Altruism in Coalition Formation Games

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Abstract

Nguyen et al. (2016) introduced altruistic hedonic games in which agents' utilities depend not only on their own preferences but also on those of their friends that are in the same coalition. We propose to extend their model to coalition formation games in general, considering also the preferences of friends in other coalitions. In this model, an agent might not get any immediate advantage from a friend being happy in another coalition but might still be happy for that friend, on purely altruistic grounds. We study the common stability notions for this model and provide a computational analysis of the associated verification and existence problems.

1 Introduction

We consider coalition formation games where agents have to form coalitions based on their preferences. Inspired by the work of Nguyen et al. (2016) who introduced altruistic hedonic games, we extend their idea of altruism to a more general case. In their model, agents gain utility not only from their own satisfaction but also from their friends' satisfaction. However, Nguyen et al. (2016) specifically considered hedonic games only, which require that an agent's utility only depends on his or her own coalition. In their interpretation of altruism, the utility of an agent is composed of the agent's own valuation of his or her coalition and the valuations of all this agent's friends in this coalition.

By contrast, we propose a model where agents behave altruistically to *all their friends*, not only to the friends in their own coalition. We think that this better reflects the idea of altruism. An agent might not get any immediate advantage from a friend being happy in another coalition but might still be happy for that friend, on purely altruistic grounds. Such behavior might nurture and cement their friendship, and perhaps such an unselfish agent might, unintentionally of course, benefit from this friendship in a subsequent coalition formation process. In fact, examples show that our model crucially differs from the model due to Nguyen et al. (2016) and provides a more intuitive interpretation of altruism.

As a motivating example of altruism, consider the following scenario. Let $1, 2, 3$, and 4 be agents, where 1 and 2 are friends of each other, 2 and 3 are friends, and 3 and 4 are friends. Furthermore, assume that all other pairs of players are enemies of each other. This scenario is represented by the corresponding *network of friends* in Figure 1.

$$1 \text{ --- } 2 \text{ --- } 3 \text{ --- } 4$$

Figure 1: A network of friends

Possible partitions of the players into coalitions are, for example, $\Gamma = \{\{1, 2, 3\}, \{4\}\}$ and $\Delta = \{\{1, 2\}, \{3, 4\}\}$. If player 2 were selfish, it is clear that she would prefer Γ to Δ because she would be together with both her friends in Γ but only with one friend in Δ. If, however, player 2 were truly altruistic, she might prefer Δ to Γ for the following reason. Player 2 has two friends, 1 and 3, and she wants them to be happy. In Γ, players 1 and 3 both are together with one of their friends and with one of their enemies. But in Δ, both 1 and 3 are together with just one friend. This is obviously better than to also have enemies in one's coalition. Hence, player 2 might prefer Δ to Γ if she cares more for her friends' satisfaction than for her own.

2 Preliminaries

We will now provide some basic definitions. First, we will define *coalition formation games*. After that, we will explain the *"friends and enemies" encoding* due to Dimitrov et al. (2006) (see also the work of Sung and Dimitrov (2007)), which we will use for the representation of the players' preferences. Finally, we provide some basic background on graph theory which will be needed later.

2.1 Coalition Formation Games

Let $N = \{1, \dots, n\}$ be a set of *agents* (or *players*). Each subset of N is called a *coalition*. We denote the set of all possible coalitions containing an agent $i \in N$ by $\mathcal{N}^i = \{C \subseteq N \mid i \in C\}$. A *coalition structure* Γ is a partition of N. The set of all possible coalition structures for a set of agents N is denoted by $\mathcal{C}_N$. For a player $i \in N$ and a coalition structure $\Gamma \in \mathcal{C}_N$, $\Gamma(i)$ is the coalition in Γ containing i.

The objective of a *coalition formation game* is to form a coalition structure based on the agents' preferences over all possible coalitions they might be in. Hence, a *coalition formation game* is a pair $(N, \succeq)$, where $N = \{1, \dots, n\}$ is a set of agents, $\succeq = (\succeq_1, \dots, \succeq_n)$ is a profile of preferences, and every preference $\succeq_i$ is a complete weak order over $\mathcal{N}^i$.

2.2 The "Friends and Enemies" Encoding

Since $|\mathcal{N}^i|$, the number of coalitions containing agent i, is exponential in the number of agents, it is not reasonable to ask every agent for his or her complete preference over $\mathcal{N}^i$. Instead, many ways of how to compactly represent the agents' preferences have been proposed in the literature, such as the *additive encoding* (Sung and Dimitrov 2007; 2010; Aziz, Brandt, and Seedig 2013; Woeginger 2013), the *singleton encoding* due to Cechlárová and Romero-Medina (2001) and further studied by Cechlárová and Hajduková (2003; 2004), the *"friends and enemies" encoding* due to Dimitrov et al. (2006), and FEN-hedonic games due to Lang et al. (2015) and also used by Rothe, Schadrack, and Schend (2018) and Kerkmann and Rothe (2019).

Here, we will consider the friend-and-enemy encoding due to Dimitrov et al. (2006). We focus on the friend-oriented model and will later adapt it to our altruistic model.

In the friend-oriented model, the preferences of the agents N are given by a network of friends, i.e., a (simple) graph $G = (N, A)$ whose vertices are the players and where two players $i, j \in N$ are connected by an edge $\{i, j\} \in A$ exactly if they are each other's friends. Agents not connected by an edge consider each other as enemies. For an agent $i \in N$, we denote the set of i's friends by $F_i = \{j \in N \mid \{i, j\} \in A\}$ and the set of i's enemies by $E_i = \{j \in N \mid j \neq i \wedge \{i, j\} \notin A\} = N \setminus (F_i \cup \{i\})$.

Based on the network of friends, each agent $i \in N$ values a coalition $C \in \mathcal{N}^i$ with $v_i(C) = n|C \cap F_i| - |C \cap E_i|$. Note that $-(n-1) \leq v_i(C) \leq n(n-1)$ and $v_i(C) > 0$ if and only if there is at least one friend of i's in C (i.e., $|C \cap F_i| > 0$). For a given coalition structure $\Gamma \in \mathcal{C}_N$, we also write $v_i(\Gamma)$ for player i's value of $\Gamma(i)$. We, furthermore, denote the sum of the values of i's friends by $\mathrm{sum}_i^F(\Gamma) = \sum_{f \in F_i} v_f(\Gamma)$.

2.3 Some Fundamentals of Graph Theory

An *undirected graph* is a pair $G = (V, E)$, where V is a set of vertices and $E \subseteq \{\{u, v\} \mid u, v \in V\}$ is a set of edges.

