mathcal{M}_{\mathcal{F}}$ such that all formulas $\varphi \in \Sigma$ are valid in $\mathcal{M}_{\mathcal{F}}$. We write $\mathcal{F} \models \varphi$ to mean that φ is valid in each model based on frame $\mathcal{F}$.

The interpretation of modal operators in $\mathsf{C}\mathcal{L}$ follows the approach called *minimal models* [5], cf. clause 4). From condition $I)$ on relation R, a structure for $\mathsf{C}\mathcal{L}$ is serial with respect to all the modalities.

Axioms for Coalition Logic:

(1) $\neg[C]\bot$; (2) $[C]\top$

(3) $\neg[\emptyset]\neg\varphi \to [N]\varphi$; (4) $[C](\varphi \wedge \psi) \to ([C]\varphi \wedge [C]\psi$

(5) let $C_1 \cap C_2 = \emptyset$

$$([C_1]\varphi_1 \wedge [C_2]\varphi_2) \to [C_1 \cup C_2](\varphi_1 \wedge \varphi_2)$$

$\mathsf{C}\mathcal{L}$ is determined by the class of coalition structures [9].

3 The Action Logic $\mathsf{A}\mathcal{L}$

We consider the multi-agent modal logic given in [3]. This language has the characteristics of merging modalities and quantifiers to model true concurrency, and was shown to be complete and decidable with respect to Kripke semantics. For our task, we will apply the fragment of this system selected below and that we dub $\mathsf{A}\mathcal{L}$.

Fix a set *PropId* of proposition identifiers, with $\bot \in PropId$, and a disjoint set of variables, *Var*. We will use p_i for proposition identifiers and x_i for variables.

Definition 3.1 (Modality Marker for $\mathsf{A}\mathcal{L}$)
A modality marker *for* $\mathsf{A}\mathcal{L}$ *is a* k*-column of quantified variables*

$$\begin{matrix} Q_1 x_1 \\ Q_2 x_2 \\ \vdots \\ Q_k x_k \end{matrix} \qquad Q_i \in \{\forall, \exists\}$$

To maintain a direct connection[1] *with [3], no variable can occur more than once in a modality marker.*

The set of formulas for $\mathsf{A}\mathcal{L}$ is the smallest set satisfying the following clauses:

a. all elements of $PropId$ are formulas (*atomic formulas*)
b. $\varphi \vee \psi$ and $\neg\varphi$ are formulas if φ and ψ are formulas
c. $[M]\varphi$ is a formula if M is a modality marker and φ is a formula

(From this, all $\mathsf{A}\mathcal{L}$ formulas turn out to be closed.)

There is an obvious bijection between the set of modality markers and the set of modal operators in the language. We will use the two notions indifferently.

Definition 3.2 (k-Action) *Given a set Act of actions, a* k-action *is any column of* k *elements* $\begin{matrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{matrix}$ *with* $\alpha_i \in$ *Act.*[2]

Definition 3.3 (Multi-agent Kripke Frame for $\mathsf{A}\mathcal{L}$)
A Multi-agent Kripke Frame for $\mathsf{A}\mathcal{L}$ *is a triple* $\mathcal{K} = \langle W, Act, R\rangle$ *where:*

- W *is a non-empty set* (the set of states),
- *Act is a non-empty set* (the set of actions)*, and*
- R *is an* (accessibility) *relation mapping* k*-actions* A*, over Act, to binary relations on* W: $R(A) \subseteq W \times W$.

Definition 3.4 (Multi-agent Kripke Structure for $\mathsf{A}\mathcal{L}$)
A Multi-agent Kripke Structure for $\mathsf{A}\mathcal{L}$ *is a 4-tuple* $\mathcal{M}_{\mathcal{K}} = \langle W, Act, R, [\![\cdot]\!]\rangle$ *where:*

- $\langle W, Act, R\rangle$ *is a multi-agent Kripke frame* $\mathcal{K}$;
- $[\![\cdot]\!]$ *is a valuation function PropId* $\mapsto \mathcal{P}(W)$ *s.t.* $[\![\bot]\!] = \emptyset$.

In the remaining of the paper, we use the terms Kripke frames (structures, respectively) to refer to multi-agent Kripke frames (structures).

Definition 3.5 (Instances) *Fix a* k*-action* A *and a modality marker* M. $A(i)$ *(*$M(i)$*, resp.) is the* i*-th entry of* A *(*M*). For each* j*, we say that* $A(j)$ instantiates *the* j*-th variable of* M.

[1]This connection is a central motivation for this work (section 7) and justifies our use of columns as modality markers.

[2]In [3] these are called "basic k-actions."

