

GENERALIZED LEVITIN–POLYAK WELL-POSEDNESS IN CONSTRAINED OPTIMIZATION*

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Abstract. In this paper, we consider Levitin–Polyak-type well-posedness for a general constrained optimization problem. We introduce generalized Levitin–Polyak well-posedness and strongly generalized Levitin–Polyak well-posedness. Necessary and sufficient conditions for these types of well-posedness are given. Relations among these types of well-posedness are investigated. Finally, we consider convergence of a class of penalty methods and a class of augmented Lagrangian methods under the assumption of strongly generalized Levitin–Polyak well-posedness.

Key words. constrained optimization, generalized minimizing sequence, generalized Levitin–Polyak well-posedness, penalty-type methods

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1. Introduction. The study of well-posedness originates from Tykhonov [26] in dealing with unconstrained optimization problems. Its extension to the constrained case was developed by Levitin and Polyak [18]. Since then, various notions of well-posedness have been defined and extensively studied (see, e.g., [22, 6, 24, 28, 29, 9, 24, 30]). It is worth noting that recent research on well-posedness has been extended to vector optimization problems (see, e.g., [3, 20, 21, 12, 13, 7]).

Let (X, d_1) and (Y, d_2) be two metric spaces, and let $X_1 \subset X$ and $K \subset Y$ be two nonempty and closed sets. Consider the following constrained optimization problem:

$$(P) \quad \min f(x) \\ \text{s.t. } x \in X_1, \quad g(x) \in K,$$

where $f : X \rightarrow R^1$ is a lower semicontinuous function and $g : X \rightarrow Y$ is a continuous function. Denote by X_0 the set of feasible solutions of (P), i.e.,

$$X_0 = \{x \in X_1 : g(x) \in K\}.$$

Denote by \bar{X} and \bar{v} the optimal solution set and the optimal value of (P), respectively. Throughout the paper, we always assume that $X_0 \neq \emptyset$ and $\bar{v} > -\infty$.

Let (Z, d) be a metric space and $Z_1 \subset Z$. We denote by $d_{Z_1}(z) = \inf\{d(z, z') : z' \in Z_1\}$ the distance from the point z to the set Z_1 .

Levitin–Polyak (LP) well-posedness of (P) in the usual sense (when the optimal set of (P) is not necessarily a singleton) says that, for any sequence $\{x_n\} \subset X_1$ satisfying (i) $d_{X_0}(x_n) \rightarrow 0$ and (ii) $f(x_n) \rightarrow \bar{v}$, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and some $\bar{x} \in \bar{X}$ such that $x_{n_k} \rightarrow \bar{x}$.

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It should be noted that many optimization algorithms, such as penalty-type methods, e.g., penalty function methods and augmented Lagrangian methods, terminate when the constraint is approximately satisfied; i.e., $d_K(g(\bar{x})) \leq \epsilon$ for some $\epsilon > 0$ sufficiently small, and \bar{x} is taken as an approximate solution of (P). These methods may generate sequences $\{x_n\} \subset X_1$ that satisfy $d_K(g(x_n)) \rightarrow 0$, not necessarily $d_{X_0}(x_n) \rightarrow 0$, as shown in the following simple example.

Example 1.1. Let $\alpha > 0$. Let $X = R^1$, $X_1 = R_+^1$, $K = R_-^1$, and

$$f(x) = \begin{cases} -x^\alpha & \text{if } x \in [0, 1]; \\ -1/x^\alpha & \text{if } x \geq 1, \end{cases}$$

$$g(x) = \begin{cases} x & \text{if } x \in [0, 1]; \\ 1/x^2 & \text{if } x \geq 1. \end{cases}$$

Consider the following penalty problem:

$$(PP_\alpha(n)) \quad \min_{x \in X_1} f(x) + n [\max\{0, g(x)\}]^\alpha, \quad n \in N.$$

It is easily verified that $x_n = 2^{1/\alpha} n^{1/\alpha}$ is the unique global solution to $(PP_\alpha(n))$ for each $n \in N$. Note that $X_0 = \{0\}$. It follows that we have $d_K(g(x_n)) = 1/(2^{2/\alpha} n^{2/\alpha}) \rightarrow 0$, while $d_{X_0}(x_n) = 2^{1/\alpha} n^{1/\alpha} \rightarrow +\infty$.

Thus, it is useful to consider sequences that satisfy $d_K(g(x_n)) \rightarrow 0$ instead of $d_{X_0}(x_n) \rightarrow 0$ as $n \rightarrow \infty$ in order to study convergence of penalty-type methods.

The sequence $\{x_n\}$ satisfying (i) and (ii) above is called an LP minimizing sequence. In what follows, we introduce two more types of generalized LP well-posedness.

DEFINITION 1.1. (P) is called LP well-posedness in the generalized sense if, for any sequence $\{x_n\} \subset X_1$ satisfying (i) $d_K(g(x_n)) \rightarrow 0$ and (ii) $f(x_n) \rightarrow \bar{v}$, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and some $\bar{x} \in \bar{X}$ such that $x_{n_k} \rightarrow \bar{x}$. The sequence $\{x_n\}$ is called a generalized LP minimizing sequence.

DEFINITION 1.2. (P) is called LP well-posedness in the strongly generalized sense if, for any sequence $\{x_n\} \subset X_1$ satisfying (i) $d_K(g(x_n)) \rightarrow 0$ and (ii) $\limsup_{n \rightarrow +\infty} f(x_n) \leq \bar{v}$, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and some $\bar{x} \in \bar{X}$ such that $x_{n_k} \rightarrow \bar{x}$. The sequence $\{x_n\}$ is called a weakly generalized LP minimizing sequence.