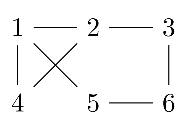

Figure 2: An undirected graph G

A *path* in an undirected graph $G = (V, E)$ is a sequence $(v_1, e_1, v_2, \dots, e_{k-1}, v_k)$ of vertices $v_1, \dots, v_k \in V$ and edges $e_1, \dots, e_{k-1} \in E$, where $e_i = \{v_i, v_{i+1}\}$ for all $i, 1 \leq i \leq k-1$. The *length of a path* is the number of edges on it. For example, $(4, \{4, 2\}, 2, \{2, 1\}, 1)$ is a path of length 2 in the graph of Figure 2.

A graph $G = (V, E)$ is *connected* if there exists a path between every pair $u, v \in V$ of vertices. For a subset $V' \subseteq V$ of the vertices, the *subgraph of G induced by V'* is defined by $G[V'] = (V', \{\{u, v\} \in E | u, v \in V'\})$. For example, the subgraph of the graph G from Figure 2 induced by $V' = \{1, 2, 3, 4, 6\}$ is shown in Figure 3.

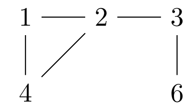

Figure 3: Subgraph $G[V']$ of G from Figure 2 induced by $V' = \{1, 2, 3, 4, 6\}$

A *connected component of graph* $G = (V, E)$ is a nonextendable subset of the vertices $V' \subseteq V$ such that $G[V']$ is connected. Here, "nonextendable" means that adding any further vertex v to V' will result in $G[V' \cup \{v\}]$ being disconnected. A subset $V' \subseteq V$ of the vertices is a *clique* if and only if for each two distinct vertices $u, v \in V'$, there is an edge $\{u, v\} \in E$. The *distance between two vertices* $u, v \in V$ is the length of a shortest path between u and v, or ∞ if there is no path between u and v. For example, 1 and 6 have a distance of 2 in Figure 2, yet a distance of 3 in Figure 3. Let $k \in \mathbb{N}$. A subset $V' \subseteq V$ of the vertices is a *k-clan* if and only if for each two vertices $u, v \in V'$, the distance between u and v in $G[V']$ is less than or equal to k.

Note that several of the just defined properties can be checked in polynomial time. In particular, the following statements will be used later on. The connected components of a graph can be determined in polynomial time. It is easy to verify whether a given subset of the vertices is a clique. Checking whether a subset of the vertices is a k-clan is possible in polynomial time.[1] More details about graph theory and graph algorithms can be found, e.g., in the textbooks by McHugh (1990) and Krumke and Noltemeier (2012).

3 Altruism in Coalition Formation Games

The main question in coalition formation games is which coalition structures might form. There are several notions of stability, each indicating whether a given coalition structure would be accepted by the agents or if there are other coalition structures that are more likely to form. To evaluate stability notions based on the agents' preferences, we first need to define when an agent prefers one coalition structure to another. Therefore, we now define the utility that an agents gains from a given coalition structure. Since we focus on altruistic agents, the utility will be composed of the agent's value for the coalition structure and her friends' values.

3.1 The Three Degrees of Altruism

Nguyen et al. (2016) introduced three degrees of altruism: selfish first, equal treatment, and altruistic treatment. We will also distinguish these three degrees, but adapt them to our model, extending the agents' altruism to all their friends.

[1] We can determine the shortest paths between all pairs of vertices in polynomial time. For more details, see the ALL-PAIRS-SHORTEST-PATH-PROBLEM that can be efficiently solved, for example, by the algorithm of Dijkstra (1959), the algorithm of Bellman (1958) and Ford Jr. (1956), or the algorithm of Floyd (1962) and Warshall (1962), see also Seidel (1995).

- **Selfish First:** Under this model, agents rank different coalition structures mainly based on their own valuations. Only in the case of a tie between two coalition structures, their friends' valuations are considered as well.
- **Equal Treatment:** Under equal treatment, agents treat themselves and their friends the same. That means that an agent $i \in N$ and all of i's friends have the same impact on i's utility.
- **Altruistic Treatment:** Under altruistic treatment, agents rank coalition structures based on their friends' valuations. They only consider their one valuation in the case of a tie.

Formally, for an agent $i \in N$ and a coalition structure $\Gamma \in \mathcal{C}_N$, we denote i's utility for Γ under selfish-first preferences by $u_i^{SF}(\Gamma)$, under equal treatment by $u_i^{EQ}(\Gamma)$, and under altruistic treatment by $u_i^{AL}(\Gamma)$. They are defined as

$$
\begin{aligned}
u_i^{SF}(\Gamma) &= M \cdot v_i(\Gamma) + \mathrm{sum}_i^F(\Gamma) \text{ with } M \geq n^3,\\
u_i^{EQ}(\Gamma) &= v_i(\Gamma) + \mathrm{sum}_i^F(\Gamma), \text{ and}\\
u_i^{AL}(\Gamma) &= v_i(\Gamma) + M \cdot \mathrm{sum}_i^F(\Gamma) \text{ with } M \geq n^2.
\end{aligned}
$$

For any coalition structures $\Gamma, \Delta \in \mathcal{C}_N$, agent $i \in N$ *weakly prefers* Γ to Δ under the selfish-first model (under equal treatment; under altruistic treatment), denoted by $\Gamma \succeq_i^{SF} \Delta$ ($\Gamma \succeq_i^{EQ} \Delta$; $\Gamma \succeq_i^{AL} \Delta$), if $u_i^{SF}(\Gamma) \geq u_i^{SF}(\Delta)$ ($u_i^{EQ}(\Gamma) \geq u_i^{EQ}(\Delta)$; $u_i^{AL}(\Gamma) \geq u_i^{AL}(\Delta)$). Analogously, i *prefers* Γ to Δ, denoted by $\Gamma \succ_i^{SF} \Delta$ ($\Gamma \succ_i^{EQ} \Delta$; $\Gamma \succ_i^{AL} \Delta$), if $u_i^{SF}(\Gamma) > u_i^{SF}(\Delta)$ ($u_i^{EQ}(\Gamma) > u_i^{EQ}(\Delta)$; $u_i^{AL}(\Gamma) > u_i^{AL}(\Delta)$). The factor M, which is used for the selfish-first model and for altruistic treatment, ensures that an agent's utility is first determined by the agent's own valuation in the selfish-first model and by the friends' valuations in the altruistic model. Propositions 1 and 2 are shown similarly as the corresponding properties by Nguyen et al. (2016); these proofs are omitted due to space constraints.

Proposition 1. *For $M \geq n^3$, $v_i(\Gamma) > v_i(\Delta)$ implies $\Gamma \succ_i^{SF} \Delta$.*

Proposition 2. *For $M \geq n^2$, $\mathrm{sum}_i^F(\Gamma) > \mathrm{sum}_i^F(\Delta)$ implies $\Gamma \succ_i^{AL} \Delta$.*

An *altruistic coalition formation game (ACFG)* is a coalition formation game where the agents' preferences were obtained by a network of friends via one of these three cases of altruism. Since an ACFG is completely determined by the underlying network of friends, we usually just specify this graph.