Fix a Kripke structure $\mathcal{M}_\mathcal{K}$ and a state s. Relation $\models$ is defined recursively for $\mathcal{M}_\mathcal{K}$ as follows:

1. $\mathcal{M}_\mathcal{K}, s \models p$ if $s \in [\![p]\!]$ (for p atomic)
2. $\mathcal{M}_\mathcal{K}, s \models \neg\varphi$ if $\mathcal{M}_\mathcal{K}, s \not\models \varphi$
3. $\mathcal{M}_\mathcal{K}, s \models \varphi \vee \psi$ if $\mathcal{M}_\mathcal{K}, s \models \varphi$ or $\mathcal{M}_\mathcal{K}, s \models \psi$
4. Let $x_1, \ldots, x_r$ ($r \geq 0$) be all the existentially quantified variables in M and let x_j occur at $M(i_j)$, $\mathcal{M}_\mathcal{K}, s \models [M]\varphi$ if $\exists \alpha_1, \ldots, \alpha_r \in Act$ (α_m, α_n not necessarily distinct) such that for all k-action A with $A(i_j) = \alpha_j$ (i.e., α_j instantiates x_j in M) for all $j \leq r$, if $(s, s') \in R(A)$ then $\mathcal{M}_\mathcal{K}, s' \models \varphi$.

We write $\mathcal{M}_\mathcal{K} \models \varphi$ to mean that formula φ is *valid in* $\mathcal{M}_\mathcal{K}$, that is, it is true at each state of $\mathcal{M}_\mathcal{K}$. A *Kripke model* for a set of formulas Σ in $\mathsf{A}\mathcal{L}$ is a structure $\mathcal{M}_\mathcal{K}$ such that all formulas $\varphi \in \Sigma$ are valid in $\mathcal{M}_\mathcal{K}$. We write $\mathcal{K} \models \varphi$ to mean that φ is valid in each model based on frame $\mathcal{K}$.

Let k be the number of agents in the multi-agent system one wants to model. Fix an arbitrary order of the agents and let $\mathcal{A}_1$ be the first agent, $\mathcal{A}_2$ the second agent, ..., $\mathcal{A}_k$ the k-th agent. Technically, a formula $[M]p$ with existential entries $i_1, i_2, \ldots$, is true in a state s when there exist values for the existentially quantified variables such that no matter the values selected for the universally quantified variables, the corresponding k-action brings (through R) only to states verifying p. From the point of view of a multi-agent system, a formula $[M]p$ is true if the agents $\mathcal{A}_{i_1}, \mathcal{A}_{i_2}, \ldots$ (those whose positions correspond to existential entries of M) can perform a set of actions that force p to be verified no matter the actions executed by the remaining agents. (The general perspective behind $\mathsf{A}\mathcal{L}$ is presented in [2].)

4 The Coalition Action Logic $\mathsf{CA}\mathcal{L}$

In this section, we analyze the relationship between $\mathsf{C}\mathcal{L}$ and $\mathsf{A}\mathcal{L}$. A byproduct of this comparison is the clarification of a common misunderstanding on the notion of coalition in $\mathsf{C}\mathcal{L}$.

At the core of Coalition Logic there is the notion of *effectiveness* which is taken from the theory of social choice [7]. This notion captures the case of a group of agents that can force the system to evolve to states where some given property holds, no matter what the other agents do. The actions of these agents are thus *effective* in achieving the goal. Adopting such an interpretation, one assumes that the realization of the property is a common goal for the agent in the coalition and that they are capable of and intend to coordinate their actions. As a slogan, one can say that Coalition Logic studies the *existence of a joint strategy for a common goal* (the strategy itself is unspecified and possibly unknown). This approach is part and parcel of the view proposed by Coalition Logic, i.e., that of a system as a whole where the references to agents are more a way of speaking at the informal level than an effective (no pun intended) necessity.

Recall the informal reading of $[C]$ in $\mathsf{C}\mathcal{L}$: formula $[C]\,p$ stands for "the group C of agents has the power to enforce a state where p holds." Assume that there are k ($k \geq 1$) agents in the multi-agent system and that an ordering of the agents has been fixed. Let agents $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_r$ form group C. From the semantics of $\mathsf{A}\mathcal{L}$, it is natural to associate $[C]\,p$ with the following formula of $\mathsf{A}\mathcal{L}$ (let p be a formula in both $\mathsf{C}\mathcal{L}$ and $\mathsf{A}\mathcal{L}$):

$$\begin{bmatrix} \exists x_1 \\ \exists x_2 \\ \vdots \\ \exists x_r \\ \forall x_{r+1} \\ \vdots \\ \forall x_k \end{bmatrix} p \qquad (1)$$

Thus, we can identify this set C with the set of indices $\{1, \ldots, r\}$. The modality in (1) is not normal, due to the existential quantifiers occurring in it, although all its instances are. This fact is at the core of the connection between $\mathsf{C}\mathcal{L}$ and $\mathsf{A}\mathcal{L}$. However, things do not work quite right. Exploiting the previous correspondence, we connect modality $[\emptyset]$ (for the empty coalition) to

$$\begin{bmatrix} \forall x_1 \\ \forall x_2 \\ \vdots \\ \forall x_k \end{bmatrix} \qquad (2)$$

