Equivalence classes of Dyck paths modulo some statistics

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Abstract

We investigate new equivalence relations on the set $\mathcal{D}$ of Dyck paths relatively to the three statistics of double rises, peaks and valleys. Two Dyck paths are r-equivalent (respectively p-equivalent and v-equivalent) whenever the positions of their double rises (respectively peaks and valleys) are the same. Then, we provide generating functions for the numbers of r-, p- and v-equivalence classes of $\mathcal{D}$.

Keywords: Dyck path, equivalence relation, statistics, rise, peak, valley, Catalan, Motzkin and Fibonacci sequences.

1 Introduction and notations

A large number of various classes of combinatorial objects are enumerated by the Catalan numbers (A000108 in the on-line encyclopedia of integer sequences [10]). It is the case, among others, for planar trees, Young tableaux, stack sortable permutations, Dyck paths, and so on (see [12]).

A *Dyck path* of semilength n, $n \geq 0$, is a lattice path starting at $(0,0)$, ending at $(2n,0)$, and never going below the x-axis, consisting of up steps $U=(1,1)$ and down steps $D=(1,-1)$. Let $\mathcal{D}$ be the set of all Dyck paths. Any non-empty Dyck path $P \in \mathcal{D}$ has a unique first return decomposition of the form $P = U\alpha D\beta$ where α and β are two Dyck paths (see [1]).

A *double rise* of a Dyck path is an occurrence UU of two consecutive up steps. A *peak* (resp. *valley*, resp. *zigzag*) is an occurrence of UD (resp.

DU, resp. DUD). More generally, a *pattern* consists of consecutive steps of a Dyck path. We will say that a pattern is at position i, $i \geq 1$, in a Dyck path whenever the first step of the pattern appears at the i-th step of the Dyck path. The *height* of a pattern is the minimal ordinate reached by this pattern. For instance, the path $P = UUDUDUUDDD$ contains two double rises at positions 1 and 6, three peaks at positions 2, 4, 7, and one zigzag of height one at position 3.

In the literature, many statistics on Dyck paths have been studied. Almost always, it is shown how we can enumerate Dyck paths according to several parameters, such as length, number of peaks or valleys, number of double rises, number of returns to the x-axis (see for instance [1, 2, 3, 4, 5, 6, 7, 8, 9, 11]). Here, we take a new approach to study these statistics. We define three equivalence relations on $\mathcal{D}$.

Two Dyck paths of the same semilength are r-equivalent (resp. p-equivalent and v-equivalent) whenever the positions of their double rises (resp. peaks and valleys) are the same.

For instance, the path $UD\mathbf{UU}DDUD$ is r-equivalent to $UD\mathbf{UU}DUDD$ since they coincide on the unique double rise at position 3. The paths $U\mathbf{UD}U\mathbf{UD}DD\mathbf{UD}U\mathbf{UD}D$ and $U\mathbf{UD}D\mathbf{UD}UU\mathbf{UD}D\mathbf{UD}D$ are p-equivalent since they coincide on their four peaks at positions 2, 5, 9 and 12.

In this paper, we provide generating functions for the numbers of r-, p- and v-equivalence classes in $\mathcal{D}$, with respect to the semilength (see Table 1). The general method used in Sections 2,3 and 4 consists of exhibiting one-to-one correspondences between some subsets of Dyck paths and the different sets of equivalence classes by using combinatorial reasonings, and then, evaluating algebraically the generating functions for these subsets.

Pattern	Sequence	Sloane	$a_n, 1 \leq n \leq 9$
$\{UU\}, \{DD\}$	$\frac{1-x+\sqrt{1-2x-3x^2}}{1-3x+x^2+x^3+(1-x^2)\sqrt{1-2x-3x^2}}$	New	$1, 2, 4, 9, 22, 56, 147, 393, 1065$
$\{UD\}$	$\frac{1-6x+12x^2-8x^3+x^4}{(1-2x)^2(1-3x+x^2)}$	New	$1, 2, 5, 14, 41, 121, 354, 1021, 2901$
$\{DU\}$	$\frac{1-2x}{1-3x+x^2}$	A001519	$1, 2, 5, 13, 34, 89, 233, 610, 1597$

Table 1: Number of equivalence classes for Dyck paths.