3.2 Comparison to Altruism in Hedonic Games

As mentioned before, Nguyen et al. (2016) focus on altruism in hedonic games where an agent's utility only depends on his or her own coalition. That means that, in their model, agents only consider friends that are in their own coalition. The following example shows that our model crucially differs from the model of Nguyen et al. (2016).

Example 3. *Consider an ACFG with five agents, $N = \{1,2,3,4,5\}$, and the network of friends in Figure 4.*

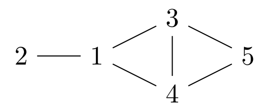

Figure 4: Network of friends for Example 3

The following table shows the values $v_i(\Gamma)$, $1 \leq i \leq 5$, and the sum of the values of player 1's friends for coalition structures $\Gamma_1 = \{\{1,2,3,4\},\{5\}\}$ and $\Gamma_2 = \{\{1,2\},\{3,4,5\}\}$:

$v_i(\Gamma)$	1	2	3	4	5	$\mathrm{sum}_1^F(\Gamma)$
Γ_1	15	3	9	9	0	21
Γ_2	5	5	10	10	10	25

Under the selfish-first model, agent 1 prefers Γ_1 to Γ_2 ($\Gamma_1 \succ_1^{SF} \Gamma_2$) because in Γ_1 she is together with all her friends while in Γ_2 she is only together with one friend.

Under altruistic treatment, however, she prefers Γ_2 to Γ_1 ($\Gamma_2 \succ_1^{AL} \Gamma_1$). Here, she would rather form a coalition with only 2 than being with all her friends. This is because 2 doesn't like 3 and 4, and 3 and 4 don't like 2. Since 1 is altruistic to all *her friends, she prefers the coalition structure that is valued higher by her friends.*

This shows that our model crucially differs from the model of Nguyen et al. (2016). In their model, agents only consider friends in their own coalition. Thus, under their model of altruistic treatment, 1 prefers Γ_1 to Γ_2 and would rather be with 2, 3, and 4 because the average valuation of her friends in her coalition would then be $\frac{21}{3} = 7$ instead of $\frac{5}{1} = 5$ when being alone with 2. Their model does not pay attention to the fact that all of agent 1's friends are even happier when agents 3 and 4 are in a coalition with 5. Since considering all friends and increasing their happiness better reflects the idea of altruism, we think that this example gives a good motivation of why our definitions make sense.

4 Computing the Utilities

For any altruistic coalition formation game (given by a network of friends), the players' utilities can be computed in polynomial time by the following proposition the proof of which again is omitted due to space constraints.

Proposition 4. *Let $G = (N, A)$ be a network of friends, $i \in N$ a player, and Γ a coalition structure. Let further λ be the number of friends that i has in her coalition (i.e., $\lambda = \left|\{\{i,j\} \in A \mid j \in \Gamma(i)\}\right| = \left|\Gamma(i) \cap F_i\right|$), μ be the number of edges between friends of i that are together in a coalition (i.e., $\mu = |\{\{j,k\} \in A \mid j \in F_i,\ k \in F_i,\ k \in \Gamma(j)\}| = \frac{1}{2}\sum_{j \in F_i} |F_i \cap F_j \cap \Gamma(j)|$), and ν be the number of edges between friends of i and enemies of i that are together in a coalition (i.e., $\nu = |\{\{j,k\} \in A \mid j \in F_i,\ k \notin F_i,\ k \in \Gamma(j)\}| = \sum_{j \in F_i} |E_i \cap F_j \cap \Gamma(j)|$). Then, i's utility can be computed by using $v_i(\Gamma) = (n+1)\lambda - |\Gamma(i)| + 1$ and $\mathrm{sum}_i^F(\Gamma) = (n+2)|F_i| + (n+1)(2\mu + \nu) - \sum_{f \in F_i} |\Gamma(f)|$.*

5 Stability

There are several stability concepts that are well-studied for hedonic games, a class of coalition formation games where an agent's utility depends on his or her own coalition only; see, e.g., the book chapter by Aziz and Savani (2016). Although we consider more general coalition formation games, we can easily adapt these definitions to our framework.

5.1 Stability Notions

Let $(N, \succeq)$ be an altruistic coalition formation game with agents $N = \{1, \dots, n\}$ and preferences $\succeq = (\succeq_1, \dots, \succeq_n)$ obtained from a network of friends via one of the three degrees of altruism. A coalition structure Γ is said to be

- *Nash stable* if no player prefers moving to another coalition; formally: $(\forall i \in N)(\forall C \in \Gamma \cup \{\emptyset\})[\Gamma \succeq_i \Gamma_{i \to C}]$, where $\Gamma_{i \to C}$ denotes the coalition structure that arises from Γ when moving i to C, i.e., $\Gamma_{i\to C} = \Gamma \setminus \{\Gamma(i), C\} \cup \{\Gamma(i) \setminus \{i\}, C \cup \{i\}\}$;
- *individual rational* if no player would prefer being alone: $(\forall i \in N)[\Gamma \succeq_i \Gamma_{i\to\emptyset}]$;
- *individually stable* if no player prefers moving to another coalition and could deviate to it without harming any player in that coalition: $(\forall i \in N)(\forall C \in \Gamma \cup \{\emptyset\})\big[\Gamma \succeq_i \Gamma_{i\to C} \vee (\exists j \in C)[\Gamma \succ_j \Gamma_{i\to C}]\big]$;
- *contractually individually stable* if no player prefers another coalition and could deviate to it without harming any player in the new or the old coalition: $(\forall i \in N)(\forall C \in \Gamma \cup \{\emptyset\})\big[\Gamma \succeq_i \Gamma_{i\to C} \vee (\exists j \in C)[\Gamma \succ_j \Gamma_{i\to C}] \vee (\exists k \in \Gamma(i))[\Gamma \succ_k \Gamma_{i\to C}]\big]$;
- *totally individually stable* if no player prefers another coalition and could deviate to it without harming any other player: $(\forall i \in N)(\forall C \in \Gamma \cup \{\emptyset\})\big[\Gamma \succeq_i \Gamma_{i\to C} \vee (\exists l \in N \setminus \{i\})[\Gamma \succ_l \Gamma_{i\to C}]\big]$;
- *core stable* if no nonempty coalition blocks Γ: $(\forall C \subseteq N, C \neq \emptyset)(\exists i \in C)[\Gamma \succeq_i \Gamma_{C\to\emptyset}]$, where $\Gamma_{C\to\emptyset}$ denotes the coalition structure that arises from Γ when all players in C leave their coalitions to form the new coalition C, i.e., $\Gamma_{C\to\emptyset} = \Gamma \setminus \{\Gamma(j) | j \in C\} \cup \{\Gamma(j) \setminus C | j \in C\} \cup \{C\}$;
- *strictly core stable* if no coalition weakly blocks Γ: $(\forall C \subseteq N)(\exists i \in C)[\Gamma \succ_i \Gamma_{C\to\emptyset}] \vee (\forall i \in C)[\Gamma \sim_i \Gamma_{C\to\emptyset}]$;
- *popular* if for every other coalition structure Δ, at least as many players prefer Γ to Δ as there are players who prefer Δ to Γ: $(\forall \Delta \in \mathcal{C}_N, \Delta \neq \Gamma)\big[|\{i \in N \mid \Gamma(i) \succ_i \Delta(i)\}| \geq |\{i \in N \mid \Delta(i) \succ_i \Gamma(i)\}|\big]$;
- *strictly popular* if for every other coalition structure Δ, more players prefer Γ to Δ than there are players who prefer Δ to Γ: $(\forall \Delta \in \mathcal{C}_N, \Delta \neq \Gamma)\big[|\{i \in N \mid \Gamma(i) \succ_i \Delta(i)\}| > |\{i \in N \mid \Delta(i) \succ_i \Gamma(i)\}|\big]$; and
- *perfect* if there is no player who prefers any coalition structure to Γ: $(\forall i \in N)(\forall \Delta \in \mathcal{C}_N)[\Gamma \succeq_i \Delta]$.

