Upper and lower bounds for finite domain constraints to realize skeptical c-inference over conditional knowledge bases

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Abstract

Skeptical c-inference for a knowledge base containing conditionals of the form *If A then usually B* is defined by taking the set of all c-representations into account. C-representations are ranking functions induced by impact vectors encoding the conditional impact of the conditionals on each possible world. We deal with the question of determining a maximal impact factor $u \in \mathbb{N}$ such that c-inference can be implemented by a finite domain constraint problem with solutions bounded by u. While in general, determining a sufficient upper bound for these CSPs is an open problem, we prove that for a knowledge base with n conditionals with a world verifying all conditionals it follows that $n-1$ is a sufficient maximal impact factor. Furthermore, we show that the conjecture supported by previous work that the number of conditionals is sufficient does not hold. By constructing suitable knowledge bases with n conditionals we establish that the exponential lower bound 2^{n-1} is needed as possible impact factor for solutions of these finite domains problems to fully realize skeptical c-inference.

1 Introduction

In the area of knowledge representation and reasoning, rules play a prominent role. Nonmonotonic reasoning investigates default rules of the form "If A then normally/usually/preferably B", and various semantical approaches have been proposed for inductive inferences based on knowledge bases of such rules. Calculi to compute inductive inferences like Adam's system P (Adams 1975), probabilistic approaches like p-entailment (Goldszmidt and Pearl 1991), or possibilistic inference methods (Dubois and Prade 2015) have been developed, as well as inductive methods based on Spohn's ordinal conditional functions (OCFs) (Spohn 1988; 2012) like Pearl's system Z (Pearl 1990) or c-representations (Kern-Isberner 2001; 2004). OCFs assign a degree of surprise to each world ω inducing a non-monotonic inference relation (Dubois and Prade 2015; Pearl 1990; Spohn 1988). C-representations are special ranking functions exhibiting desirable inference properties (Kern-Isberner 2001; 2004). In this paper we focus on skeptical c-inference which is introduced in (Beierle, Eichhorn, and Kern-Isberner 2016) as skeptical inference relation taking all c-representations into account. The authors show that c-inference can be reduced to solve a constraint satisfaction problem (CSP). In (Beierle et al. 2018) c-inference under a maximal impact factor is introduced as skeptical inference operation taking c-representations as solutions of a finite domain CSP into account. Let us follow (Beierle and Kutsch 2017) and call $u \in \mathbb{N}$ *sufficient* for $\mathcal{R}$ if c-inference under a maximal impact u fully realizes skeptical c-inference and $l \in \mathbb{N}$ *minimally sufficient* if this property is not fulfilled for $l-1$. The present paper deals with upper and lower bounds for $u \in \mathbb{N}$ to be sufficient and minimally sufficient, respectively. We provide the following main contributions.

- We formulate and prove a criterion generalizing (Beierle and Kutsch 2019, Proposition 19) such that $u = |\mathcal{R}| - 1$ is sufficient for $\mathcal{R}$: If there is a world verifying all conditionals from $\mathcal{R}$ (Proposition 13).
- We prove that for every given verification/ falsification behaviour of conditionals on worlds there is a knowledge base realizing this behaviour (Proposition 14).
- All experiments made with a reasoning platform InfOCF in (Beierle and Kutsch 2019) supported the conjecture that a maximal impact $u = |\mathcal{R}|$ is sufficient for $\mathcal{R}$. However, here we construct a knowledge base with n conditionals such that 2^{n-1} is minimally sufficient. Consequently there is no polynomial bound for $u \in \mathbb{N}$ to be minimally sufficient for all knowledge bases with n conditionals (Proposition 16).

The rest of the paper is organized as follows. After briefly recalling the basics of conditional logic, ranking functions, skeptical c-inference and its formulation as a constraint satisfaction problem (CSP) we deal in Section 3 with resource bounded c-inference and the concept of sufficient and regular bounds for finite domain CSPs. The topic of Section 4 is a criterion on a knowledge base $\mathcal{R}$ such that $|\mathcal{R}| - 1$ is a sufficient bound for $\mathcal{R}$. In Section 5, we deal with the construction of knowledge bases whose existence will establish the exponential lower bound for skeptical inference under maximal impact u to be equivalent to skeptical c-inference for all knowledge bases. In the final section we conclude and point out future work.

2 Conditional logic, OCFS, c-representations and the constraint satisfaction problem

Conditional Logic and OCFs Let $\Sigma = \{v_1, ..., v_m\}$ be a propositional alphabet. A *literal* is the positive (v_i) or negated ($\overline{v_i}$) form of a propositional variable, $\dot{v}_i$ stands for either v_i or $\overline{v_i}$. From these we obtain the propositional language $\mathcal{L}$ as the set of formulas of Σ closed under negation $\neg$, conjunction $\wedge$, and disjunction $\vee$. For shorter formulas, we abbreviate conjunction by juxtaposition (i.e., AB stands for $A \wedge B$), and negation by overlining (i.e., $\overline{A}$ is equivalent to $\neg A$). Let Ω_Σ denote the set of possible worlds over $\mathcal{L}$; Ω_Σ will be taken here simply as the set of all propositional interpretations over $\mathcal{L}$ and can be identified with the set of all complete conjunctions over Σ; we will often just write Ω instead of Ω_Σ. For $\omega \in \Omega$, $\omega \models A$ means that the propositional formula $A \in \mathcal{L}$ holds in the possible world ω.