Note that $[\emptyset]$ can bring about non-trivial consequences.[3] Since in $\mathsf{A}\mathcal{L}$ only agents have this power and since $[\emptyset]$ corresponds to a coalition with no member, the correspondence given by (2) falls short to model the $\mathsf{C}\mathcal{L}$ modality.[4] There are two possible ways out: to enrich the formalization of $\mathsf{C}\mathcal{L}$ by constraining further modality $[\emptyset]$ or to accept (and motivate) the above peculiarity of $\mathsf{C}\mathcal{L}$ by capturing it in the translation. The two choices have different consequences. Here we follow the second option. (For the first, proceed according the method given below using operators (1) and (2).)

The correspondence we formalize is between a $\mathsf{C}\mathcal{L}$ system with k agents and an $\mathsf{A}\mathcal{L}$ system with $k+1$ agents as follows

$$[\emptyset] \;\cong\; \begin{bmatrix} \exists x_0 \\ \forall x_1 \\ \vdots \\ \forall x_k \end{bmatrix} \quad ; \quad [\{i_1, .., i_r\}] \;\cong\; \begin{bmatrix} \exists x_0 \\ Q_1 x_1 \\ \vdots \\ Q_k x_k \end{bmatrix} \qquad (3)$$
$$Q_j = \exists \text{iff } j \in \{i_1, .., i_r\}$$

In this way, the new agent (associated to x_0 in row 0) is always in the coalition group. If the coalition C is empty, then the agent at row 0 is the *only* member of the corresponding coalition in $\mathsf{CA}\mathcal{L}$. What is the role of the new agent? Formally, it makes the translation possible since it allows to discriminate applications of the $[\emptyset]$ operator. Informally, it justifies the fact that an "empty coalition" can bring non-trivial consequences. The 0-th agent plays the role of the *environment* of the system (or *nature* for a more suggestive reading) whose power corresponds to that of the empty coalition in $\mathsf{C}\mathcal{L}$. In short, we can say that the axiomatization of $\mathsf{C}\mathcal{L}$ makes sure that a coalition can force some state *provided* the environment plays on its side.

Since the agents in a coalition, say $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_r$, act according to a *joint* strategy, the choices at the existential entries (here entries at columns 0 to r) have to be made allowing full coordination. This means that the existentially quantified variables $x_0, x_1, \ldots, x_r$ are instantiated at the same time,

[3] This observation does not apply when modality $[\emptyset]$ is forced to be universal, i.e., if for all s, $(s, \emptyset, X) \in R$ implies $X = W$.

[4] An orthogonal issue is the normality of modality (2) which, generally speaking, might not be a desired property for $[\emptyset]$, although it is forced by the $\mathsf{C}\mathcal{L}$ system of section 2.

i.e., sequentially as they were forming an unbroken prefix list bounded by a unique existential quantifier. Furthermore, the joint strategy of the agents does not depend on the actions performed by the remaining agents since the agents are acting simultaneously. That is, the instantiation of variables $x_0, x_1, \ldots, x_r$ does not depend on the values chosen for variables $x_{r+1}, \ldots, x_k$. This argument illustrates that the semantics inherited from $\mathsf{C}\mathcal{L}$ through the translation given above, corresponds to the semantics of $\mathsf{A}\mathcal{L}$. Needless to say that $\mathsf{C}\mathcal{L}$ and $\mathsf{A}\mathcal{L}$ are equivalent on the (common) propositional fragment of their languages.

Fix $k+1$ variables: $x_0, x_1, \ldots, x_k$. We call $\mathsf{CA}\mathcal{L}$ the language $\mathsf{A}\mathcal{L}$ (for $k+1$ agents) restricted to operators $[M]$ such that $M(0) = \exists x_0$ and $M(i) = Q_i x_i$ with $Q_i \in \{\forall, \exists\}$. (Note that the first entry in a modality for $\mathsf{CA}\mathcal{L}$ has index 0. In this way, we preserve the correspondence of the agents $\mathcal{A}_1, \ldots, \mathcal{A}_k$ in $\mathsf{C}\mathcal{L}$ and in $\mathsf{CA}\mathcal{L}$.) The other characteristics of $\mathsf{CA}\mathcal{L}$ are easily inherited from section 3.