6 How Hard are Verification and Existence?

We now study the associated *verification* and *existence problems* and conduct a computational analysis for them. Given a stability concept α, these problems are defined as follows:

α-VERIFICATION	
Given:	An ACFG $(N, \succeq)$ and a coalition structure $\Gamma \in \mathcal{C}_N$.
Question:	Does Γ satisfy α?

α-EXISTENCE	
Given:	An ACFG $(N, \succeq)$.
Question:	Does there exist a coalition structure $\Gamma \in \mathcal{C}_N$ that satisfies α?

6.1 Individual Rationality

Verifying individual rationality is easy since we just need to iterate all agents and compare two coalition structures in each iteration. Since players' utilities can be computed in polynomial time, individual rationality can be verified in polynomial time (polynomial in the number of agents). The existence problem is trivial, since $\Gamma = \{\{1\}, \dots, \{n\}\}$ is always individually rational. Furthermore, we give the following characterization for individual rationality.

Theorem 5. *Let $(N, \succeq)$ be an ACFG where the preferences were obtained from a network of friends via one of the three degrees of altruism and let Γ be a coalition structure. Γ is individually rational if and only if it holds for all players $i \in N$ that $\Gamma(i)$ contains a friend of i's or i is alone, formally: $(\forall i \in N)[\Gamma(i) \cap F_i \neq \emptyset \vee \Gamma(i) = \{i\}]$.*

Proof. Γ is individually rational if and only if $(\forall i \in N)[\Gamma \succeq_i \Gamma_{i\to\emptyset}]$. In the selfish-first model, $\Gamma \succeq_i^{SF} \Gamma_{i\to\emptyset}$ is equivalent to

$$M\Big(v_i(\Gamma) - v_i(\Gamma_{i\to\emptyset})\Big) + \Big(\mathrm{sum}_i^F(\Gamma) - \mathrm{sum}_i^F(\Gamma_{i\to\emptyset})\Big) \geq 0;$$

for equal treatment, $\Gamma \succeq_i^{EQ} \Gamma_{i\to\emptyset}$ is equivalent to

$$\Big(v_i(\Gamma) - v_i(\Gamma_{i\to\emptyset})\Big) + \Big(\mathrm{sum}_i^F(\Gamma) - \mathrm{sum}_i^F(\Gamma_{i\to\emptyset})\Big) \geq 0;$$

and for altruistic treatment, $\Gamma \succeq_i^{AL} \Gamma_{i\to\emptyset}$ is equivalent to

$$\Big(v_i(\Gamma) - v_i(\Gamma_{i\to\emptyset})\Big) + M\Big(\mathrm{sum}_i^F(\Gamma) - \mathrm{sum}_i^F(\Gamma_{i\to\emptyset})\Big) \geq 0.$$

For all $f \in F_i$ with $f \notin \Gamma(i)$, we have $v_f(\Gamma) = v_f(\Gamma_{i\to\emptyset})$ because f's coalition stays the same when i deviates to a new coalition. For all $f \in F_i \cap \Gamma(i)$, it holds that $v_f(\Gamma_{i\to\emptyset}) = v_f(\Gamma) - n$ because $\Gamma_{i\to\emptyset}(f)$ contains exactly one friend of f less than $\Gamma(f)$, namely i. Hence, we have $\mathrm{sum}_i^F(\Gamma) - \mathrm{sum}_i^F(\Gamma_{i\to\emptyset}) = \sum_{f\in F_i}\Big(v_f(\Gamma) - v_f(\Gamma_{i\to\emptyset})\Big) = \sum_{f\in F_i\cap\Gamma(i)} n = |F_i \cap \Gamma(i)| \cdot n$. Furthermore, it holds that $v_i(\Gamma_{i\to\emptyset}) = 0$ since $\Gamma_{i\to\emptyset}(i) = \{i\}$ contains no friends and no enemies of i's. We then have

- $\Gamma \succeq_i^{SF} \Gamma_{i\to\emptyset}$ if and only if $M \cdot v_i(\Gamma) + |F_i \cap \Gamma(i)| \cdot n \geq 0$;
- $\Gamma \succeq_i^{EQ} \Gamma_{i\to\emptyset}$ if and only if $v_i(\Gamma) + |F_i \cap \Gamma(i)| \cdot n \geq 0$;
- $\Gamma \succeq_i^{AL} \Gamma_{i\to\emptyset}$ if and only if $v_i(\Gamma) + M \cdot |F_i \cap \Gamma(i)| \cdot n \geq 0$.

Note that $v_i(\Gamma) > 0$ is equivalent to $\Gamma(i) \cap F_i \neq \emptyset$, and $v_i(\Gamma) = 0$ is equivalent to $\Gamma(i) = \{i\}$. It is easy to see that all three inequalities above are equivalent to $\Gamma(i) \cap F_i \neq \emptyset \vee \Gamma(i) = \{i\}$, which completes the proof. ❑

6.2 Nash Stability

Since there are at most n coalitions in a coalition structure $\Gamma \in \mathcal{C}_N$, we can verify Nash stability in polynomial time: We just iterate all agents $i \in N$ ($|N| = n$) and all coalitions $C \in \Gamma \cup \{\emptyset\}$ (at most $n+1$) and check whether $\Gamma \succeq_i \Gamma_{i \to C}$. Since we can check a player's preference over two coalition structures in polynomial time and since we have at most a quadratic number of iterations ($n \cdot (n+1)$), verification is in P for Nash stability.