Remark 1.1. (i) The study of well-posedness for optimization problems with explicit constraints dates back to [17] when the abstract set X_1 does not appear. In [17], it was assumed that X is a Banach space and Y is a Banach space ordered by a closed and convex cone with some special properties; see [17] for details. What is worth emphasizing is that [17] studied only the case when (P) is a convex program. However, it is well known that penalty-type methods such as penalization methods and augmented Lagrangian methods are mostly developed for constrained nonconvex optimization problems. This is the main motivation of this paper.

(ii) The LP well-posedness in the strongly generalized sense defined above was called well-posedness in the strongly generalized sense in [17], while a weakly generalized LP minimizing sequence in the above definition is called a generalized minimizing sequence in [17].

(iii) It is obvious that LP well-posedness in the strongly generalized sense implies LP well-posedness in the generalized sense because a generalized LP minimizing sequence is a weakly generalized LP minimizing sequence.

(iv) If there exists some $\delta_0 > 0$ such that g is uniformly continuous on the set

$$\{x \in X_1 : d_{X_0}(x) \leq \delta_0\},$$

then it is not difficult to see that LP well-posedness in the generalized sense implies LP well-posedness.

(v) Any one type of (generalized) LP well-posedness defined above implies that the optimal set \bar{X} of (P) is nonempty and compact.

The paper is organized as follows. In section 2, we investigate characterizations and criteria for the three types of (generalized) LP well-posednesses. In section 3, we establish relations among the three types of (generalized) LP well-posednesses. In section 4, we obtain convergence of a class of penalty methods and a class of augmented Lagrangian methods under the assumption of strongly generalized LP well-posedness.

2. Necessary and sufficient conditions for three types of (generalized) LP well-posedness. In this section, we present some criteria and characterizations for the three types of (generalized) LP well-posedness defined in section 1.

Consider the following statement:

(1)

[$\bar{X} \neq \emptyset$ and, for any LP minimizing sequence (resp., generalized LP minimizing sequence, weakly generalized LP minimizing sequence) $\{x_n\}$, we have $d_{\bar{X}}(x_n) \rightarrow 0$].

The proof of the following proposition is elementary and thus omitted.

PROPOSITION 2.1. *If (P) is LP well-posed (resp., LP well-posed in the generalized sense and LP well-posed in the strongly generalized sense), then (1) holds. Conversely, if (1) holds and \bar{X} is compact, then (P) is LP well-posed (resp., LP well-posed in the generalized sense and LP well-posed in the strongly generalized sense).*

Consider a real-valued function $c = c(t, s)$ defined for $t, s \geq 0$ sufficiently small, such that

$$(2) \quad c(t, s) \geq 0 \quad \forall t, s, \quad c(0, 0) = 0,$$

$$(3) \quad s_k \rightarrow 0, t_k \geq 0, c(t_k, s_k) \rightarrow 0 \text{ imply } t_k \rightarrow 0.$$

THEOREM 2.1. *If (P) is LP well-posed, then there exists a function c satisfying (2) and (3) such that*

$$(4) \quad |f(x) - \bar{v}| \geq c(d_{\bar{X}}(x), d_{X_0}(x)) \quad \forall x \in X_1.$$

Conversely, suppose that \bar{X} is nonempty and compact, and (4) holds for some c satisfying (2) and (3). Then (P) is LP well-posed.

Proof. Define

$$c(t, s) = \inf\{|f(x) - \bar{v}| : x \in X_1, d_{\bar{X}}(x) = t, d_{X_0}(x) = s\}.$$

It is obvious that $c(0, 0) = 0$. Moreover, if $s_n \rightarrow 0$, $t_n \geq 0$ and $c(t_n, s_n) \rightarrow 0$, then there exists a sequence $\{x_n\} \subset X_1$ with

$$(5) \quad d_{\bar{X}}(x_n) = t_n,$$

$$(6) \quad d_{X_0}(x_n) = s_n$$

such that

$$(7) \quad |f(x_n) - \bar{v}| \rightarrow 0.$$

Note that $s_n \rightarrow 0$. Equations (6) and (7) jointly imply that $\{x_n\}$ is an LP minimizing sequence. By Proposition 2.1, we have $t_n \rightarrow 0$. This completes the proof of the first half of the theorem. Conversely, let $\{x_n\}$ be an LP minimizing sequence. Then, by (4), we have

$$(8) \quad |f(x_n) - \bar{v}| \geq c(d_{\bar{X}}(x_n), d_{X_0}(x_n)) \quad \forall x \in X_1.$$

Let

$$t_n = d_{\bar{X}}(x_n), \quad s_n = d_{X_0}(x_n).$$

Then $s_n \rightarrow 0$. In addition, $|f(x_n) - \bar{v}| \rightarrow 0$. These facts together with (8) as well as the properties of the function c imply that $t_n \rightarrow 0$. By Proposition 2.1, we see that (P) is LP well-posed. \square

THEOREM 2.2. *If (P) is LP well-posed in the generalized sense, then there exists a function c satisfying (2) and (3) such that*

$$(9) \quad |f(x) - \bar{v}| \geq c(d_{\bar{X}}(x), d_K(g(x))) \quad \forall x \in X_1.$$

Conversely, suppose that \bar{X} is nonempty and compact, and (9) holds for some c satisfying (2) and (3). Then (P) is LP well-posed in the generalized sense.

Proof. The proof is almost the same as that of Theorem 2.1. The only difference lies in the proof of the first part of Theorem 2.1. Here we define

$$c(t, s) = \inf\{|f(x) - \bar{v}| : x \in X_1, d_{\bar{X}}(x) = t, d_K(g(x)) = s\}. \quad \square$$

Next we give a necessary and sufficient condition in the form of Furi and Vignoli [10] to characterize the LP well-posedness in the strongly generalized sense.