A *conditional* $(B|A)$ with $A, B \in \mathcal{L}$ encodes the defeasible rule "if A then normally B" and is a trivalent logical entity with the evaluation (de Finetti 1937; Kern-Isberner 2001) (with u for *unknown* or *indeterminate*)

$$[\![(B|A)]\!]_\omega = \begin{cases} 1 & \text{iff} \quad \omega \models AB \quad \text{(verification)} \\ 0 & \text{iff} \quad \omega \models A\overline{B} \quad \text{(falsification)} \\ u & \text{iff} \quad \omega \models \overline{A} \quad \text{(not applicable)} \end{cases} \tag{1}$$

An *Ordinal Conditional Function* (OCF, ranking function) (Spohn 1988; 2012) is a function $\kappa : \Omega \to \mathbb{N}_0 \cup \{\infty\}$ that assigns to each world $\omega \in \Omega$ an implausibility rank $\kappa(\omega)$: the higher $\kappa(\omega)$, the more surprising ω is. OCFs have to satisfy the normalization condition that there has to be a world that is maximally plausible, i.e., $\kappa^{-1}(0) \neq \emptyset$. The rank of a formula A is defined by $\kappa(A) = \min\{\kappa(\omega) \mid \omega \models A\}$. An OCF κ *accepts* a conditional $(B|A)$, denoted by $\kappa \models (B|A)$, iff the verification of the conditional is less surprising than its falsification, i.e., iff $\kappa(AB) < \kappa(A\overline{B})$. This can also be understood as a nonmonotonic inference relation between the premise A and the conclusion B: We say that A *κ-entails* B, written $A \mathrel{|\!\sim}^{\kappa} B$, iff $A = \perp$ or κ accepts the conditional $(B|A)$: $\kappa \models (B|A)$ iff $\kappa(AB) < \kappa(A\overline{B})$ iff $A \mathrel{|\!\sim}^{\kappa} B$. The acceptance relation is extended as usual to a set $\mathcal{R}$ of conditionals, called a *knowledge base*, by defining $\kappa \models \mathcal{R}$ iff $\kappa \models (B|A)$ for all $(B|A) \in \mathcal{R}$. This is synonymous to saying that κ is *admissible* with respect to $\mathcal{R}$ (Goldszmidt and Pearl 1996), or that κ is a *ranking model* of $\mathcal{R}$. $\mathcal{R}$ is *consistent* iff it has a ranking model.

Among the models of $\mathcal{R}$, c-representations are special models obtained by assigning an individual impact to each conditional and generating the world ranks as the sum of impacts of falsified conditionals. C-inference is an inference relation taking all c-representations of $\mathcal{R}$ into account.

Definition 1 (c-representation (Kern-Isberner 2001; 2004))**.** *A* c-representation *of a knowledge base $\mathcal{R}$ is a ranking function $\kappa_{\vec{\eta}}$ constructed from $\vec{\eta} = (\eta_1, \ldots, \eta_n)$ with integer impacts $\eta_i \in \mathbb{N}_0, i \in \{1, \ldots, n\}$ assigned to each conditional $(B_i|A_i)$ such that κ accepts $\mathcal{R}$ and is given by:*

$$\kappa_{\vec{\eta}}(\omega) = \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i\overline{B}_i}} \eta_i \tag{2}$$

We will denote the set of all c-representations of $\mathcal{R}$ by $\mathcal{O}(CR(\mathcal{R}))$.

Definition 2 (c-inference, $\mathrel{|\!\sim}^{c}_{\mathcal{R}}$ (Beierle, Eichhorn, and Kern-Isberner 2016))**.** *Let $\mathcal{R}$ be a knowledge base and let A, B be formulas. B is* a (skeptical) c-inference from A in the context of $\mathcal{R}$, *denoted by $A \mathrel{|\!\sim}^{c}_{\mathcal{R}} B$, iff $A \mathrel{|\!\sim}^{\kappa} B$ holds for all c-representations κ for $\mathcal{R}$.*

In (Beierle, Eichhorn, and Kern-Isberner 2016), a modeling of c-representations as solutions of a constraint satisfaction problem $CR(\mathcal{R})$ is given and shown to be correct and complete with respect to the set of all c-representations of $\mathcal{R}$.

Definition 3 (***CR(R)*** (Beierle, Eichhorn, and Kern-Isberner 2013))**.** *Let $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\}$. The constraint satisfaction problem for c-representations of $\mathcal{R}$, denoted by $CR(\mathcal{R})$, on the constraint variables $\{\eta_1, \ldots, \eta_n\}$ ranging over $\mathbb{N}_0$ is given by the conjunction of the constraints, for all $i \in \{1, \ldots, n\}$:*

$$\eta_i \geqslant 0 \tag{3}$$

$$\eta_i > \min_{\omega \models A_iB_i} \sum_{\substack{j \neq i \\ \omega \models A_j\overline{B_j}}} \eta_j - \min_{\omega \models A_i\overline{B}_i} \sum_{\substack{j \neq i \\ \omega \models A_j\overline{B_j}}} \eta_j \tag{4}$$

A solution of $CR(\mathcal{R})$ is an n-tuple $(\eta_1, \ldots, \eta_n) \in \mathbb{N}_0^n$. For a constraint satisfaction problem CSP, the set of solutions is denoted by $Sol(CSP)$. Thus, with $Sol(CR(\mathcal{R}))$ we denote the set of all solutions of $CR(\mathcal{R})$. Let us recall the soundness and completeness of constructing c-representations by integer impacts from solutions of $CR(\mathcal{R})$.

Proposition 4 ((Beierle, Eichhorn, and Kern-Isberner 2016; Beierle et al. 2018))**.** *Let $\mathcal{R} = \{(B_i|A_i), i = 1, \ldots, n\}$ be a knowledge base. Then we have*

$$\mathcal{O}(CR(\mathcal{R})) = \{\kappa_{\vec{\eta}} \mid \vec{\eta} \in Sol(CR(\mathcal{R}))\} \tag{5}$$

where $\kappa_{\vec{\eta}}$ is defined as in (2).

3 Resource bounded c-inference

If a knowledge base $\mathcal{R}$ is consistent, there are in general infinitely many c-representations accepting $\mathcal{R}$, including inferentially equivalent ones.

Definition 5 ($\equiv_{|\sim}$)**.** *Two ranking functions κ, κ' are* inferentially equivalent*, denoted by $\kappa \equiv_{|\sim} \kappa'$ iff for all $(B|A)$ it is the case that $\kappa \models (B|A)$ iff $\kappa' \models (B|A)$.*

For instance, if there is a $k \in \mathbb{N}$ such that $\kappa'(\omega) = k \cdot \kappa(\omega)$ for all worlds ω, then $\kappa \equiv_{|\sim} \kappa'$; in general, two ranking functions are inferentially equivalent iff they induce the same total preorder on worlds.