Assume $\mathsf{C}\mathcal{L}$ and $\mathsf{CA}\mathcal{L}$ have the same proposition identifiers *PropId* (with $\perp \in PropId$). According to (3) we define a one-to-one function τ from $\mathsf{C}\mathcal{L}$ to $\mathsf{CA}\mathcal{L}$ as follows

1_τ) if $p \in PropId$, then $\tau(p) = p$;

2_τ) if $\varphi = \psi \vee \chi$, then $\tau(\varphi) = \tau(\psi) \vee \tau(\chi)$;

3_τ) if $\varphi = \neg\psi$, then $\tau(\varphi) = \neg\tau(\psi)$;

4_τ) if $\varphi = [\{i_1, \ldots, i_r\}]\psi$, then $\tau(\varphi) = [M]\tau(\psi)$ where $M(j)$ is existential iff $j \in \{0, i_1, \ldots, i_r\}$.[5]

We sometimes write $[C_\tau]$ for the τ-image of operator $[C]$ of $\mathsf{C}\mathcal{L}$, i.e., $\tau([C]) = [C_\tau]$.

5 From $\mathsf{C}\mathcal{L}$-Frames to $\mathsf{CA}\mathcal{L}$-Frames

The next step is to show how this translation behaves with respect to the truth-values of formulas. We take care of this aspect providing a procedure that, given a frame for $\mathsf{C}\mathcal{L}$, determines a frame for $\mathsf{CA}\mathcal{L}$ with the following properties: (a) the frames have the same states and (b) for all $\mathsf{C}\mathcal{L}$-formulas φ, φ is true at a state s in the $\mathsf{C}\mathcal{L}$-frame (for some interpretation for the propositional variables) if and only if $\tau(\varphi)$ is true at the corresponding state s in the $\mathsf{CA}\mathcal{L}$-frame (for the same interpretation.)

Fix a set $N = \{1, \ldots, k\}$ and the language of coalition logic $\mathsf{C}\mathcal{L}$. The language $\mathsf{CA}\mathcal{L}$ has the same propositional identifiers of $\mathsf{C}\mathcal{L}$ and variables $x_0, x_1, \ldots, x_k$ only.

Fix a $\mathsf{C}\mathcal{L}$-frame for N, say, $\mathcal{F} = \langle W, R^{\mathcal{F}} \rangle$. The algorithm we are going to describe consists of two parts: subalgorithm A defines the set *Act* and subalgorithm B the relation $R^{\mathcal{K}}$. These are the elements needed to isolate a Kripke frame $\mathcal{K} = \langle W, Act, R^{\mathcal{K}} \rangle$ for $\mathsf{CA}\mathcal{L}$.

`[Subalgorithm A]` With each state $s \in W$ associate a set[6] D^s ($D^s \cap D^t = \emptyset$ for all states s, t). For any state s and set D^s, proceed as follows.

Basic case:

$(A.0)$ For all $(s, \emptyset, V) \in R^{\mathcal{F}}$, fix a pair $< \alpha^V, (s, \emptyset, V) >$ where $\alpha^V \in D^s$ and $\alpha^V = \alpha^{V'}$ if and only if $V = V'$.

Let D^s_0 be the set of constants α^V selected at this step.

General case:

$(A.m)$ For all $1 \leq i_1 < \ldots < i_m \leq k$ and all triple $(s, \{i_1, .., i_m\}, V) \in R^{\mathcal{F}}$, fix a pair $< \alpha^V_{i_1,\ldots,i_m}, (s, \{i_1, .., i_m\}, V) >$ where $\alpha^V_{i_1,\ldots,i_m} \in D^s \setminus D^s_{m-1}$ and $\alpha^V_{i_1,\ldots,i_m} = \alpha^{V'}_{i'_1,\ldots,i'_m}$ iff $V = V'$ and $i_j = i'_j$ for $j \leq m$.

Let D^s_m be D^s_{m-1} plus the set of constants $\alpha^V_{i_1,\ldots,i_m}$ selected at this step.

Put $\bigcup_{s \in W} D^s_k = Act$. This will be the set of actions in the frame of $\mathsf{CA}\mathcal{L}$.

We now define relation $R^{\mathcal{K}} \subseteq \{k+1\text{-actions}\} \times W \times W$ via subalgorithm B. Let P be the set of pairs $<\alpha, (s, C, V)>$ isolated at steps $(A.0), \ldots, (A.k)$. As usual, we write $s' \in R^{\mathcal{K}}(s)(A)$ or $(s, s') \in R^{\mathcal{K}}(A)$ to mean $(A, s, s') \in R^{\mathcal{K}}$.

`[Subalgorithm B]` For $\alpha \in Act$ and $I = \{i_1, \ldots, i_m\}$, let $M_{\alpha,I}$ be any $k+1$-action A such that $A(j) = \alpha$ for $j \in \{0\} \cup I$. Let M_{α,I^-} be any $k+1$-action A such that $A(0) = \alpha$ and $A(j) \neq \alpha$ for some $j \in I$. We start with $R^{\mathcal{K}} = \emptyset$.