Let

$$\Omega(\epsilon) = \{x \in X_1 : f(x) \leq \bar{v} + \epsilon, d_K(g(x)) \leq \epsilon\}.$$

Let (X, d_1) be a complete metric space. Recall that the Kuratowski measure of noncompactness for a subset A of X is defined as

$$\alpha(A) = \inf \left\{ \epsilon > 0 : A \subset \bigcup_{1 \leq i \leq n} C_i, \text{ for some } C_i, \text{diam}(C_i) \leq \epsilon \right\},$$

where $\text{diam}(C_i)$ is the diameter of C_i defined by

$$\text{diam}(C_i) = \sup\{d_1(x_1, x_2) : x_1, x_2 \in C_i\}.$$

The next theorem can be proved analogously to [17, Theorem 5.5].

THEOREM 2.3. *Let (X, d_1) be a complete metric space and f be bounded below on X_0 . Then (P) is LP well-posed in the strongly generalized sense if and only if*

$$\alpha(\Omega(\epsilon)) \rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$

DEFINITION 2.1. *Let Z be a topological space and $Z_1 \subset Z$ be nonempty. Suppose that $h : Z \rightarrow \mathbb{R}^1 \cup \{+\infty\}$ is an extended real-valued function. h is said to be level-compact on Z_1 if, for any $s \in \mathbb{R}^1$, the subset $\{z \in Z_1 : h(z) \leq s\}$ is compact.*

For any $\delta \geq 0$, define

$$(10) \quad X_1(\delta) = \{x \in X_1 : d_K(g(x)) \leq \delta\}.$$

The following proposition gives sufficient conditions that guarantee LP well-posedness in the strongly generalized sense.

PROPOSITION 2.2. *Let one of the following conditions hold.*

- (i) *There exists $\delta_0 > 0$ such that $X_1(\delta_0)$ is compact.*
- (ii) *f is level-compact on X_1 .*
- (iii) *X is a finite dimensional normed space and*

$$(11) \quad \lim_{x \in X_1, \|x\| \rightarrow +\infty} \max\{f(x), d_K(g(x))\} = +\infty.$$

- (iv) *There exists $\delta_0 > 0$ such that f is level-compact on $X_1(\delta_0)$.*

Then (P) is LP well-posed in the strongly generalized sense.

Proof. Let $\{x_n\} \subset X_1$ be a weakly generalized LP minimizing sequence. Then

$$(12) \quad \limsup_{n \rightarrow +\infty} f(x_n) \leq \bar{v},$$

$$(13) \quad d_K(g(x_n)) \rightarrow 0.$$

The proof of (i) is elementary. It is obvious that condition (ii) implies (iv). Now we show that (iii) implies (iv). Indeed, we need only to show that for any $s \in \mathbb{R}^1$ and any $\delta > 0$, the set

$$A = \{x \in X_1(\delta) : f(x) \leq s\}$$

is bounded since X is a finite dimensional space. Suppose to the contrary that there exist $\delta > 0$, $s > 0$, and $\{x'_n\} \subset X_1(\delta)$ such that

$$\|x'_n\| \rightarrow +\infty \text{ and } f(x'_n) \leq s.$$

By $\{x'_n\} \subset X_1(\delta)$, we have $\{x'_n\} \subset X_1$ and

$$d_K(g(x'_n)) \leq \delta.$$

As a result,

$$\max\{f(x'_n), d_K(g(x'_n))\} \leq \max\{s, \delta\},$$

contradicting (11).

Thus, we need only to prove that if (iv) holds, then (P) is LP well-posed in the strongly generalized sense. By (13), it is apparent that we can assume without loss of generality that $\{x_n\} \subset X_1(\delta_0)$. By (12), we can assume without loss of generality that

$$\{x_n\} \subset \{x \in X_1 : f(x) \leq \bar{v} + 1\}.$$

By the level-compactness of f on $X_1(\delta_0)$, we deduce that there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\bar{x} \in X_1$ such that $x_{n_k} \rightarrow \bar{x}$. It is obvious from (13) that $\bar{x} \in X_0$. Furthermore, from (12), we deduce that $f(\bar{x}) \leq \bar{v}$. So we have $f(\bar{x}) = \bar{v}$. That is, $\bar{x} \in \bar{X}$. Hence, (P) is LP well-posed in the strongly generalized sense. \square

Now we consider the case when Y is a normed space and K is a closed and convex cone with nonempty interior $\text{int}K$. Arbitrarily fix an $e \in \text{int}K$. Let $t \geq 0$ and consider the following perturbed problem of (P):

$$(14) \quad \begin{aligned} (P_t) \quad & \min f(x) \\ & \text{s.t. } x \in X_1, \quad g(x) \in K - te. \end{aligned}$$

Let

$$(15) \quad X_2(t) = \{x \in X_1 : g(x) \in K - te\}.$$

PROPOSITION 2.3. *Let one of the following conditions hold.*

- (i) *There exists $t_0 > 0$ such that $X_2(t_0)$ is compact.*
- (ii) *f is level-compact on X_1 .*
- (iii) *X is a finite dimensional normed space and*

$$\lim_{x \in X_1, \|x\| \rightarrow +\infty} \max\{f(x), d_K(g(x))\} = +\infty.$$

- (iv) *There exists $t_0 > 0$ such that f is level-compact on $X_2(t_0)$.*

Then (P) is LP well-posed in the strongly generalized sense.

Proof. The proof is similar to that of Proposition 2.2. \square

Now we make the following assumption.