Proposition 6 ((Beierle et al. 2018))**.** *For ranking functions κ and κ', we have $\kappa \equiv_{|\sim} \kappa'$ iff for all $\omega_1, \omega_2 \in \Omega$ it is the case that $\kappa(\omega_1) \leqslant \kappa(\omega_2)$ iff $\kappa'(\omega_1) \leqslant \kappa'(\omega_2)$.*

For a set $\mathcal{O}$ of OCFs, $\mathcal{O}_{/\equiv_{|\sim}}$ denotes the set of induced equivalence classes. Recently, it has been suggested to take inferential equivalence of c-representations into account and to sharpen $CR(\mathcal{R})$ by introducing an upper bound for the impact values η_i.

Definition 7 (***CR^u(R)*** (Beierle et al. 2018))**.** *Let $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ and $u \in \mathbb{N}$. The finite domain constraint satisfaction problem $CR^u(\mathcal{R})$ on the constraint* *variables* $\{\eta_1, \ldots, \eta_n\}$ *ranging over* $\mathbb{N}_0$ *is given by the conjunction of the constraints, for all* $i \in \{1, \ldots, n\}$*:*

$$\eta_i \geqslant 0 \tag{6}$$

$$\eta_i > \min_{\omega \models A_i B_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B_j}}} \eta_j - \min_{\omega \models A_i \overline{B}_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B_j}}} \eta_j \tag{7}$$

$$\eta_i \leqslant u \tag{8}$$

A solution of $CR^u(\mathcal{R})$ is an n-tuple $(\eta_1, \ldots, \eta_n) \in \mathbb{N}_0^n$, its set of solutions is denoted by $Sol(CR^u(\mathcal{R}))$. For $\overrightarrow{\eta} \in Sol(CR^u(\mathcal{R}))$ and κ as in equation (2), κ is the ***OCF induced by*** $\overrightarrow{\eta}$, denoted by $\kappa_{\overrightarrow{\eta}}$, and the set of all induced OCFs is denoted by $\mathcal{O}(CR^u(\mathcal{R})) = \{\kappa_{\overrightarrow{\eta}} \mid \overrightarrow{\eta} \in Sol(CR^u(\mathcal{R}))\}$.

C-inference defined with respect to a maximal impact value can be viewed as a kind of resource-bounded inference operation.

Definition 8 (c-inference under maximal impact value, $\mathrel{|\!\!\sim}^{c,u}_{\mathcal{R}}$ (Beierle et al. 2018))**.** *Let* $\mathcal{R}$ *be a knowledge base,* $u \in \mathbb{N}$*, and let* A*,* B *be formulas.* B *is* a (skeptical) c-inference from A in the context of $\mathcal{R}$ under maximal impact value u*, denoted by* $A \mathrel{|\!\!\sim}^{c,u}_{\mathcal{R}} B$*, iff* $A \mathrel{|\!\!\sim}^{\kappa} B$ *holds for all c-representations* κ *with* $\kappa \in \mathcal{O}(CR^u(\mathcal{R}))$.

The following definition introduces a criterion for a maximal impact value ensuring that $\mathrel{|\!\!\sim}^{c,u}_{\mathcal{R}}$ fully realizes skeptical c-inference. For an OCF κ, the definition uses the total preorder $\preccurlyeq_\kappa$ on worlds given by $\omega_1 \preccurlyeq_\kappa \omega_2$ iff $\kappa(\omega_1) \leqslant \kappa(\omega_2)$.

Definition 9 (regular, minimally regular (Beierle et al. 2018; Beierle and Kutsch 2017))**.** *For* $\mathcal{R}$ *let* $\hat{u} \in \mathbb{N}$ *be the smallest number such that* $|\{\preccurlyeq_\kappa \mid \kappa \in \mathcal{O}(CR^{\hat{u}}(\mathcal{R}))\}| = |\{\preccurlyeq_\kappa \mid \kappa \in \mathcal{O}(CR(\mathcal{R}))\}|$*. Then* $CR^u(\mathcal{R})$ *is called* regular *iff* $u \geqslant \hat{u}$*, and* $CR^{\hat{u}}(\mathcal{R})$ *is* minimally regular*; we also say that* u *is* regular for $\mathcal{R}$ *and* $\hat{u}$ *is* minimally regular for $\mathcal{R}$.

While $CR(\mathcal{R})$ correctly and completely models the set of all c-representations for $\mathcal{R}$ (Beierle, Eichhorn, and Kern-Isberner 2016), every regular $CR^u(\mathcal{R})$ is correct and complete when taking inferential equivalence into account (Beierle et al. 2018). Thus, for regular u, $\mathrel{|\!\!\sim}^{c}_{\mathcal{R}}$ and $\mathrel{|\!\!\sim}^{c,u}_{\mathcal{R}}$ coincide.

Proposition 10 ((Beierle et al. 2018))**.** *Let* $\mathcal{R}$ *be a knowledge base,* $CR^u(\mathcal{R})$ *regular, and* A*,* B *be formulas. Then* $A \mathrel{|\!\!\sim}^{c}_{\mathcal{R}} B$ *iff* $A \mathrel{|\!\!\sim}^{c,u}_{\mathcal{R}} B$*.*

When we are not interested in capturing all c-representations as done by a regular $CR^u(\mathcal{R})$, but aim at capturing c-inference instead, we can specify a maximal impact value from this perspective in order to obtain a finite domain CSP.

Definition 11 (sufficient, minimally sufficient (Beierle et al. 2018; Beierle and Kutsch 2017))**.** *Let* $\mathcal{R}$ *be a knowledge base and let* $u \in \mathbb{N}$*. Then* $CR^u(\mathcal{R})$ *is called* sufficient *iff for all formulas* A*,* B *we have*

$$A \mathrel{|\!\!\sim}^{c}_{\mathcal{R}} B \quad \textit{iff} \quad A \mathrel{|\!\!\sim}^{c,u}_{\mathcal{R}} B. \tag{9}$$

If $CR^u(\mathcal{R})$ *is sufficient, we will also call* u sufficient for $\mathcal{R}$*. If* $\hat{u}$ *is sufficient for* $\mathcal{R}$ *and* $\hat{u} - 1$ *is not sufficient for* $\mathcal{R}$*, then* $\hat{u}$ *is* minimally sufficient for $\mathcal{R}$.

The condition that $CR^l(\mathcal{R})$ is regular is only a sufficient condition for (9) but not necessary, see (Beierle and Kutsch 2019, Proposition 5). Let us introduce the following concept of a minimal solution.