Basic case:

$(B.0)$ given pair $< \alpha, (s, \emptyset, V) > \in P$, for all $M_{\alpha,\emptyset}$
- put $(s, s') \in R^{\mathcal{K}}(M_{\alpha,\emptyset})$ for all $s' \in V$,
- put $(t, u) \in R^{\mathcal{K}}(M_{\alpha,\emptyset})$ for all $t, u \in W$ with $t \neq s$

General case:

$(B.m)$ given pair $< \alpha, (s, I, V) > \in P$, for all $M_{\alpha,I}, M_{\alpha,I^-}$
- put $(s, s') \in R^{\mathcal{K}}(M_{\alpha,I})$ for all $s' \in V$,
- put $(t, u) \in R^{\mathcal{K}}(M_{\alpha,I})$ for all $t, u \in W$ with $t \neq s$,
- put $(t, u) \in R^{\mathcal{K}}(M_{\alpha,I^-})$ for all $t, u \in W$

From section 2, for all s in W and $C \subseteq \{1, \ldots, k\}$, there exists $V \subseteq W$, $V \neq \emptyset$, such that $(s, C, V) \in R^{\mathcal{F}}$. Since all elements of *Act* have been introduced at some step $(A.0), \ldots, (A.k)$ for some $s \in W$ and because of the second condition in $(B.m)$, it follows that for any state s and any $k+1$-action A, there exists s' such that $(s, s') \in R^{\mathcal{K}}(A)$. In modal logic lingo, this property tell us that any multi-agent Kripke frame resulting from this algorithm is *serial*.

Having completed the definition of $R^{\mathcal{K}}$, we can finally put the pieces together. The frame for $\mathsf{CA}\mathcal{L}$ is given by $\mathcal{K}_{\mathcal{F}} = \langle W, Act, R^{\mathcal{K}} \rangle$ which is a Kripke frame for the multi-agent logic. Our next task is to verify the properties of frames $\mathcal{F}$ and $\mathcal{K}_{\mathcal{F}}$. For this task, we will need the following lemma which follows by induction from the previous algorithm (note the special role of action α_0).

Lemma 5.1 $R^{\mathcal{K}} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_k \end{pmatrix} = \bigcup_{\substack{\gamma_1, \ldots, \gamma_k \text{ with} \\ \gamma_i = \alpha_0 \text{ if } \alpha_i = \alpha_0}} R^{\mathcal{K}} \begin{pmatrix} \alpha_0 \\ \gamma_1 \\ \vdots \\ \gamma_k \end{pmatrix}$

Clearly, the two frames $\mathcal{F} = \langle W, R^{\mathcal{F}} \rangle$ and $\mathcal{K}_{\mathcal{F}} = \langle W, Act, R^{\mathcal{K}} \rangle$ have same support. Since the languages have

[5] To fully determine τ, fix $k+1$ variables, say $x_0, x_1, \ldots, x_k$, and use these at positions $0, 1, \ldots, k$ (resp.ly) in all modalities.

[6] The size is bounded above by $2^k \times |W|$.

the same propositional variables, without loss of generality we can use the same valuation function $[\![\cdot]\!]$ for both systems.

Fix a formula φ of $\mathsf{C}\mathcal{L}$. If φ does not contain modal operators, then $\varphi = \tau(\varphi)$ and the two relations $R^{\mathcal{F}}, R^{\mathcal{K}}$ are inconsequential for the truth-value of φ.

To show that $\langle \mathcal{F}, [\![\cdot]\!]\rangle, s \models \varphi$ if and only if $\langle \mathcal{K}_{\mathcal{F}}, [\![\cdot]\!]\rangle, s \models \tau(\varphi)$ for each sentence φ of $\mathsf{C}\mathcal{L}$, we proceed by induction. Assume $\langle \mathcal{F}, [\![\cdot]\!]\rangle, s \models \varphi$ iff $\langle \mathcal{K}_{\mathcal{F}}, [\![\cdot]\!]\rangle, s \models \tau(\varphi)$ with φ well-formed formula in $\mathsf{C}\mathcal{L}$. We show that for all operators $[C]$ in $\mathsf{C}\mathcal{L}$: $\langle \mathcal{F}, [\![\cdot]\!]\rangle, s \models [C]\varphi \quad\Leftrightarrow\quad \langle \mathcal{K}_{\mathcal{F}}, [\![\cdot]\!]\rangle, s \models [C_\tau]\tau(\varphi)$, where the relations $\models$ refer to $\mathsf{C}\mathcal{L}$ and to $\mathsf{CA}\mathcal{L}$, respectively.