ASSUMPTION 2.1. *X is a finite dimensional normed space, Y is a normed space, $X_1 \subset X$ is a nonempty, closed, and convex set, $K \subset Y$ is a closed, and convex cone with nonempty interior $\text{int}K$ and $e \in \text{int}K$, f and g are continuous on X_1 , f is a convex function on X_1 , and g is K -concave on X_1 (namely, for any $x_1, x_2 \in X_1$ and any $\theta \in (0, 1)$, there holds that $g(\theta x_1 + (1 - \theta)x_2) - \theta g(x_1) - (1 - \theta)g(x_2) \in K$).*

It is obvious that under Assumption 2.1, (P) is a convex program.

The next lemma can be proved similarly to that of [16, Proposition 2.4].

LEMMA 2.1. *Let Assumption 2.1 hold. Then the following two statements are equivalent.*

- (i) *The optimal set \bar{X} of (P) is nonempty and compact.*
- (ii) *For any $t \geq 0$, f is level-compact on the set $X_2(t)$.*

THEOREM 2.4. *Let Assumption 2.1 hold. Then (P) is LP well-posed in the strongly generalized sense if and only if the optimal set \bar{X} of (P) is nonempty and compact.*

Proof. The sufficiency part follows directly from Lemma 2.1 and Proposition 2.3, while the necessity part is obvious by Remark 1.1. \square

The next two lemmas will be used to derive Theorem 2.5.

LEMMA 2.2 (see [1]). *Let (Z, d) be a complete metric space and $h : Z \rightarrow R^1 \cup \{+\infty\}$ be lower semicontinuous and bounded below. Let $\epsilon > 0$. Suppose that $z_0 \in Z$ satisfies $h(z_0) \leq \inf\{h(z) : z \in Z\} + \epsilon$. Then there exists $z_\epsilon \in Z$ such that*

- (i) $h(z_\epsilon) \leq h(z_0)$;
- (ii) $d(z_\epsilon, z_0) \leq \sqrt{\epsilon}$;
- (iii) $h(z_\epsilon) < h(z) + \sqrt{\epsilon}d(z, z_\epsilon) \quad \forall z \in Z \setminus \{z_\epsilon\}$.

LEMMA 2.3. *Let Y be a normed space and $K \subset Y$ be a closed and convex cone with $\text{int}K \neq \emptyset$ and $e \in \text{int}K$. Suppose that $\{y_n\} \subset Y$. Then $d_K(y_n) \rightarrow 0$ if and only if there exists a sequence $\{t_n\} \subset R_+^1$ with $t_n \rightarrow 0$ such that $y_n \in K - t_n e$.*

Proof. For the necessity part, from $d_K(y_n) \rightarrow 0$, we have $\{u_n\} \subset K$ such that $\|y_n - u_n\| \rightarrow 0$. Let $y'_n = y_n - u_n$. Then $\|y'_n\| \rightarrow 0$. Let $t_n = \sqrt{\|y'_n\|}$. Then $\{t_n\} \subset R_+^1$,

$t_n \rightarrow 0$ and $y'_n/t_n \rightarrow 0$. Since $e \in \text{int}K$, it follows that $e + y'_n/t_n \in K$ when n is sufficiently large. Consequently, $y'_n \in K - t_n e$. Hence, $y_n = u_n + y'_n \in K - t_n e$.

For the sufficiency part, as $y_n \in K - t_n e$, we have $y_n + t_n e \in K$. Thus,

$$d_K(y_n) \leq \|y_n - (y_n + t_n e)\| = t_n \|e\|.$$

Hence, $d_K(y_n) \rightarrow 0$. \square

Suppose that K is a cone. We denote by K^* the positive polar cone of K , i.e.,

$$K^* = \{\mu \in Y^* : \mu(u) \geq 0 \forall u \in K\}.$$

THEOREM 2.5. *Assume that X is a Banach space, Y is a normed space, and $X_1 \subset X$ is nonempty, closed, and convex. $K \subset Y$ is a closed and convex cone with $\text{int}K \neq \emptyset$ and $e \in \text{int}K$. Suppose that $f : X \rightarrow R^1$ is convex and continuously differentiable on X_1 and $g : X \rightarrow Y$ is K -concave and continuously differentiable on X_1 . Let Slater constraint qualification for (P) hold: there exists $x_0 \in X_1$ such that $g(x_0) \in \text{int}K$. Assume that the optimal set \bar{X} of (P) is nonempty. Further assume that there exists a convergent subsequence of $\{x_n\}$ for any sequences $\{x_n\} \subset X_1$ and $\{\mu_n\} \subset K^*$ satisfying the following.*

(i) $\lim_{n \rightarrow +\infty} d_K(g(x_n)) = 0$.

(ii) *There exists a subsequence $\{\mu_{n_k}\}$ such that $\mu_{n_k} = 0 \forall k$ or $\lim_{n \rightarrow +\infty} \mu_n(g(x_n))/\|\mu_n\| = 0$.*

(iii) $\lim_{n \rightarrow +\infty} d_{(-N_{X_1}(x_n))}(\nabla f(x_n) - \mu_n(\nabla g(x_n))) = 0$, where $N_{X_1}(x_n)$ is the normal cone of X_1 at x_n .

Then, (P) is LP well-posed in the strongly generalized sense.