Definition 12 (minimal solution)**.** *Let* $\mathcal{R}$ *be a knowledge base and let* $\overrightarrow{\eta} = (\eta_1, \ldots, \eta_n)$ *be a solution to the constraint satisfaction problem* $CR(\mathcal{R})$*. Then* $\overrightarrow{\eta}$ *is called* minimal solution *to* $CR(\mathcal{R})$ *if for every solution* $(\eta'_1, \ldots, \eta'_n)$ *to* $CR(\mathcal{R})$ *we have* $\eta_i \leqslant \eta'_i$ *for all* $i \in \{1, \ldots, n\}$*.*

It follows immediately from the definition that a minimal solution to $CR(\mathcal{R})$, if such a solution exists, is uniquely determined. Further a minimal solution in the sense of Definition 12 is also cw-minimal, ind-minimal and sum-minimal in the sense of (Beierle, Eichhorn, and Kutsch 2017).

4 A criterion such that $CR^{n-1}(\mathcal{R})$ is sufficient

The scope of this section is to prove for a knowledge base $\mathcal{R} = \{r_1, \ldots, r_n\}$ with a world $\omega \in \Omega$ verifying all conditionals from $\mathcal{R}$ we have that $CR^{n-1}(\mathcal{R})$ is sufficient.

Proposition 13. *Let* $n \in \mathbb{N}, n > 1$*, and let* $\mathcal{R} = \{(B_i|A_i), i = 1, \ldots, n\}$ *be a knowledge base. We assume that there exists* $\omega \in \Omega$ *with* $\omega \models A_i B_i$ *for all* $i \in \{1, \ldots n\}$*. Then the knowledge base is consistent and* $CR^{n-1}(\mathcal{R})$ *is sufficient.*

Proof. Choose $\widetilde{\omega} \in \Omega$ accepting all conditionals from $\mathcal{R}$. Then

$$\min_{\omega \models A_i B_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B_j}}} \eta_j - \min_{\omega \models A_i \overline{B_i}} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B_j}}} \eta_j = - \min_{\omega \models A_i \overline{B_i}} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B_j}}} \eta_j$$

for all $i \in \{1, \ldots, n\}$. Thus, the constraint (4) reduces to

$$\eta_i > - \min_{\omega \models A_i \overline{B_i}} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B_j}}} \eta_j \tag{10}$$

for all $i \in \{1, \ldots, n\}$.

The implication " $\Longrightarrow$ " from (9) is obvious since $\mathcal{O}(CR^{n-1}(\mathcal{R})) \subseteq \mathcal{O}(CR(\mathcal{R}))$. For the proof of the other implication " $\Longleftarrow$ " fix formulas A, B such that

$$A \mathrel{|\!\!\sim}^{c,n-1}_{\mathcal{R}} B. \tag{11}$$

We have to show $A \mathrel{|\!\!\sim}^{c}_{\mathcal{R}} B$. Due to Proposition 4 this requires $\kappa_{\overrightarrow{\eta}}(AB) < \kappa_{\overrightarrow{\eta}}(A\overline{B})$ for all $\kappa_{\overrightarrow{\eta}}$ (defined in (2)) where $\overrightarrow{\eta} = (\eta_1, \ldots, \eta_n) \in Sol(CR(\mathcal{R}))$. That is, in turn, equivalent to

$$\forall \omega^0 \in \Omega_{A\overline{B}}\ \exists \omega^1 \in \Omega_{AB} \text{ with } \kappa_{\overrightarrow{\eta}}(\omega^1) < \kappa_{\overrightarrow{\eta}}(\omega^0). \tag{12}$$

Fix any c-representation $\kappa_{\overrightarrow{\eta}} \in \mathcal{O}(CR(\mathcal{R}))$ with $\overrightarrow{\eta} = (\eta_1, \ldots, \eta_n)$ and $\eta_i \geqslant 0, i \in \{1, \ldots, n\}$. Further, fix $\omega^0 \in \Omega_{A\overline{B}}$. Let us define the set of all indices such that $\eta_i > 0$ and the corresponding conditional is falsified by ω^0 as

$$J := \{i \in \{1, \ldots, n\}; \eta_i > 0 \text{ and } \omega^0 \models A_i\overline{B_i}\}. \tag{13}$$

Our goal is to construct $\omega^1 \in \Omega_{AB}$ such that

$$\Big\{i \in \{1\,,\dots\,,n\}\,;\eta_i > 0 \text{ and } \omega^1 \models A_i\overline{B_i}\Big\} \subsetneq \Big\{i \in \{1\,,\dots\,,n\}\,;\eta_i > 0 \text{ and } \omega^0 \models A_i\overline{B_i}\Big\}\,. \tag{14}$$

Indeed, assume that (14) is proven. From (14) we get immediately

$$\kappa_{\vec{\eta}}(\omega^1) = \sum_{\substack{i\in\{1\,,\dots\,,n\}\\ \omega^1\models A_i\overline{B_i}}} \eta_i < \sum_{\substack{i\in\{1\,,\dots\,,n\}\\ \omega^0\models A_i\overline{B_i}}} \eta_i = \kappa_{\vec{\eta}}(\omega^0)$$

implying that (12) is fulfilled. Altogether, to finish the proof we have to show the existence of $\omega^1 \in \Omega_{AB}$ with (14). We distinguish the cases as follows.

Case (i). First, let us consider the case $|J| = n$. Choose arbitrary $\eta'_i \in \{1\,,\dots\,,n-1\}$ for $i \in \{1\,,\dots\,,n\}$. Since $\eta'_i > 0$ it follows that (10) holds (which is equivalent to (4)) and so $\vec{\eta}'$ fulfils (3), (4). Due to Proposition 4 it follows $\kappa_{\vec{\eta}'} \in \mathcal{O}(CR^{n-1}(\mathcal{R}))$. From (11) we get $\omega^1 \in \Omega_{AB}$ with $\kappa_{\vec{\eta}'}(\omega^1) < \kappa_{\vec{\eta}'}(\omega^0)$. Thus $\sum_{\substack{i\in\{1\,,\dots\,,n\}\\ \omega^1\models A_i\overline{B_i}}} \eta'_i < \sum_{i=1}^{n} \eta'_i$. However, this yields

$$\big\{i \in \{1\,,\dots\,,n\}\,;\,\omega^1 \models A_i\overline{B_i}\big\} \subsetneq \{1\,,\dots\,,n\}\,.$$

Consequently (14) holds.