Without loss of generality, let $[C] = [\{1, \ldots, r\}]$. Let us write $Z = \begin{pmatrix} z_0 \\ z_1 \\ \vdots \\ z_k \end{pmatrix}$ with z_i new variables ranging over *Act*. Then, $\langle \mathcal{K}_{\mathcal{F}}, [\![\cdot]\!]\rangle, s \models [C_\tau]\tau(\varphi) \Leftrightarrow \exists z_0 \ldots z_r \, \forall z_{r+1} \ldots z_k$ if $(s, s') \in R^{\mathcal{K}}(Z)$ then $\langle \mathcal{K}_{\mathcal{F}}, [\![\cdot]\!]\rangle, s' \models \tau(\varphi)$. By inductive hypothesis, the latter holds $\Leftrightarrow \exists z_0 \ldots z_r \, \forall z_{r+1} \ldots, z_k$ if $(s, s') \in R^{\mathcal{K}}(Z)$ then $\langle \mathcal{F}, [\![\cdot]\!]\rangle, s' \models \varphi$. Now, put $V = \{t \mid \langle \mathcal{F}, [\![\cdot]\!]\rangle, t \models \varphi\}$. To prove our claim, it suffices to show:

$$V \in R^{\mathcal{F}}(s)(\{1, \ldots, r\}) \text{ iff } \exists z_0 \ldots z_r \, \forall z_{r+1} \ldots z_k \, . \, R^{\mathcal{K}}(s)(Z) \subseteq V$$

Definition 5.1 *Given an operator $[M]$, we write $U_s(\alpha, r)$, where $\alpha \in Act$ and $r \geq 0$, for the set of* states reachable *from s through $k+1$-actions A for which $A(j) = \alpha$ ($0 \leq j \leq r$)*

$$U_s(\alpha, r) =_{def} \bigcup\nolimits_{\beta_{r+1}, \ldots, \beta_k} R^{\mathcal{K}}(s) \begin{pmatrix} \alpha \\ \vdots \\ \alpha \\ \beta_{r+1} \\ \vdots \\ \beta_k \end{pmatrix}.$$

Note that, from the definition of *Act* and subalgorithm B, $U_s(\alpha, r)$ is non-empty for all $\alpha \in Act$. By Lemma 5.1, it suffices that: $V \in R^{\mathcal{F}}(s)(\{1, \ldots, r\}) \Leftrightarrow \exists \alpha \, . \, U_s(\alpha, r) \subseteq V$.

First, consider case $V = \emptyset$. From Def 2.1, condition I), $\emptyset \notin R^{\mathcal{F}}(s)(C)$. On the other hand, we already observed that $U_s(\alpha, r) \neq \emptyset$ for all $r \leq k$.

Consider now $V \neq \emptyset$.

From left to right. By recursion on the size of C we prove a stronger claim, that is, there exists α such that $U_s(\alpha, r) = V$.

Case $C = \emptyset$. Let $V \in R^{\mathcal{F}}(s)(\emptyset)$. By construction, step $(A.0)$, there exists $\alpha \in Act$ such that for all $\beta_i \in Act$, $V \subseteq R^{\mathcal{K}}(s) \begin{pmatrix} \alpha \\ \beta_1 \\ \vdots \\ \beta_k \end{pmatrix}$. Fix such an α, then $V \subseteq U_s(\alpha, r)$. We need to show $V \supseteq U_s(\alpha, r)$. If $t \in U_s(\alpha, r)$, then (s, t) has been included in the definition of $R^{\mathcal{K}} \begin{pmatrix} \alpha \\ \beta_1 \\ \vdots \\ \beta_k \end{pmatrix}$ for some $\beta_1, \ldots, \beta_k$. From steps $(A.0)$ and $(B.0)$, the minimality of $R^{\mathcal{K}}$ and the use of a different action at row 0 at each step $(A.m)$, we have $t \in V$. Thus, $U_s(\alpha, r) = V$.

Assume the statement holds for $|C| < r \leq k$. We show it holds for $|C| = r$. Without loss of generality, let $C = \{1, \ldots, r\}$ and fix $V \in R^{\mathcal{F}}(s)(\{1, \ldots, r\})$. By construction, step $(A.r)$, there exists action α such that $V \subseteq U_s(\alpha, r)$. Let $t \in U_s(\alpha, r)$. Since at each step $(A.m)$, a different action is used, (s, t) must have been included while considering pair $< \alpha, (s, \{1, \ldots, r\}, V) >$. Thus, $t \in U_s(\alpha, r)$ implies $t \in V$, i.e., $U_s(\alpha, r) = V$.

From right to left. Without loss of generality, we consider sets $\{1, \ldots, r\}$ with $0 \leq r \leq k$. Let α be such that $U_s(\alpha, r) \subseteq V$, we show $V \in R^{\mathcal{F}}(s)(\{1, \ldots, r\})$. Here the recursion proceeds backward on the size of C from $C = \{1, \ldots, k\}$ to $C = \emptyset$. Since we apply a backword recursion, at step r ($0 \leq r < k$) it suffices to consider sets V such that $U_s(\alpha, k - r) \subseteq V$ and $U_s(\alpha, k - (r+1)) \not\subseteq V$.