Proof. Suppose that $\bar{x} \in \bar{X}$. Since Slater constraint qualification holds, we have $\bar{\mu} \in K^*$ such that

$$(16) \quad f(\bar{x}) \leq f(x) - \bar{\mu}(g(x)) \quad \forall x \in X_1$$

and

$$(17) \quad \bar{\mu}(g(\bar{x})) = 0.$$

Let $\{x_n\} \subset X_1$ be a weakly generalized LP minimizing sequence for (P). Then, by Lemma 2.3,

$$(18) \quad \limsup_{n \rightarrow +\infty} f(x_n) \leq \bar{v}$$

and

$$(19) \quad g(x_n) \in K - t_n e$$

for some $\{t_n\} \subset R^1_+$ with $t_n \rightarrow 0$. From (16), we have

$$f(\bar{x}) \leq f(x) - \bar{\mu}(g(x)) \quad \forall x \in X_2(t_n).$$

Note that

$$-\bar{\mu}(g(x)) \leq t_n \bar{\mu}(e) \quad \forall x \in X_2(t_n).$$

Thus,

$$(20) \quad f(\bar{x}) \leq f(x) + t_n \bar{\mu}(e) \quad \forall x \in X_2(t_n).$$

Hence,

$$(21) \quad \inf_{x \in X_2(t_n)} f(x) > -\infty.$$

The combination of (19) and (20) gives

$$f(\bar{x}) \leq f(x_n) + t_n \bar{\mu}(e).$$

Consequently,

$$f(\bar{x}) \leq \liminf_{n \rightarrow +\infty} f(x_n).$$

This together with (18) yields

$$(22) \quad \lim_{n \rightarrow +\infty} f(x_n) = f(\bar{x}).$$

This combined with (20) implies that there exists $\epsilon_n \rightarrow 0^+$ such that

$$f(x_n) \leq f(x) + \epsilon_n \quad \forall x \in X_2(t_n).$$

Note that $X_2(t_n) \subset X$ is nonempty and closed. $(X_2(t_n), \|\cdot\|)$ can be seen as a complete (metric) subspace of X . Applying Lemma 2.2, we obtain

$$(23) \quad x'_n \in X_2(t_n)$$

such that

$$(24) \quad \|x_n - x'_n\| \leq \sqrt{\epsilon_n}$$

and

$$(25) \quad f(x'_n) \leq f(x) + \sqrt{\epsilon_n} \|x - x'_n\| \quad \forall x \in X_2(t_n).$$

Note that Slater constraint qualification also holds for the following constrained optimization problem:

$$(P_n) \quad \min f(x) + \sqrt{\epsilon_n} \|x - x'_n\| \\ \text{s.t. } x \in X_1, \quad g(x) \in K - t_n e,$$

and by (25), x'_n is an optimal solution of (P_n) . Hence, there exists $\mu_n \in K^*$ such that

$$(26) \quad 0 \in \nabla f(x'_n) - \mu_n (\nabla g(x'_n)) + \sqrt{\epsilon_n} B^* + N_{X_1}(x'_n)$$

and

$$(27) \quad \mu_n (g(x'_n) + t_n e) = \mu_n (g(x'_n)) + t_n \mu_n(e) = 0,$$

where B^* is the closed unit ball of X^* . Equation (26) implies that

$$(28) \quad \lim_{n \rightarrow +\infty} d_{(-N_{X_1}(x'_n))}(\nabla f(x'_n) - \mu_n (\nabla g(x'_n))) = 0.$$

From (27), we see that if there does not exist a subsequence $\{\mu_{n_k}\}$ such that $\mu_{n_k} = 0 \forall k$, then

$$(29) \quad \lim_{n \rightarrow +\infty} \mu_n (g(x_n)) / \|\mu_n\| = 0.$$

The combination of (24), (28), and (29) implies that $\{x'_n\}$ and $\{\mu_n\}$ satisfy conditions (i)–(iii) of the theorem. Thus, $\{x'_n\}$ has a subsequence $\{x'_{n_k}\}$ which converges to some $\bar{x}' \in X_0$. From (24), we deduce that $x_{n_k} \rightarrow \bar{x}' \in X_0$. This combined with (22) implies $\bar{x}' \in \bar{X}$. Hence, (P) is LP well-posed in the strongly generalized sense. \square

Remark 2.1. Conditions (i)–(iii) of Theorem 2.5 can be seen as the well-known Palais–Smale condition (C) [1] in the case of constrained optimization.

3. Relations among three types of (generalized) LP well-posedness.

Simple relationships among the three types of LP well-posedness were mentioned in Remark 1.1. Now we investigate further relationships among them.

The proof of next theorem is elementary and is omitted.

THEOREM 3.1. *Suppose that there exist $\delta > 0$, $\alpha > 0$, and $c > 0$ such that*

$$(30) \quad d_{X_0}(x) \leq cd_K^\alpha(g(x)) \quad \forall x \in X_1(\delta),$$

where $X_1(\delta)$ is defined by (10). If (P) is LP well-posed, then (P) is LP well-posed in the generalized sense.

Remark 3.1. Equation (30) is an error bound condition for the set X_0 in terms of the residual function

$$r(x) = d_K(g(x)) \quad \forall x \in X_1.$$

When $X = R^l$, $Y = R^m$, $X_1 = X$, and $X_0 \neq \emptyset$, by Theorem 5 of [23], (30) holds if and only if, for any $y \in R^m$ with $\|y\| \leq \delta$,

$$\Psi(y) \subset \Psi(0) + c\|y\|^\alpha B,$$

where

$$\Psi(y) = \{x \in R^l : g(x) \in K + y\}, \quad y \in R^m,$$

and B is the closed unit ball of Y . Sufficient conditions guaranteeing (30) were given in numerous papers on error bounds for systems of inequalities and metric regularity of set-valued maps (when (30) holds locally with $\alpha = 1$) in finite and infinite dimensional spaces (see, e.g., [5, 8, 18] and the references therein).