Case (ii). Let us consider the case $|J| < n$. We define $\vec{\eta}' = (\eta'_1\,,\dots\,,\eta'_n)$ by

$$\eta'_i := \begin{cases} 0\,, & \text{if } i \in \{1\,,\dots n\} \text{ with } \eta_i = 0\,,\\ 1\,, & \text{if } i \in J\,,\\ n-1\,, & \text{otherwise}\,. \end{cases} \tag{15}$$

Since $\eta'_i \geqslant 0\,, i \in \{1\,,\dots\,,n\}$ it remains to prove the constraint (4) which we know is equivalent to (10). If $i \in \{1\,,\dots\,,n\}$ such that $\eta'_i > 0$ then obviously (10) holds. Therefore let us consider $i \in \{1\,,\dots\,,n\}$ such that $\eta'_i = 0$. Since $\eta_i = 0$ we know that (10) holds. Thus

$$0 > -\min_{\omega\models A_i\overline{B_i}} \sum_{\substack{j\neq i\\ \omega\models A_j\overline{B_j}}} \eta_j$$

and so $\min_{\omega\models A_i\overline{B_i}} \sum_{\substack{j\neq i\\ \omega\models A_j\overline{B_j}}} \eta_j \in \mathbb{N}^\infty$ (i.e. $\neq 0$). By (15) we have $\eta_j > 0$ implies $\eta'_j > 0$ and so $\min_{\omega\models A_i\overline{B_i}} \sum_{\substack{j\neq i\\ \omega\models A_j\overline{B_j}}} \eta'_j \in \mathbb{N}^\infty$. Consequently (4) is also satisfied for $\eta'_i = 0$. Making use of Proposition 4 it follows that $\kappa_{\vec{\eta}'} \in \mathcal{O}(CR^{n-1}(\mathcal{R}))$. By (11) we know $A \vdash^{\kappa_{\vec{\eta}'}} B$. Therefore, there is $\omega^1 \in \Omega_{AB}$ with $\kappa_{\vec{\eta}'}(\omega^1) < \kappa_{\vec{\eta}'}(\omega^0)$. Thus

$$\sum_{\substack{i\in\{1\,,\dots\,,n\}\\ \omega^1\models A_i\overline{B_i}}} \eta'_i < \sum_{\substack{i\in\{1\,,\dots\,,n\}\\ \omega^0\models A_i\overline{B_i}}} \eta'_i\,. \tag{16}$$

By definition of J and $\vec{\eta}'$, see (13), (15), it holds for all $i \in \{1\,,\dots\,,n\}$ with $\eta_i > 0$ and $\omega^0 \models A_i\overline{B_i}$ that $\eta'_i = 1$. Therefore $\sum_{\substack{i\in\{1\,,\dots\,,n\}\\ \omega^0\models A_i\overline{B_i}}} \eta'_i = |J|$. By (16) we get

$$\sum_{\substack{i\in\{1\,,\dots\,,n\}\\ \omega^1\models A_i\overline{B_i}}} \eta'_i < |J| \leqslant n-1. \tag{17}$$

We finish the proof of (14) by contradiction. Suppose that (14) is wrong. Due to (16) the two sets in (14) can not be identical. Therefore, there must be an $k \in \{1\,,\dots\,,n\}$ such that $\eta_k > 0$ and $\omega^1 \models A_k\overline{B_k}$ but not $k \in J$. Since $k \notin J$, by (15), there holds $\eta'_k = n-1$. But then

$$\sum_{\substack{i\in\{1\,,\dots\,,n\}\\ \omega^1\models A_i\overline{B_i}}} \eta'_i = n-1 + \sum_{\substack{i\neq k\\ \omega^1\models A_i\overline{B_i}}} \eta'_i \geqslant n-1\,,$$

contradicting (17). Altogether the existence of ω^1 with (14) is also proven in the case (ii). The proof is complete. ☐

Remark. *In (Beierle and Kutsch 2019) the knowledge base $\mathcal{R}_n = \{(a_1|\top)\,,\dots\,,(a_n|\top)\}$ of n conditionals facts over $\Sigma_n = \{a_1\,,\dots\,,a_n\}$ is introduced and it is proven that $CR^{n-1}(\mathcal{R}_n)$ is sufficient for $n > 1$. Since $\omega = (a_1\,,\dots\,,a_n)$ accepts all conditionals from $\mathcal{R}_n$, our more general Proposition 13 yields the same conclusion in that case but makes no use of the special structure of the conditionals from $\mathcal{R}_n$.*

5 Existence of a knowledge base such that 2^{n-1} is minimally sufficient and minimally regular

In this section we deal with the construction of a knowledge base $\mathcal{R} = \{(B_i|A_i)\,, i = 1\,,\dots\,,n\}$ where 2^{n-1} is minimally sufficient and minimally regular. Further we will clarify the construction and present an explicit knowledge base for $n = 5$.

Let $\Omega = \{\omega_i\,;\, i = 1\,,\dots\,,m\}$. For conditionals $(B_j|A_j)_{j=1\,,\dots\,,n}$ a matrix $(m_{i,j})$ with $m_{i,j} \in \{v\,,f\,,-\}$ describing the evaluation according to (1) can be defined by

$$m_{i,j} = [[(B_j|A_j)]]_{\omega_i}\,. \tag{18}$$

In (18) $m_{i,j} = v$ means that ω_i verifies $(B_j|A_j)$, the meaning of $m_{i,j} = f$ is that ω_i falsifies $(B_j|A_j)$ and we write $m_{i,j} = -$ if $\omega_i \models \overline{A_j}$.

In the following proposition we tackle the "inverse problem": For a given evaluation matrix $(m_{i,j})_{i=1,\dots,m\,;\,j=1\,,\dots\,,n}$ we construct a (not necessarily consistent) knowledge base $\mathcal{R} = \{(B_j|A_j)\,, j = 1,\dots,n\}$ such that the evaluation is just given by (18).