Case $C = \{1, \ldots, k\}$. Consider set V with $U_s(\alpha, k) \subseteq V$ and $U_s(\alpha, k-1) \not\subseteq V$. From the construction and the Lemma 5.1, α must have been introduced at step $(A.k)$. Let V' be the set for which α has been introduced. By construction, $(s, \{1, \ldots, k\}, V') \in R^{\mathcal{F}}$ so that $V' \subseteq U_s(\alpha, k)$. Thus, $V' \subseteq V$ and, by property IV) of $R^{\mathcal{F}}$ (see definition 2.1), $(s, \{1, \ldots, k\}, V) \in R^{\mathcal{F}}$, i.e., $V \in R^{\mathcal{F}}(s)(\{1, \ldots, k\})$.

Assume now that the statement holds for $C = \{1, \ldots, r+1\}$. We show it holds for $C = \{1, \ldots, r\}$.

Let $U_s(\alpha, r) \subseteq V$ and $U_s(\alpha, r-1) \not\subseteq V$. This condition implies α has been introduced at step $(A.r)$ for some set V'. $V' \subseteq U_s(\alpha, r)$ follows by construction, thus $V' \subseteq V$. Finally, $V \in R^{\mathcal{F}}(s)(\{1, \ldots, r\})$ is obtained by condition IV) of $R^{\mathcal{F}}$ (see Definition 2.1) as before.

Case $C = \emptyset$ is analogous.

Theorem 5.1 *For any frame $\mathcal{F}$, valuation function $[\![\cdot]\!]$, state s, and $\mathsf{C}\mathcal{L}$-formula φ*

$$\langle \mathcal{F}, [\![\cdot]\!]\rangle, s \models \varphi \;\text{ if and only if }\; \langle \mathcal{K}_{\mathcal{F}}, [\![\cdot]\!]\rangle, s \models \tau(\varphi)$$

For any frame $\mathcal{F}$ and $\mathsf{C}\mathcal{L}$-formula φ

$$\mathcal{F} \models \varphi \;\text{ if and only if }\; \mathcal{K}_{\mathcal{F}} \models \tau(\varphi)$$

6 From $\mathsf{CA}\mathcal{L}$-frames to $\mathsf{C}\mathcal{L}$-frames

Let $\mathcal{K}$ be a frame for $\mathsf{CA}\mathcal{L}$ satisfying (1_τ)-(5_τ), i.e., the τ-images of axioms (1)-(5).[7] A frame $\mathcal{F}_{\mathcal{K}}$ for $\mathsf{C}\mathcal{L}$ is obtained by defining $R^{\mathcal{F}}$ to be minimal such that if $V = \{s' \mid (s, s') \in R^{\mathcal{K}}(A)\}$ and $A = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_k \end{pmatrix}$, then $(s, C, V) \in R^{\mathcal{F}}$ where $C = \{i \mid 1 \leq i \leq k \text{ and } A(i) = \alpha_0\}$. It is easy to verify that

Theorem 6.1 *For any frame $\mathcal{K}$ satisfying (1_τ)-(5_τ), valuation function $[\![\cdot]\!]$, state s, and $\mathsf{C}\mathcal{L}$-formula φ*

$$\langle \mathcal{K}, [\![\cdot]\!]\rangle, s \models \tau(\varphi) \;\text{ if and only if }\; \langle \mathcal{F}_{\mathcal{K}}, [\![\cdot]\!]\rangle, s \models \varphi$$

For any frame $\mathcal{K}$ and $\mathsf{C}\mathcal{L}$-formula φ

$$\mathcal{K} \models \tau(\varphi) \;\text{ if and only if }\; \mathcal{F}_{\mathcal{K}} \models \varphi$$

7 Discussion and Conclusions

We presented a translation between two logics of agency that have been developed independently: a logic of *coalition power* ($\mathsf{C}\mathcal{L}$), where actions are only *implicit*, and (a fragment of) a logic for *groups of agents with individual actions* ($\mathsf{A}\mathcal{L}$). $\mathsf{A}\mathcal{L}$ models in detail the concurrent and interactive activity of the agents while $\mathsf{C}\mathcal{L}$ coarsens the description by focusing on groups and by disregarding individual actions. Our work provided a way to switch from one representation to the other

[7] Note that (2_τ) and (4_τ) are always satisfied in $\mathsf{CA}\mathcal{L}$-frames. Axiom (1_τ) gives seriality on all the modalities. The remaining axioms are specific to the interaction of coalitions.

and can also be used to refine proposals like [10]. Technically, we first isolated a fragment of $\mathsf{A}\mathcal{L}$, called $\mathsf{CA}\mathcal{L}$, which is isomorphic to the language of $\mathsf{C}\mathcal{L}$ (sect. 4). Second, we showed that the Kripke semantics of $\mathsf{CA}\mathcal{L}$ and the minimal semantics of $\mathsf{C}\mathcal{L}$ agree (sect. 5). Finally, we showed that for each model for $\mathsf{C}\mathcal{L}$ there is a corresponding $\mathsf{CA}\mathcal{L}$-model and vice versa (sect. 5 and 6). We observe that $\mathsf{CA}\mathcal{L}$ is a fragment of a language determined by the class of multi-agent Kripke frames [3] while $\mathsf{C}\mathcal{L}$ is determined by its class of minimal frames. Although $\mathsf{CA}\mathcal{L}$ itself is not determined by the class of $\mathcal{K}$ frames satisfying (1_τ)-(5_τ) (note that the last axiom fails due to the existential in row 0), it provides the needed connection between these two determined systems.