DEFINITION 3.1 (see [4]). *Let W be a topological space and $F : W \rightarrow 2^X$ be a set-valued map. F is said to be upper Hausdorff semicontinuous (u.H.c.) at $w \in W$ if, for any $\epsilon > 0$, there exists a neighborhood U of w such that $F(U) \subset B(F(w), \epsilon)$, where, for $Z \subset X$ and $r > 0$,*

$$B(Z, r) = \{x \in X : d_Z(x) \leq r\}.$$

DEFINITION 3.2 (see [1]). *Let W be a topological space and $F : W \rightarrow 2^X$ be a set-valued map. F is said to be upper semicontinuous (u.s.c.) in the Berge's sense at $w \in W$ if, for any neighborhood Ω of $F(w)$, there exists a neighborhood U of w such that $F(U) \subset \Omega$.*

It is obvious that the notion of u.s.c. (in Berge's sense) is stronger than u.H.c.

Clearly, $X_1(\delta)$ given by (10) can be seen as a set-valued map from R_+^1 to X . The next two theorems use conditions similar to those for the general stability results presented in section 3 of [4], where the uniform continuity of the objective function around the feasible set and the u.H.c. of the perturbation set-valued map were considered.

THEOREM 3.2. *Assume that the set-valued map $X_1(\delta)$ defined by (10) is u.H.c. at $0 \in R_+^1$. If (P) is LP well-posed, then (P) is LP well-posed in the generalized sense.*

Proof. Let $\{x_n\} \subset X_1$ be a generalized LP minimizing sequence. That is,

$$(31) \quad f(x_n) \rightarrow \bar{v},$$

$$(32) \quad d_K(g(x_n)) \rightarrow 0.$$

Equation (32), together with the u.H.c. of $X_1(\delta)$ at 0, implies that $d_{X_0}(x_n) \rightarrow 0$. This fact combined with (31) implies that $\{x_n\}$ is an LP minimizing sequence. Thus,

there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and some $\bar{x} \in \bar{X}$ such that $x_{n_k} \rightarrow \bar{x}$. Hence, (P) is LP well-posed in the generalized sense. \square

THEOREM 3.3. *Assume that there exists $\epsilon_0 > 0$ such that f is uniformly continuous on $B(X_0, \epsilon_0)$ and the set-valued map $X_1(\delta)$ is u.H.c. at 0. If (P) is LP well-posed, then it is LP well-posed in the strongly generalized sense.*

Proof. Let $\{x_n\}$ be a weakly generalized LP minimizing sequence. That is,

$$(33) \quad \limsup_{n \rightarrow +\infty} f(x_n) \leq \bar{v},$$

$$(34) \quad d_K(g(x_n)) \rightarrow 0.$$

Note that $X_1(\delta)$ is u.H.c. at 0. This fact together with (34) implies that $d_{X_0}(x_n) \rightarrow 0$. Note that f is uniformly continuous on $B(X_0, \epsilon_0)$. It follows that

$$(35) \quad \liminf_{n \rightarrow +\infty} f(x_n) \geq \bar{v}.$$

The combination of (33) and (35) yields that

$$f(x_n) \rightarrow \bar{v}.$$

Hence, $\{x_n\}$ is an LP minimizing sequence. Thus, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and some $\bar{x} \in \bar{X}$ such that $x_{n_k} \rightarrow \bar{x}$. So, (P) is LP well-posed in the strongly generalized sense. \square

Let $\delta \geq 0$. Consider the perturbed problem of (P):

$$(P_\delta) \quad \begin{aligned} &\min f(x) \\ &\text{s.t. } x \in X_1, \quad d_K(g(x)) \leq \delta. \end{aligned}$$

Denote by $v_1(\delta)$ the optimal value of (P_δ) . Clearly, $v_1(0) = \bar{v}$.

THEOREM 3.4. *Consider problems (P) and (P_δ) . Suppose that (P) is LP well-posed in the generalized sense and*

$$(36) \quad \liminf_{\delta \rightarrow 0^+} v_1(\delta) = \bar{v}.$$

Then (P) is LP well-posed in the strongly generalized sense.

Proof. Let $\{x_n\} \subset X_1$ be a weakly generalized LP minimizing sequence. Then

$$(37) \quad \limsup_{n \rightarrow +\infty} f(x_n) \leq \bar{v}$$

and

$$\lim_{n \rightarrow +\infty} d_K(g(x_n)) = 0.$$

Let $\delta_n = d_K(g(x_n))$. Then x_n is feasible for (P_{δ_n}) . Thus,

$$v_1(\delta_n) \leq f(x_n).$$

Passing to the lower limit, we get

$$\liminf_{n \rightarrow +\infty} v_1(\delta_n) \leq \liminf_{n \rightarrow +\infty} f(x_n).$$

This together with (37) and (36) yields

$$\lim_{n \rightarrow +\infty} f(x_n) = \bar{v}.$$

It follows that $\{x_n\}$ is a generalized LP minimizing sequence. Thus, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and some $\bar{x} \in \bar{X}$ such that $x_{n_k} \rightarrow \bar{x}$. So, (P) is LP well-posed in the strongly generalized sense. \square

Remark 3.2. If the set-valued map $X_1(\delta)$ defined by (10) is u.s.c. at $0 \in R_+^1$, by Theorem 4.2.3 (1) of [2], (36) holds. In this case, the generalized LP well-posedness of (P) implies the strongly generalized LP well-posedness of (P).