Proposition 14. *Let $n\,,m \in \mathbb{N}$, let $\Sigma = \{v_1,\dots,v_m\}$ be a propositional alphabet and let $\Omega = \{\omega_1,\dots,\omega_{2^m}\}$. For all $i \in \{1,\dots,2^m\}$ and $j \in \{1,\dots,n\}$ let $m_{i,j} \in \{v\,,f\,,/\}$ be*

worlds	r_1	r_2	r_3	$\dots$	r_n
ω_1	$m_{1,1}$	$m_{1,2}$	$m_{1,3}$	$\dots$	$m_{1,n}$
ω_2	$m_{2,1}$	$m_{2,2}$	$m_{2,3}$	$\dots$	$m_{2,n}$
ω_3	$m_{3,1}$	$m_{3,2}$	$m_{3,3}$	$\dots$	$m_{3,n}$
$\dots$					
ω_{2^m}	$m_{2^m,1}$	$m_{2^m,2}$	$m_{2^m,3}$	$\dots$	$m_{2^m,n}$

Table 1: Evaluation tableau

given. Then there exists a (not necessarily consistent) knowledge base $\mathcal{R} = \{(B_j|A_j), j = 1, \dots, n\}$ such that the following holds:

$$\text{If } m_{i,j} = v \text{ then } \omega_i \models A_j B_j\,, \tag{19}$$

$$\text{If } m_{i,j} = f \text{ then } \omega_i \models A_j \overline{B_j}\,, \tag{20}$$

$$\text{If } m_{i,j} = - \text{ then } \omega_i \models \overline{A_j}\,. \tag{21}$$

Proof. We have to construct $r_j = (B_j|A_j)$ such that $m_{i,j} = [[(B_i|A_i)]]_{\omega_i}$. To do so we define the formulas

$$A_j := \bigvee_{\{\omega_i \in \Omega;\, m_{i,j} \in \{+,-\}\}} \omega_i\,, \tag{22}$$

$$B_j := \bigvee_{\{\omega_i \in \Omega;\, m_{i,j} \in \{+\}\}} \omega_i \tag{23}$$

for all $j \in \{1, \dots, n\}$. Due to construction we see that (19), (20), (21) hold. □

Proposition 14 can be summarized as follows: Every evaluation table (see Table 1), described by an evaluation matrix $(m_{i,j})_{i=1,\dots,m\,;\,j=1,\dots,n}$, can be generated by a knowledge base $\mathcal{R} = \{(B_j|A_j), j = 1, \dots, n\}$.

Proposition 15. *There exists a consistent knowledge base $\mathcal{R} = \{(B_i|A_i), i = 1, \dots, n\}$, such that the constraint satisfaction problem $CR(\mathcal{R})$ is given by the conjunction of the constraints*

$$\eta_i > \sum_{j=1}^{i-1} \eta_j \tag{24}$$

for all $i \in \{1, \dots, n\}$. (If $i = 1$ then $\sum_{j=1}^{i-1} \eta_j = 0$ and so (24) means $\eta_1 > 0$.) The constraint satisfaction problem (24) has the minimal solution

$$\vec{\eta} = (1, 2, 4, 8, \dots, 2^{n-1})\,. \tag{25}$$

Proof. Consider disjoint subsets $\Omega_+, \Omega_- \subseteq \Omega$ where $|\Omega_+| = n$ and $|\Omega_-| = n$. Let us write $\Omega = \Omega_+ \cup \Omega_- \cup \Omega_{\text{rest}}$ where

$$\Omega_- = \{\omega_1^-, \omega_2^-, \dots, \omega_n^-\}\,,$$
$$\Omega_+ = \{\omega_1^+, \omega_2^+, \dots, \omega_n^+\}\,,$$
$$\Omega_{\text{rest}} := \Omega \setminus (\Omega_- \cup \Omega_+)\,.$$

Looking at Proposition 14 there exists a knowledge base $\mathcal{R} = \{r_i = (B_i|A_i), i = 1, \dots, n\}$ fulfilling the following evaluation tableau:

worlds	r_1	r_2	r_3	$\dots$	r_{n-1}	r_n
ω_1^+	v	$-$	$-$	$\dots$	$-$	$-$
ω_2^+	f	v	$-$	$\dots$	$-$	$-$
ω_3^+	f	f	v	$\dots$	$-$	$-$
$\dots$						
ω_{n-1}^+	f	f	f	$\dots$	v	$-$
ω_n^+	f	f	f	$\dots$	f	v
ω_1^-	f	$-$	$-$	$\dots$	$-$	$-$
ω_2^-	$-$	f	$-$	$\dots$	$-$	$-$
ω_3^-	$-$	$-$	f	$\dots$	$-$	$-$
$\dots$						
ω_{n-1}^-	$-$	$-$	$-$	$\dots$	f	$-$
ω_n^-	$-$	$-$	$-$	$\dots$	$-$	f
all other worlds	$-$	$-$	$-$	$\dots$	$-$	$-$

By (22), (23) in the proof of Proposition 14 it follows that we can choose the knowledge base $\mathcal{R} = \{r_i = (B_i|A_i), i = 1, \dots, n\}$ in the following way

$$A_i := \bigvee_{\{\omega \in \Omega;\, [[r_i]]_\omega \in \{+,-\}\}} \omega = \omega_i^- \vee \omega_i^+ \vee \dots \vee \omega_n^+\,,$$

$$B_i := \bigvee_{\{\omega \in \Omega;\, [[r_i]]_\omega \in \{+\}\}} \omega = \omega_i^+$$

for all $i \in \{1, \dots, n\}$. Consequently we obtain

$$r_i = \left(\omega_i^+ \mid \omega_i^- \vee \omega_i^+ \vee \dots \vee \omega_n^+\right).$$

Fix $i \in \{1, \dots, n\}$. Due to construction

$$\min_{\omega \models A_i B_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B_j}}} \eta_j - \min_{\omega \models A_i \overline{B}_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B_j}}} \eta_j = \sum_{j=1}^{i-1} \eta_j$$

for all $i \in \{1, \dots, n\}$. Consequently (4) transforms to (24) and so $CR(\mathcal{R})$ is given by (24) for all $i \in \{1, \dots, n\}$. (Inequality (24) implies (3).) It follows immediately that there exists a minimal solution $\vec{\eta} = (\eta_1, \dots, \eta_n)$ to $CR(\mathcal{R})$ satisfying

$$\eta_i = 1 + \sum_{j=1}^{i-1} \eta_j$$

for all $i \in \{1, \dots, n\}$. The proof of the representation (25) is by induction. For $i = 1$ we have $\eta_1 = 1 = 2^{1-1}$. The proof of the inductive step follows from

$$\eta_i = 1 + \sum_{j=1}^{i-1} \eta_j = 1 + \sum_{j=0}^{i-2} 2^j = 1 + \frac{1 - 2^{i-1}}{1 - 2} = 2^{i-1}\,.$$

Consequently (25) holds. □

The following remarkable lemma states that all c-representations of $\mathcal{R}$ are inferentially equivalent.