Existence is trivially in P for Nash stability; indeed, the same example that Nguyen et al. (2016) gave for altruistic hedonic games works here as well. Specifically, for $C = \{i \in N \mid F_i = \emptyset\} = \{c_1, \dots, c_k\}$ the coalition structure $\{\{c_1\}, \dots, \{c_k\}, N \setminus C\}$ is Nash stable.

Theorem 6. *Let $(N, \succeq)$ be an ACFG where the preferences were obtained by a network of friends via one of the three degrees of altruism and let Γ be a coalition structure. Γ is Nash stable if and only if*

$$(\forall i \in N)(\forall C \in \Gamma \cup \{\emptyset\})\Big[|\Gamma(i) \cap F_i| > |C \cap F_i| \\ \vee \big(|\Gamma(i) \cap F_i| = |C \cap F_i| \wedge |\Gamma(i) \cap E_i| \leq |C \cap E_i|\big)\Big],$$

The proof of Theorem 6 can be conducted similarly to the proof of Theorem 5 and is also omitted here.

6.3 Individual Stability

For individual stability, contractual individual stability, and total individual stability, existence is trivially in P. Nash stability implies all these three concepts, hence, the Nash stable coalition structure from above is also (contractually; totally) individually stable.

Verification is also in P for these concepts. Similarly to Nash stability, we iterate all players and all coalitions and check the particular conditions in polynomial time.

6.4 Core Stability and Strict Core Stability

The next theorem shows that the existence problem is trivial for (strict) core stability if the preferences are obtained via the selfish-first model.

Theorem 7. *Let $(N, \succeq^{SF})$ be an ACFG where the preferences $\succeq^{SF}$ were obtained from a network of friends G via the selfish-first model. Let further $C_1, \dots, C_k$ be the vertex sets of the connected components of G. Then $\Gamma = \{C_1, \dots, C_k\}$ is strictly core stable (and thus core stable).*

Proof. For the sake of contradiction, assume that Γ is not strictly core stable, i.e., that there is a coalition $D \neq \emptyset$ that weakly blocks Γ. We then have

$$(\forall i \in D)[\Gamma \preceq_i^{SF} \Gamma_{D \to \emptyset}] \wedge (\exists j \in D)[\Gamma \prec_j^{SF} \Gamma_{D \to \emptyset}].$$

Since every player i in D weakly prefers deviating from $\Gamma(i)$ to D, there have to be at least as many friends of i's in D as in $\Gamma(i)$. Since $\Gamma(i)$ contains all of i's friends, D also has to contain all friends of i's. Additionally, D cannot contain any players $k \notin \Gamma(i)$ because these players are enemies of i's and i wouldn't like to deviate to a coalition with the same number of friends but more enemies than in $\Gamma(i)$. Hence, D is a subset of $\Gamma(i)$ ($D \subseteq \Gamma(i)$). Since one player $j \in D$ strictly prefers deviating from $\Gamma(j)$ to D, D cannot equal $\Gamma(i)$ and thus has to be a strict subset of it ($D \subset \Gamma(i)$). However, since the subgraph of G that is induced by the vertices in $\Gamma(i)$ is connected, there is an edge between a vertex k in D and a vertex ℓ in $\Gamma(i) \setminus D$. Then, $k \in D$ does not have all his friends in D, which is a contradiction to the fact that k weakly prefers deviating to D. ❑

We now show that the coalition structure from the proof of Theorem 7 is not always core stable under equal treatment or altruistic treatment.

Example 8. *Let $N = \{1, \dots, 10\}$ and let the preferences be given by the network of friends G shown in Figure 5.*

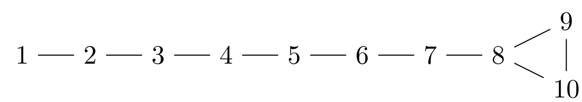

Figure 5: Network of friends for Example 8

Consider the coalition structure consisting of the connected components of G, i.e., $\Gamma = \{\{1, \dots, 10\}\}$, and the coalition $C = \{8, 9, 10\}$. It holds that C blocks Γ under equal treatment and altruistic treatment. To see this, we consider how players 7 to 10 value Γ and $\Gamma_{C \to \emptyset}$ and compute the utilities for players 8, 9, and 10.

Omitting the details due to space constraints, for all $i \in C = \{8, 9, 10\}$ we have that $u_i^{EQ}(\Gamma) < u_i^{EQ}(\Gamma_{C \to \emptyset})$, which implies $\Gamma \prec_i^{EQ} \Gamma_{C \to \emptyset}$, and that $\mathrm{sum}_i^F(\Gamma) < \mathrm{sum}_i^F(\Gamma_{C \to \emptyset})$, which implies $u_i^{AL}(\Gamma) < u_i^{AL}(\Gamma_{C \to \emptyset})$ and $\Gamma \prec_i^{AL} \Gamma_{C \to \emptyset}$. Hence, C blocks Γ for equal treatment and altruistic treatment.

Lemma 9. *Let $(N, \succeq)$ be an ACFG where the preferences were obtained from a network of friends G via one of the three degrees of altruism. Let Γ be a coalition structure. If there is a coalition $C \in \Gamma$ that does not induce a connected subgraph of G, Γ is not core stable.*

Proof. Assume that $C \in \Gamma$ does not induce a connected subgraph in the network of friends G. Let $D \subset C$ be a maximal subset of C such that D induces a connected subgraph in G. It then holds that D blocks Γ: For all $i \in D$, i has the same number of friends in D and C but $|C| - |D|$ enemies less in D. Hence, it holds that $v_i(\Gamma_{D \to \emptyset}) = v_i(\Gamma) + |C| - |D| > v_i(\Gamma)$ for all $i \in D$. This directly implies $\Gamma_{D \to \emptyset} \succ_i^{SF} \Gamma$ for all $i \in D$. This completes the proof for the selfish-first model.

Moreover, all friends f of $i \in D$ are either in D or not in C. (There are no edges between D and $C \setminus D$.) For $f \in D$, we already know that $v_f(\Gamma_{D \to \emptyset}) > v_f(\Gamma)$. For $f \notin C$, $v_f(\Gamma_{D \to \emptyset}) = v_f(\Gamma)$ because f's coalition does not change when D deviates. Hence, $\mathrm{sum}_i^F(\Gamma_{D \to \emptyset}) \geq \mathrm{sum}_i^F(\Gamma)$ for all $i \in D$. This implies $v_i(\Gamma_{D \to \emptyset}) + \mathrm{sum}_i^F(\Gamma_{D \to \emptyset}) \geq v_i(\Gamma) + \mathrm{sum}_i^F(\Gamma)$ for all $i \in D$. Thus $\Gamma_{D \to \emptyset} \succ_i^{EQ} \Gamma$ for all $i \in D$. This completes the proof for equal treatment.