Now let Y be a normed space and $y \in Y$. Consider the following perturbed problem of (P):

$$(P_y) \quad \begin{aligned} &\min f(x) \\ &\text{s.t. } x \in X_1, \quad g(x) \in K + y. \end{aligned}$$

Denote by

$$(38) \quad X_3(y) = \{x \in X_1 : g(x) \in K + y\}$$

the feasible set of (P_y) and $v_3(y)$ the optimal value of (P_y) . Here we note that if $X_3(y) = \emptyset$, we set $v_3(y) = +\infty$. It is obvious that $X_3(y)$ can be seen as a set-valued map from Y to X . Corresponding to Theorems 3.2–3.4, respectively, we have the following theorems.

THEOREM 3.5. *Assume that Y is a normed space and that the set-valued map $X_3(y)$ is u.H.c. at $0 \in Y$. If (P) is LP well-posed, then (P) is LP well-posed in the generalized sense.*

THEOREM 3.6. *Assume that Y is a normed space and that there exists $\epsilon_0 > 0$ such that f is uniformly continuous on $B(X_0, \epsilon_0)$ and the set-valued map $X_3(y)$ is u.H.c. at $0 \in Y$. If (P) is LP well-posed, then it is LP well-posed in the strongly generalized sense.*

THEOREM 3.7. *Assume that Y is a normed space. Consider problems (P) and (P_y) . Suppose that (P) is LP well-posed in the generalized sense and*

$$(39) \quad \liminf_{y \rightarrow 0} v_3(y) = \bar{v}.$$

Then (P) is LP well-posed in the strongly generalized sense.

Similar to Remark 3.2, when the set-valued map X_3 is u.s.c. at $0 \in Y$, then (39) holds. Thus, the generalized LP well-posedness of (P) implies its strongly generalized LP well-posedness.

In the special case when K is a closed and convex cone with nonempty interior $\text{int}K$, arbitrarily fix an $e \in \text{int}K$. It is obvious that $X_2(t)$ defined by (15) can be seen as a set-valued map from R_+^1 to X . Denote by $v_2(t)$ the optimal value of (P_t) .

THEOREM 3.8. *Assume that K is a closed and convex cone with nonempty interior $\text{int}K$ and that the set-valued map $X_2(t)$ is u.H.c. at $0 \in R_+^1$. If (P) is LP well-posed, then (P) is LP well-posed in the generalized sense.*

THEOREM 3.9. *Assume that K is a closed and convex cone with nonempty interior $\text{int}K$ and that there exists $\epsilon_0 > 0$ such that f is uniformly continuous on $B(X_0, \epsilon_0)$ and the set-valued map $X_2(t)$ is u.H.c. at $0 \in R_+^1$. If (P) is LP well-posed, then it is LP well-posed in the strongly generalized sense.*

THEOREM 3.10. *Assume that K is a closed and convex cone with nonempty interior $\text{int}K$. Consider problems (P) and (P_t) . Suppose that (P) is LP well-posed in the generalized sense and*

$$(40) \quad \liminf_{t \rightarrow 0^+} v_2(t) = \bar{v}.$$

Then (P) is LP well-posed in the strongly generalized sense.

Again, as noted in Remark 3.2, when the set-valued map X_2 is u.s.c. at $0 \in R_+^1$, then (39) holds. Thus, the generalized LP well-posedness of (P) implies its strongly generalized LP well-posedness.

4. Applications to penalty-type methods. In this section, we consider the convergence of a class of penalty methods and a class of augmented Lagrangian methods under the assumption of strongly generalized LP well-posedness of (P).

4.1. Penalty methods. Let $\alpha > 0$. Consider the following penalty problem:

$$(PP_\alpha(r)) \quad \min_{x \in X_1} f(x) + rd_K^\alpha(g(x)), \quad r > 0.$$

Denote by $v_4(r)$ the optimal value of $(PP_\alpha(r))$. It is clear that

$$(41) \quad v_4(r) \leq \bar{v} \quad \forall r > 0.$$

Remark 4.1. When $\alpha \in (0, 1)$, $X = R^l$, $Y = R^m$, $K = R_-^{m_1} \times \{0_{m-m_1}\}$, where $m \geq m_1$ and 0_{m-m_1} is the origin of the space R^{m-m_1} , this class of penalty functions was applied to the study of mathematical programs with equilibrium constraints [19]. Necessary and sufficient conditions for the exact penalization of this class of penalty functions were derived in [14]. This class of penalty methods was also applied to mathematical programs with complementarity constraints [27] and nonlinear semidefinite programs [15]. An important advantage of this class of penalty methods is that it requires weaker conditions to guarantee its exact penalization property than the usual l_1 penalty function method (see [19]).

THEOREM 4.1. *Let $0 < r_n \rightarrow +\infty$. Consider problems (P) and $(PP_\alpha(r_n))$. Assume that there exist $\bar{r} > 0$ and $m_0 \in R^1$ such that*

$$(42) \quad f(x) + \bar{r}d_K^\alpha(g(x)) \geq m_0 \quad \forall x \in X_1.$$

Let $0 < \epsilon_n \rightarrow 0$. Suppose that each $x_n \in X_1$ satisfies

$$(43) \quad f(x_n) + r_n d_K^\alpha(g(x_n)) \leq v_4(r_n) + \epsilon_n.$$

Further assume that (P) is LP well-posed in the strongly generalized sense. Then there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and some $\bar{x} \in \bar{X}$ such that $x_{n_k} \rightarrow \bar{x}$.