An Example

Consider a $\mathsf{C}\mathcal{L}$ system with one agent and two states, say t and n. At state t, the actual state, the agent is thirsty. At state n the agent is not. Assume that the empty coalition $[\emptyset]$ (corresponding to the environment in $\mathsf{CA}\mathcal{L}$) has the power of making impossible to reach n, e.g., blocking tap water. Thus in $\mathsf{C}\mathcal{L}$, $(t, \emptyset, \{t\}) \in R$ and, by II) in Def. 2.1, $(t, \emptyset, \{t, n\}) \in R$. The other coalition $C = [1]$ (that is, the agent plus the environment) has the power to reach n, e.g., making tap water available (by the environment) and drinking (by the agent). Let us assume this coalition has no other power. Then, $(t, \{1\}, \{n\}) \in R$ and, by II) again, $(t, \{1\}, \{t, n\}) \in R$. By subalgorithm A, we get four actions at this state, say $Act_t = \{\alpha, \beta, \gamma, \delta\}$, for the triples above (in the given order). From subalgorithm B, (1) $\{t\} = R(\frac{\alpha}{x})(t)$ for any $x \in Act_t$; (2) $\{t, n\} = R(\frac{\beta}{x})(t)$ for any $x \in Act_t$; (3) $\{n\} = R(\frac{\gamma}{\gamma})(t)$ and $\{t, n\} = R(\frac{\gamma}{x})(t)$ if $x \neq \gamma$; (4) $\{t, n\} = R(\frac{\delta}{x})(t)$ for any $x \in Act_t$. Two observations are in order. First, the elements of Act_t obtained from the algorithm correspond to action-plans: γ at row 0 has different meaning than γ at row 1. That is, these are not action-types, their meaning depends on the agent performing them. A renaming is needed to move from action-plans to action-types. Second, different actions, like $\frac{\beta}{\gamma}$ and $\frac{\delta}{\delta}$, may bring to the same states. However, they capture different situations: with the first the environment prevents from reaching n, with the latter they *both* act so that the output is not determined. Only in the second case the result is due to a cooperation.

Ability

One important motivation for logics of groups is to study the notion of *ability*. Ability may vary in strength: α-*ability* (effectiveness) tell us that some agents can force φ no matter what the other agents do; β-*ability* that some agents can force φ provided they know what the other agents are going to do; and *contingent ability* that some agents can force φ provided the other agents do nothing (see [11] and references therein). $\mathsf{C}\mathcal{L}$ focuses on effectiveness only and the interpretation of $\mathsf{CA}\mathcal{L}$ we have adopted corresponds to the same notion: α-ability (also known as $\exists\forall$-capability) is captured by the semantic clause 4. (section 3), which is of form: "there exist a list of actions $\vec{\alpha}$ such that for all lists of actions $\vec{\beta}, \ldots$". Contingent ability is obtained by adding a "null action" (ε) in $\mathsf{CA}\mathcal{L}$ and substituting ε for the universal quantified variables in the modal operators. Finally, β-ability ($\forall\exists$–capability) cannot be captured in the semantics of $\mathsf{CA}\mathcal{L}$ (cf. [2]).

The Translation

Translations from minimal to normal logics exist already [6]. We have not applied existing methods since they do not preserve important aspects of the structures. Indeed, the correspondence between the worlds in the models of $\mathsf{C}\mathcal{L}$ and those in the models of $\mathsf{CA}\mathcal{L}$ as well as the connections between these worlds are crucial to make explicit the relationship between agents, individual actions, and coalition power. A drawback is seen in the dimension of the target model since the size of *Act* is comparable to that of $R^{\mathcal{F}}$. Of course, one can make less distinctions. For instance at step $(A.m)$ one may require that a new action $\alpha^V_{i_1,\ldots,i_m}$ is added only if $(s, I, V) \notin R^{\mathcal{F}}$ for all $I \subset \{i_1, \ldots, i_m\}$. However, this new clause does not do justice of the notion of coalition since $[C]p$ and $[C']p$, with $C \subset C'$, are now associated to the same action. If the two coalitions enforce p using the same individual actions, the extra agents of the larger coalition C' are factually irrelevant. Instead, the algorithm of section 5 links the two coalitions to two different actions. This is important to relate $\mathsf{CA}\mathcal{L}$ (and in turn $\mathsf{C}\mathcal{L}$) to logics that can model strategies and plans since, generally speaking, bigger coalitions may have more alternative strategies and plans available.

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