If $|D| > 1$, then each $i \in D$ has at least one friend in D. Then $\mathrm{sum}_i^F(\Gamma_{D \to \emptyset}) > \mathrm{sum}_i^F(\Gamma)$ and $\Gamma_{D \to \emptyset} \succ_i^{AL} \Gamma$ follows for all $i \in D$. If $|D| = 1$, then $i \in D$ only has friends $f \notin C$ (or no friends at all). Then $\mathrm{sum}_i^F(\Gamma_{D\to\emptyset}) = \mathrm{sum}_i^F(\Gamma)$. Since $v_i(\Gamma_{D\to\emptyset}) > v_i(\Gamma)$, it follows that $\Gamma_{D\to\emptyset} \succ_i^{AL} \Gamma$. This completes the proof for altruistic treatment. ❑

Theorem 10. *(Strict) core stability verification can be done in* coNP *for all three degrees of altruism.*

Proof. Let G be a network of friends on agents N and Γ a coalition structure. Γ is not (strictly) core stable if there is a coalition $C \subseteq N$ that (weakly) blocks Γ, i.e., $(\forall i \in C)[\Gamma_{C\to\emptyset} \succ_i \Gamma]((\forall i \in C)[\Gamma_{C\to\emptyset} \succeq_i \Gamma] \wedge (\exists j \in C)[\Gamma_{C\to\emptyset} \succ_j \Gamma])$. Hence, we nondeterministically guess a coalition $C \subseteq N$ and check whether C blocks Γ. This can be done in polynomial time since $\Gamma_{C\to\emptyset} \succeq_i \Gamma$ can be verified in polynomial time for all three degrees of altruism. ❑

We can even show a lower bound for the selfish-first model. The proof is omitted due to space constraints.

Theorem 11. *Core stability verification is* coNP*-complete for the selfish-first model.*

6.5 Popularity and Strict Popularity

Under all three degrees of altruism, there does not always exist a (strictly) popular coalition structure.

Example 12. *Let $N = \{1, \dots, 7\}$ and let the preferences be given by the network of friends shown in Figure 6.*

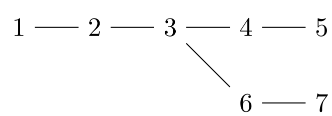

Figure 6: Network of friends for Example 12

Then there is no (strictly) popular coalition structure for any of the three degrees of altruism. Since perfectness implies popularity, there is also no perfect coalition structure.

We now show that under the selfish-first model it is hard to verify if a given coalition structure is strictly popular and it is also hard to decide whether there exists a strictly popular coalition structure for a given ACFG.

Theorem 13. *Strict popularity verification is* coNP*-complete under the selfish-first model.*

Proof. First recall that a coalition structure $\Gamma \in \mathcal{C}_N$ is not popular in an ACFG $(N, \succeq)$ (given by a network of friends G) if and only if there is a coalition structure $\Delta \in \mathcal{C}_N, \Delta \neq \Gamma$ such that $|\{i \in N \mid \Gamma(i) \succ_i \Delta(i)\}| \leq |\{i \in N \mid \Delta(i) \succ_i \Gamma(i)\}|$. Since we can nondeterministically choose the coalition structure Δ and verify the above condition in polynomial time, strict popularity verification belongs to coNP.

To show coNP-hardness, we use a restricted version of Exact Cover by 3-Sets (X3C) (Garey, Johnson, and Stockmeyer 1976), which we denote by RX3C and which is shown to remain NP-complete by Gonzalez (1985). We provide a polynomial-time many-one reduction from RX3C to the complement of strict popularity verification.

An instance $(B, \mathscr{S})$ of RX3C consists of a set $B = \{1, \dots, 3k\}$ and a collection $\mathscr{S} = \{S_1, \dots, S_{3k}\}$ of 3-element subsets of B ($S_i \subseteq B$ and $|S_i| = 3$ for $1 \leq i \leq 3k$). The instance is restricted such that each element of B occurs in exactly three sets in $\mathscr{S}$. Given this instance, the question is whether $\mathscr{S}$ contains an exact cover for B, i.e., a subset $\mathscr{S}' \subseteq \mathscr{S}$ such that every element of B occurs in exactly one set in $\mathscr{S}'$.

Given an instance $(B, \mathscr{S})$ of RX3C, we construct the following ACFG. The set of players is given by

$$N = \{\alpha_1, \dots, \alpha_{5k}\} \cup \{\beta_b | b \in B\} \cup \{\zeta_S, \eta_S | S \in \mathscr{S}\}.$$

We denote $Alpha = \{\alpha_1, \dots, \alpha_{5k}\}$, $Beta = \{\beta_b | b \in B\}$, and $Q_S = \{\zeta_S, \eta_S\}$ for each $S \in \mathscr{S}$. The network of friends is given in Figure 7, where a dashed circle around a group of players means that all these players are friends of each other:

- All players in $Alpha \cup Beta$ are friends of each other.
- For every $S \in \mathscr{S}$, ζ_S and η_S are friends.
- For every $S \in \mathscr{S}$, ζ_S is friend with every β_b with $b \in S$.

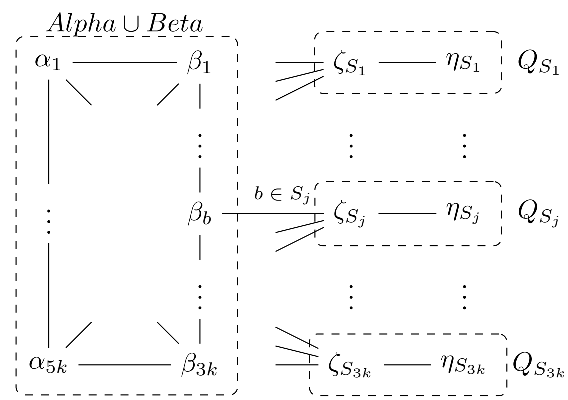

Figure 7: Network of friends in the proof of Theorem 13.

Furthermore, consider the coalition structure $\Gamma = \{Alpha \cup Beta, Q_{S_1}, \dots, Q_{S_{3k}}\}$. We show that $\mathscr{S}$ contains an exact cover for B if and only if Γ is not strictly popular.