Proof. From (41) and (43), we have

$$f(x_n) \leq \bar{v} + \epsilon_n.$$

Thus,

$$(44) \quad \limsup_{n \rightarrow +\infty} f(x_n) \leq \bar{v}.$$

Moreover, from (41)–(43), we deduce that

$$f(x_n) + \bar{r}d_K^\alpha(g(x_n)) + (r_n - \bar{r})d_K^\alpha(g(x_n)) \leq \bar{v} + \epsilon_n.$$

Thus,

$$m_0 + (r_n - \bar{r})d_K^\alpha(g(x_n)) \leq \bar{v} + \epsilon_n,$$

implying

$$d_K(g(x_n)) \leq \left[\frac{\bar{v} + \epsilon_n - m_0}{r_n - \bar{r}} \right]^{1/\alpha}.$$

Passing to the limit, we get

$$(45) \quad \lim_{n \rightarrow +\infty} d_K(g(x_n)) = 0.$$

It follows from (44) and (45) that $\{x_n\}$ is a weakly generalized LP minimizing sequence. Hence, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and some $\bar{x} \in \bar{X}$ such that $x_{n_k} \rightarrow \bar{x}$. \square

4.2. Augmented Lagrangian methods. Let (X, d_1) be a metric space, let $Y = R^m$, and let $K \subset Y$ be a nonempty, closed, and convex set. Let $\sigma : R^m \rightarrow R^1 \cup \{+\infty\}$ be an augmenting function; namely, it is a lower semicontinuous, convex function satisfying

$$\min_{y \in R^m} \sigma(y) = 0 \text{ and } \sigma \text{ attains its unique minimum at } y = 0.$$

Following Example 11.46 in [25], we define the dualizing parametrization function by setting $X = X_1$ and $\theta = \delta_K$:

$$\bar{f}(x, u) = f(x) + \delta_{X_1}(x) + \delta_K(g(x) + u),$$

where δ_A is the indicator function of a subset A of a space Z , i.e.,

$$\delta_A(a) = \begin{cases} 0 & \text{if } a \in A, \\ +\infty & \text{if } a \in Z \setminus A. \end{cases}$$

Constructing the augmented Lagrangian as in Definition 11.55 of [25], we obtain the augmented Lagrangian:

$$\bar{l}(x, y, r) = \inf_{u \in R^m} \{ \bar{f}(x, u) + r\sigma(u) - \langle y, u \rangle \}, x \in X, y \in R^m, r > 0.$$

The augmented Lagrangian problem is

$$(ALP(y, r)) \quad \min_{x \in X} \bar{l}(x, y, r), \quad y \in R^m, r > 0.$$

Denote by $v_5(y, r)$ the optimal value of $(ALP(y, r))$.

We have the following result.

THEOREM 4.2. *Let $\{y_n\} \subset R^m$ be bounded and $0 < r_n \rightarrow +\infty$. Consider (P) and $(ALP(y_n, r_n))$. Assume that there exist $(\bar{y}, \bar{r}) \in R^m \times (0, +\infty)$ and $m_0 \in R^1$ such that*

$$(46) \quad \bar{l}(x, \bar{y}, \bar{r}) \geq m_0 \quad \forall x \in X.$$

Let $0 < \epsilon_n \rightarrow 0$. Suppose that each x_n satisfies

$$(47) \quad \bar{l}(x_n, y_n, r_n) \leq v_5(y_n, r_n) + \epsilon_n,$$

$v_5(y_n, r_n) > -\infty \forall n$, and (P) is LP well-posed in the strongly generalized sense. Then there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and some $\bar{x} \in \bar{X}$ such that $x_{n_k} \rightarrow \bar{x}$.

Proof. By the definition of $\bar{l}(x, y, r)$, it is easy to see that

$$\bar{l}(x, y, r) = f(x) \quad \forall x \in X_0.$$

It follows that

$$v_5(y, r) \leq \bar{v} \quad \forall y \in R^m, r > 0.$$

Thus,

$$(48) \quad v_5(y_n, r_n) \leq \bar{v} \quad \forall n.$$

By the definition of $\bar{l}(x_n, y_n, r_n)$ and (47), $\{x_n\} \subset X_1$ and there exists $\{u_n\} \subset R^m$ satisfying

$$(49) \quad g(x_n) + u_n \in K \quad \forall n$$

such that

$$(50) \quad f(x_n) + r_n \sigma(u_n) - \langle y_n, u_n \rangle \leq v_5(y_n, r_n) + 2\epsilon_n.$$

This combined with (46) and (48) implies that

$$(51) \quad (r_n - \bar{r})\sigma(u_n) - \langle y_n - \bar{y}, u_n \rangle \leq \bar{v} + 2\epsilon_n - m_0.$$

We assert that $\{u_n\}$ is bounded. Otherwise, we assume without loss of generality that $\|u_n\| \rightarrow +\infty$. Since the lower semicontinuous and convex function σ has a unique minimum, by Proposition 3.2.5 in IV of [11] and Corollary 3.27 of [25], $\liminf_{n \rightarrow +\infty} \sigma(u_n)/\|u_n\| > 0$. As $\{y_n\}$ is bounded, (51) cannot hold. So, $\{u_n\}$ should be bounded. Assume without loss of generality that $u_n \rightarrow u_0$. We deduce from (51) that

$$\sigma(u_0) \leq \liminf_{n \rightarrow +\infty} \sigma(u_n) = 0.$$

It follows that $u_0 = 0$. We deduce from (48) and (50) that

$$f(x_n) - \langle y_n, u_n \rangle \leq \bar{v} + 2\epsilon_n.$$

Passing to the limit, we get

$$\limsup_{n \rightarrow +\infty} f(x_n) \leq \bar{v}.$$

From (49) and the fact that $u_n \rightarrow 0$, we obtain

$$\lim_{n \rightarrow +\infty} d_K(g(x_n)) = 0.$$

Thus, $\{x_n\}$ is a weakly generalized LP minimizing sequence. Hence, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and some $\bar{x} \in \bar{X}$ such that $x_{n_k} \rightarrow \bar{x}$. \square

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