Lemma 1. *Let $\mathcal{R} = \{(B_i|A_i), i = 1, \dots, n\}$ denote a knowledge base with constraint satisfaction problem $CR(\mathcal{R})$ given by (25). Then every c-representation $\kappa \in \mathcal{O}(CR(\mathcal{R}))$ is inferentially equivalent to $\kappa_{\vec{\eta}}$ (defined in (2)) where $\vec{\eta}$ is the minimal solution to $CR(\mathcal{R})$ given by $\vec{\eta} = (1, 2, 4, 8, \dots, 2^{n-1})$.*

Proof. Let $\kappa \in \mathcal{O}(CR(\mathcal{R}))$ denote an arbitrary c-representation and let $\omega_1, \omega_2 \in \Omega$. Let us define

$$J_1 := \{i \in \{1, \dots, n\}; \omega_1 \models A_i\overline{B_i}\},$$
$$J_2 := \{i \in \{1, \dots, n\}; \omega_2 \models A_i\overline{B_i}\}.$$

Let us write $\max M = 0$ if $M \subseteq \{1, \dots, n\}$ with $M = \varnothing$.

Assertion. We have

$$\kappa(\omega_1) > \kappa(\omega_2) \iff \max(J_1 \setminus J_2) > \max(J_2 \setminus J_1), \tag{26}$$

$$\kappa(\omega_1) = \kappa(\omega_2) \iff J_1 = J_2. \tag{27}$$

Proof of the assertion. Due to Proposition 4 we have $\kappa = \kappa_{\vec{\eta}'}$ with $\vec{\eta}' = (\eta'_1, \dots, \eta'_n) \in Sol(CR(\mathcal{R}))$. Since (26) implies (27) it remains to show (26). We obtain

$$\kappa(\omega_1) - \kappa(\omega_2) = \sum_{j \in J_1} \eta_j - \sum_{j \in J_2} \eta'_j = \sum_{j \in J_1 \setminus J_2} \eta'_j - \sum_{j \in J_2 \setminus J_1} \eta'_j.$$

Define $q_1 := \max J_1$ and $q_2 := \max J_2$. Assume $\max(J_1 \setminus J_2) > \max(J_2 \setminus J_1)$. Then $q_1 > q_2$ and it follows

$$\kappa(\omega_1) - \kappa(\omega_2) = \sum_{j \in J_1 \setminus J_2} \eta_j - \sum_{j \in J_2 \setminus J_1} \eta_j \geqslant \eta_{q_1} - \sum_{j \in \{1, \dots, q_1 - 1\}} \eta_j > 0$$

due to the structure of $CR(\mathcal{R})$. On the other hand if

$$\sum_{j \in J_1 \setminus J_2} \eta'_j - \sum_{j \in J_2 \setminus J_1} \eta'_j = \kappa(\omega_1) - \kappa(\omega_2) > 0$$

it follows due to the structure of $CR(\mathcal{R})$ that $\max(J_1 \setminus J_2) > \max(J_2 \setminus J_1)$. The proof of the assertion is complete.

Let $\kappa, \kappa' \in \mathcal{O}(CR(\mathcal{R}))$ be c-representations. Making use of the proven assertion we see that

$$\kappa(\omega_1) \leqslant \kappa(\omega_2) \iff \kappa'(\omega_1) \leqslant \kappa'(\omega_2)$$

for all $\omega_1, \omega_2 \in \Omega$. Due to Proposition 6 we get that κ and κ' are inferentially equivalent. The claim follows. □

Now we have all ingredients at hand to prove the main result of this section

Proposition 16. *For every $n \in \mathbb{N}$ there exists a consistent knowledge base $\mathcal{R} = \{(B_i|A_i), i = 1, \dots, n\}$ such that 2^{n-1} is minimally sufficient and minimally regular.*

Proof. Let $\mathcal{R}$ be the knowledge base whose existence is proven in Proposition 15. Due to Lemma 1 we know that $CR^{2^{n-1}}(\mathcal{R})$ is regular and, see Proposition 10, also sufficient. By (25) we have $\vec{\eta} = (1, 2, 4, 8, \dots, 2^{n-1})$ for the minimal solution of $\mathcal{R}$. Since $\eta_n = 2^{n-1}$, by the definition of a minimal solution, $Sol(CR^l(\mathcal{R}))) = \varnothing$ if $l < 2^{n-1}$. For a consistent knowledge base a regular $CR^l(\mathcal{R})$ necessarily has a non empty set of solutions $Sol(CR^l(\mathcal{R})))$. Therefore $CR^l(\mathcal{R})$ is not regular and, by Proposition 10, also not sufficient see for $l < 2^{n-1}$. Altogether 2^{n-1} is minimally sufficient and minimally regular for $\mathcal{R}$. □

Example 17. *In this example (see Proposition 16 for $n = 5$) we want to clarify and explain the construction of a knowledge base $\mathcal{R} = \{r_i = (B_i|A_i), i = 1, \dots, 5\}$ with $n = 5$ conditionals such that $2^4 = 16$ is minimally sufficient and minimally regular for $\mathcal{R}$. Looking at the proof of Proposition 16 our goal is to construct $\mathcal{R}$ such that $CR(\mathcal{R})$ is given by:*

$\eta_1 > 0$ $\qquad \eta_4 > \eta_1 + \eta_2 + \eta_3$
$\eta_2 > \eta_1$ $\qquad \eta_5 > \eta_1 + \eta_2 + \eta_3 + \eta_4$
$\eta_3 > \eta_1 + \eta_2$