Only if: Assume that there is an exact cover $\mathscr{S}' \subseteq \mathscr{S}$ for B. Since every set in $\mathscr{S}$ contains three elements of B, we have $|\mathscr{S}'| = k$. Consider the coalition structure $\Delta = \{Alpha \cup Beta \cup \bigcup_{S\in\mathscr{S}'} Q_S\} \cup \{Q_S | S \in \mathscr{S} \setminus \mathscr{S}'\}$. It holds that

- All $\beta_b, b \in B$ prefer Δ to Γ since they have $8k-1$ friends in $\Gamma(\beta_b)$ but $8k$ friends in $\Delta(\beta_b)$.
- All $\alpha_l, 1 \leq l \leq 5k$ prefer Γ to Δ because they have the same number of friends in both coalition structures but no enemies in $\Gamma(\alpha_l)$ and $2k$ enemies in $\Delta(\alpha_l)$.
- All ζ_S with $S \in \mathscr{S}'$ prefer Δ to Γ because they have one friend in $\Gamma(\zeta_S)$ but four friends in $\Delta(\zeta_S)$.
- For all ζ_S with $S \in \mathscr{S} \setminus \mathscr{S}'$, it holds that $\Delta(\zeta_S) = \Gamma(\zeta_S)$. Hence, they decide their preferences according to their friends valuations. They are friend of η_S who values Γ and Δ the same and friend of three β_b with $b \in S$ who all value Δ better than Γ. Hence ζ_S prefers Δ to Γ.

- All η_S with $S \in \mathscr{S}'$ prefer Γ to Δ because they have the same number of friends in $\Gamma(\eta_S)$ and $\Delta(\eta_S)$ but less enemies in $\Gamma(\eta_S)$.
- All η_S with $S \in \mathscr{S} \setminus \mathscr{S}'$ are indifferent between Γ and Δ because $\Delta(\eta_S) = \Gamma(\eta_S)$ and their only friend ζ_S values Γ and Δ the same.

We then have

$$\begin{aligned}\#_{\Delta\succ\Gamma} &= |\{i \in N \mid \Delta(i) \succ_i \Gamma(i)\}| \\ &= |\{\beta_1, \dots, \beta_{3k}, \zeta_{S_1}, \dots, \zeta_{S_{3k}}\}| = 6k \text{ and} \\ \#_{\Gamma\succ\Delta} &= |\{i \in N \mid \Gamma(i) \succ_i \Delta(i)\}| \\ &= |\{\alpha_1, \dots, \alpha_{5k}\} \cup \{\eta_S | S \in \mathscr{S}'\}| = 5k + k = 6k.\end{aligned}$$

Since $\#_{\Delta\succ\Gamma} = \#_{\Gamma\succ\Delta}$, Γ is not strictly popular.

If: Assume that Γ is not strictly popular, i.e., that there is a coalition structure $\Delta \in \mathcal{C}_N, \Delta \neq \Gamma$ with $\#_{\Gamma\succ\Delta} \leq \#_{\Delta\succ\Gamma}$. We first deduce the following statements:

- For every $\alpha_l, 1 \leq l \leq 5k$ it holds that $Alpha \cup Beta$ is her best valued coalition since she is together with all her friends and non of her enemies. Every other coalition is valued worse. Hence, α_l prefers Γ to every coalition structure where she is not in $Alpha \cup Beta$. Furthermore, she is indifferent between Γ and coalition structure Δ if $\Delta(\alpha_l) = Alpha \cup Beta$.
- If $Alpha \cup Beta$ was a coalition in Δ then some of the players from $Q_{S_1}, \dots, Q_{S_{3k}}$ would be partitioned in a different way than in Γ. However, this would not cause any player to be happier. There would be at least two players who prefer Γ to Δ but no player who prefers Δ to Γ. This is a contradiction to the assumption.

Hence, $Alpha \cup Beta$ is not a coalition in Δ and all $5k$ α-players prefer Γ to Δ. Furthermore, for every $\eta_S, S \in \mathscr{S}$ it holds that Q_S is her best valued coalition. Again, η_S prefers Γ to Δ if $\Delta(\eta_S) \neq Q_S$ and is indifferent between Γ and Δ if $\Delta(\eta_S) = Q_S$.

We now deduce that there is a coalition structure $\Delta' \in \mathcal{C}_N$ with $\#_{\Gamma\succ\Delta'} \leq \#_{\Delta'\succ\Gamma}$ where $Alpha \cup Beta$ are together in a coalition and for all $S \in \mathscr{S}$ Q_S are together. We obtain Δ' by slightly changing Δ:

- If $\beta_b, b \in B$ are not together in Δ then $\Delta(\beta_b), b \in B$ are unified.
- If $\alpha_l, 1 \leq l \leq 5k$ are not together with $\beta_b, b \in B$ then $\alpha_l, 1 \leq l \leq 5k$ leave their coalitions and join $\beta_b, b \in B$.
- If ζ_S and η_S are not together for any $S \in \mathscr{S}$, then η_S joins $\Delta(\zeta_S)$.

In each step, it holds that

- all players who preferred Δ to Γ do still prefer the new coalition structure Δ' to Γ ($\#_{\Delta\succ\Gamma} \leq \#_{\Delta'\succ\Gamma}$) and
- no player who didn't prefer Γ to Δ will now prefer Γ to Δ' ($\#_{\Gamma\succ\Delta'} \leq \#_{\Gamma\succ\Delta}$).

We denote the number of Q_S sets that are together with $Alpha \cup Beta$ by k'. Then $k' \leq k$ because otherwise there were at least $5k + k + 1 = 6k + 1$ players who prefer Γ to Δ' which is a contradiction to $\#_{\Gamma\succ\Delta'} \leq \#_{\Delta'\succ\Gamma}$. (There are at most $6k$ players who prefer Δ' to Γ.) Moreover $k' \geq k$ because if $k' < k$ then $\#_{\Gamma\succ\Delta} \geq 5k + k' + (3k - 3k') = 8k - 2k' > 6k' = 3k' + 3k' \geq \#_{\Delta\succ\Gamma}$ which again is a contradiction. Hence, $k' = k$. It follows that exactly k of the η-players prefer Γ to Δ. Thus, $\#_{\Gamma\succ\Delta} \geq 5k + k = 6k$. Then, all of the β-players and all ζ-players have to prefer Δ to Γ. Consequently, all β-players need to have an edge to one of the k ζ-players who are with $Alpha \cup Beta$. Then, $\{S \in \mathscr{S} | Q_S \text{ is with } Alpha \cup Beta \text{ in } \Delta'\}$ is an exact cover for B. ❑

We get bounds for two more problems by slightly changing the reduction from Theorem 13.