An inspection of the proof of Proposition 15 yields that $\mathcal{R}$ can be constructed such that the following evaluation tableau is fulfilled:

worlds	r_1	r_2	r_3	r_4	r_5
$\overline{a}bcde$	v	–	–	–	–
$a\overline{b}cde$	f	v	–	–	–
$ab\overline{c}de$	f	f	v	–	–
$abc\overline{d}e$	f	f	f	v	–
$abcd\overline{e}$	f	f	f	f	v
$a\overline{b}\overline{c}\overline{d}\overline{e}$	f	–	–	–	–
$\overline{a}b\overline{c}\overline{d}\overline{e}$	–	f	–	–	–
$\overline{a}\overline{b}c\overline{d}\overline{e}$	–	–	f	–	–
$\overline{a}\overline{b}\overline{c}d\overline{e}$	–	–	–	f	–
$\overline{a}\overline{b}\overline{c}\overline{d}e$	–	–	–	–	f
all other worlds	–	–	–	–	–

Due to (22), (23) we finally arrive at the following "explicit" knowledge base

$$
\begin{aligned}
r_1 &= (\ \overline{a}bcde \mid \overline{a}bcde \vee a\overline{b}\overline{c}\overline{d}\overline{e} \vee a\overline{b}cde \\
&\qquad \vee ab\overline{c}de \vee abc\overline{d}e \vee abcd\overline{e}\ ), \\
r_2 &= (\ a\overline{b}cde \mid a\overline{b}cde \vee \overline{a}b\overline{c}\overline{d}\overline{e} \vee ab\overline{c}de \\
&\qquad \vee abc\overline{d}e \vee abcd\overline{e}\ ), \\
r_3 &= (\ ab\overline{c}de \mid ab\overline{c}de \vee \overline{a}\overline{b}c\overline{d}\overline{e} \\
&\qquad \vee abc\overline{d}e \vee abcd\overline{e}\ ), \\
r_4 &= (\ abc\overline{d}e \mid abc\overline{d}e \vee \overline{a}\overline{b}\overline{c}d\overline{e} \vee abcd\overline{e}\ ), \\
r_5 &= (\ abcd\overline{e} \mid abcd\overline{e} \vee \overline{a}\overline{b}\overline{c}\overline{d}e\ )
\end{aligned}
$$

6 Conclusions and Further Work

We presented a criterion on a knowledge base $\mathcal{R}$ such that using $|\mathcal{R}| - 1$ as an upper bound is sufficient for realizing skeptical c-inference for $\mathcal{R}$ by a finite domain constraint system. Given any verification/ falsification behaviour of conditionals on worlds, we developed a constructive approach yielding a knowledge base realizing this behaviour. Furthermore, and in contrast to the previous conjecture that a maximal impact $u = |\mathcal{R}|$ is sufficient for $\mathcal{R}$, due to the present paper, we know that there is no polynomial bound for $u \in \mathbb{N}$ to be minimally sufficient for all knowledge bases with n conditionals for realizing skeptical c-inference over $\mathcal{R}$. The problem of proving a sufficient upper bound for all knowledge bases remains open and will be addressed in a future work.

References

Adams, E. W. 1975. *The Logic of Conditionals: An Application of Probability to Deductive Logic*. Synthese Library. Springer, Dordrecht.

Beierle, C., and Kutsch, S. 2017. Regular and sufficient bounds of finite domain constraints for skeptical c-inference. In Benferhat S, Tabia K, A. M., ed., *Advances in artificial intelligence: from theory to practice*, volume 10350 of *LNAI*, 477–487. Springer.

Beierle, C., and Kutsch, S. 2019. Computation and comparison of nonmonotonic skeptical inference relations induced by sets of ranking models for the realization of intelligent agents. *Appl Intell* 49(1):28–43.

Beierle, C.; Eichhorn, C.; Kern-Isberner, G.; and Kutsch, S. 2018. Properties of skeptical c-inference for conditional knowledge bases and its realization as a constraint satisfaction problem. *Ann. Math. Artif. Intell.* 83(3-4):247–275.

Beierle, C.; Eichhorn, C.; and Kern-Isberner, G. 2013. A declarative approach for computing ordinal conditional functions using constraint logic programming. In *19th International Conference on Applications of Declarative Programming and Knowledge Management, INAP 2011, and 25th Workshop on Logic Programming, WLP 2011, Wien, Austria, Revised Selected Papers*, volume 7773 of *LNAI*, 175–192. Springer, New York.

Beierle, C.; Eichhorn, C.; and Kern-Isberner, G. 2016. Skeptical inference based on c-representations and its characterization as a constraint satisfaction problem. In *FoIKS-2016*, volume 9616 of *LNCS*, 65–82. Springer.

Beierle, C.; Eichhorn, C.; and Kutsch, S. 2017. A practical comparison of qualitative inferences with preferred ranking models. *KI* 31:41–52.

de Finetti, B. 1937. La prévision, ses lois logiques et ses sources subjectives. *Ann. Inst. H. Poincaré* 7(1):1–68. English translation in *Studies in Subjective Probability*, ed. H. Kyburg and H.E. Smokler, 1974, 93-158. New York: Wiley & Sons.

Dubois, D., and Prade, H. 2015. Possibility theory and its applications: where do we stand? In Kacprzyk, J., and Pedrycz, W., eds., *Springer Handbook of Computational Intelligence*, 31–60. Springer, Heidelberg.

Goldszmidt, M., and Pearl, J. 1991. On the consistency of defeasible databases. *Artif. Intell.* 52(2):121–149.

Goldszmidt, M., and Pearl, J. 1996. Qualitative probabilities for default reasoning, belief revision, and causal modeling. *Artificial Intelligence* 84(1-2):57–112.

Kern-Isberner, G. 2001. *Conditionals in nonmonotonic reasoning and belief revision*, volume 2087 of *LNAI*. Springer.

Kern-Isberner, G. 2004. A thorough axiomatization of a principle of conditional preservation in belief revision. *Ann. Math. Artif. Intell.* 40(1-2):127–164.

Pearl, J. 1990. System Z: a natural ordering of defaults with tractable applications to nonmonotonic reasoning. In *Proceedings of the 3rd Conference on Theoretical Aspects of Reasoning About Knowledge (TARK1990)*, 121–135. Morgan Kaufmann Publishers Inc., San Francisco.

Spohn, W. 1988. Ordinal conditional functions: a dynamic theory of epistemic states. In Harper, W., and Skyrms, B., eds., *Causation in Decision, Belief Change, and Statistics, II*. Kluwer Academic Publishers. 105–134.

Spohn, W. 2012. *The Laws of Belief: Ranking Theory and Its Philosophical Applications*. Oxford University Press, Oxford.