Theorem 14. *Strict popularity existence is* coNP*-hard under the selfish-first model.*

Proof. We consider the same reduction as in Theorem 13 with the difference that Γ is not given by the construction. If there is an exact cover for B then there is no strictly popular coalition structure. Γ and Δ as defined above are in a tie and every other coalition structure is beaten by Γ. If there is no exact cover for B then Γ beats every other coalition structure and is strictly popular. ❑

Theorem 15. *Popularity verification is* coNP*-complete under the selfish-first model.*

Proof. The proof is very similar to the proof of Theorem 13. The only difference is that in the construction of the ACFG, we only define $5k - 1$ α-players. Then, $\mathscr{S}$ contains an exact cover for B if and only if Γ is not popular. ❑

6.6 Perfectness

Turning to perfectness, we start with the selfish-first model.

Theorem 16. *Let G be a network of friends on a set of agents N. A coalition structure $\Gamma \in \mathcal{C}_N$ is perfect under the selfish-first model if and only if it consists of the connected components of G and all them are cliques.*

Proof. From left to right, assume that the coalition structure $\Gamma \in \mathcal{C}_N$ is perfect. It then holds for all agents $i \in N$ and all coalition structures $\Delta \in \mathcal{C}_N$, $\Delta \neq \Gamma$, that i weakly prefers Γ to Δ ($\Gamma \succeq_i^{SF} \Delta$). It follows that $v_i(\Gamma) \geq v_i(\Delta)$ for all $\Delta \in \mathcal{C}_N$, $\Delta \neq \Gamma$.

We consider a coalition structure $\Delta \in \mathcal{C}_N$ where i is together with all her friends and none of her enemies (i.e., $F_i \subset \Delta(i)$ and $E_i \cap \Delta(i) = \emptyset$). Then, for Γ to satisfy $v_i(\Gamma) \geq v_i(\Delta)$, $\Gamma(i)$ also has to contain all friends and no enemies. Hence, i has an edge to every agent in $\Gamma(i)$ and to no agent outside of $\Gamma(i)$. Since this holds for every $i \in N$, all coalitions in Γ are cliques and there is no edge between any two coalitions in Γ. Thus Γ consists of the connected components of G and these components are cliques.

From right to left, note that all agents $i \in N$ value Γ with $v_i(\Gamma) = n \cdot |F_i|$, which is the maximum value possible. ❑

Since it is easy to check this characterization, perfect coalition structures can be verified in polynomial time for the selfish-first model. It follows directly from Theorem 16 that the corresponding existence problem is also in P:

Theorem 17. *Let N be a set of agents and G a network of friends on N. There exists a perfect coalition structure under the selfish-first model if and only if all connected components of G are cliques.*

We now turn to equal treatment. The proofs of Lemmas 18, 19, and 20 are omitted due to space constraints.

Lemma 18. *Let G be a network of friends on a set of agents N. If a coalition structure $\Gamma \in \mathcal{C}_N$ is perfect under equal treatment then each $i \in N$ has at least one friend in $\Gamma(i)$ or has no friends at all.*

As the following lemma shows, perfectness under equal treatment even implies that every player is together with all her friends.

Lemma 19. *Let G be a network of friends on a set of agents N. If a coalition structure $\Gamma \in \mathcal{C}_N$ is perfect under equal treatment then each agent is together with all her friends, i.e., $(\forall i \in N)[F_i \subset \Gamma(i)]$.*

Note that each agent is together with all her friends in a coalition structure Γ if and only if Γ consists of the connected components of the underlying network of friends.

Lemma 20. *Let G be a network of friends on a set of agents N. If a coalition structure $\Gamma \in \mathcal{C}_N$ is perfect under equal treatment then each coalition in Γ is a 2-clan.*

By combining Lemmas 19 and 20, we get the following.

Lemma 21. *Let G be a network of friends on a set of agents N. If a coalition structure $\Gamma \in \mathcal{C}_N$ is perfect under equal treatment then Γ consists of the connected components of G and these components are 2-clans.*

However, Lemma 21 is not an equivalence. The converse does not hold as the following example shows.

Example 22. *Let $N = \{1, \dots, 9\}$ and let the network of friends G be given by Figure 8.*

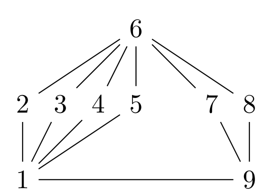

Figure 8: Network of friends for Example 22

$\Gamma = \{\{1, \dots, 9\}\}$ *consists of the only connected component of G, which is a 2-clan. However, Γ is not perfect under equal treatment because agent 1 prefers $\Delta = \{\{1, \dots, 6\}, \{7, 8, 9\}\}$ to Γ: Omitting the details, we can show that $u_1^{EQ}(\Delta) = 113 > 112 = u_1^{EQ}(\Gamma)$.*

Theorem 23. *Perfectness verification can be done in* coNP *for all three degrees of altruism.*

Proof. Let $(N, \succeq)$ be an ACFG and G the underlying network of friends. A coalition structure $\Gamma \in \mathcal{C}_N$ is not perfect if and only if there is an agent $i \in N$ and a coalition structure $\Delta \in \mathcal{C}_N$ such that $\Delta \succ_i \Gamma$. Hence, we can nondeterministically guess an agent $i \in N$ and a coalition structure $\Delta \in \mathcal{C}_N$ and, by Proposition 4, we can verify in polynomial time whether $\Delta \succ_i \Gamma$. ❑

Using Lemma 21, we can also show an upper bound for perfectness existence under equal treatment.

Theorem 24. *Perfectness existence is in* coNP *for equal treatment.*

Proof. Let $(N, \succeq)$ be an ACFG and G the underlying network of friends. Let, furthermore, $\Gamma \in \mathcal{C}_N$ be the coalition structure that consists of the connected components of G. From Lemma 21 we know that if a coalition structure does not consist of the connected components of G, it is not perfect. Hence, Γ is the only coalition structure which possibly is perfect. Therefore, there exists a coalition structure that is perfect under equal treatment if and only if Γ is perfect. Since perfectness verification is in coNP, it follows from this equivalence that existence is in coNP. ❑

7 Conclusions and Open Problems

We have proposed to extend the model of altruistic hedonic games due to Nguyen et al. (2016) to coalition formation games in general. We have compared this more general model to altruism in hedonic games and have motivated our work by giving an example with crucial differences between the models. We have studied the common stability notions and have initiated a computational analysis of the associated verification and existence problems. While these problems are solvable in polynomial for some stability notions such as individual rationality and Nash stability, for others they are coNP-complete. We were also able to give characterizations for some of the stability notions, using graph-theoretical properties of the underlying network of friends. For future work we propose to complete this analysis and get a full characterization for all stability notions. Furthermore, it would be interesting to see if those problems, for which we could only show coNP upper bounds, are also coNP-complete.

Another interesting research topic could be the consideration of altruism for other representations of the players' preferences. First, it could be interesting to consider the friends-and-enemies encoding due to Dimitrov et al. (2006) with enemy-oriented preferences. Furthermore, it might also be revealing to consider weak rankings with double thresholds as introduced by Lang et al. (2015).

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