2 Equivalence classes modulo double rises

Throughout this section, we study the r-equivalence in $\mathcal{D}$.

Let $\mathcal{A}$ be the set of Dyck paths where all occurrences of DUD are at height 0 or 1, and where the pattern $DUDD$ does not appear. For instance, the Dyck path $UU\mathbf{DUDD}UDUD$ does not belong to $\mathcal{A}$, while $UUDUDUUDDD \in \mathcal{A}$, and the Dyck paths of semilength three in $\mathcal{A}$ are $UDUDUD$, $UDUUDD$, $UUUDDD$ and $UUDDUD$.

Lemma 1 *There is a bijection between $\mathcal{A}$ and the set of r-equivalence classes of $\mathcal{D}$.*

Proof. Let P be a Dyck path in $\mathcal{D}$. Let us prove that there exists a Dyck path $P' \in \mathcal{A}$ (with the same semilength as P) such that P and P' belong to the same class.

Before describing the construction of P', it is worth to notice the following fact.

If a Dyck path P avoids $DUDD$ then for any zigzag DUD at height $h \geq 2$ there is an occurrence of DDD at height $h-1$ on its right.

Indeed, a zigzag DUD at height $h \geq 2$ is followed by a subpath $Q \in \mathcal{D}$, which is followed by the first D reaching height $h-1$ after this zigzag. Obviously, Q is neither empty nor it ends with UD, since otherwise an occurrence of $DUDD$ would appear. Hence, Q ends with DD, thus forming an occurrence of DDD at height $h-1$. This occurrence DDD will be called *the right abutment* of the zigzag DUD.

Now, we define a sequence of Dyck paths $P_0 = P, P_1, P_2, \ldots, P_{k-1}, P_k = P'$, $k \geq 1$.

For any i, $1 \leq i \leq k$, the Dyck path P_i is obtained from P_{i-1} by performing successively the two following processes (1) and (2):

(1) If $P_{i-1} = a_0 \prod_{j=1}^{k} U^{r_j} a_j$, where $a_0 = (UD)^{\lambda}$, $\lambda \geq 0$, a_j avoids UU, $j \in [k]$ and each r_j is taken to be maximal, then set $P'_{i-1} = a_0 \prod_{j=1}^{k} U^{r_j} D^{\mu_j} (UD)^{\nu_j}$, where ν_j and $\nu_j + \mu_j$ are the number of U's and the number of D's in a_j respectively.

(2) Swap the leftmost zigzag of height at least two in P'_{i-1} with its right abutment, to obtain P_i.

The process finishes because performing (1) and (2) necessarily shifts to the right the position of the leftmost zigzag of height at least two. At the end of the process, the Dyck path P' belongs to $\mathcal{A}$ since it contains neither an occurrence of $DUDD$ nor a zigzag at height of at least two. Moreover, all Dyck paths $P_0 = P, P_1, \ldots, P_k = P'$ belong to the same equivalence class.

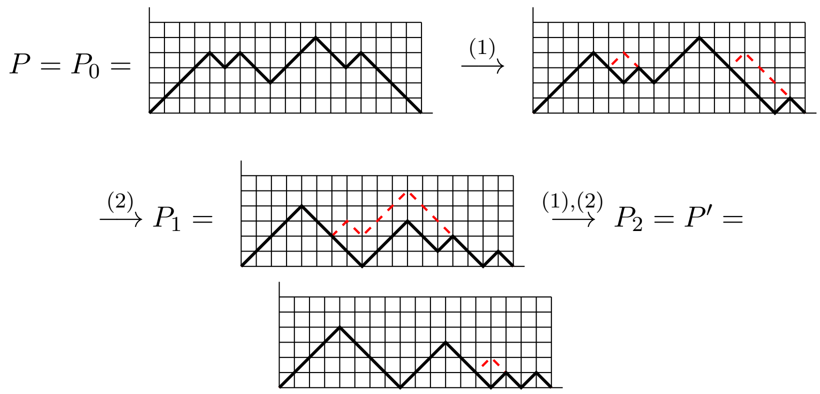


Figure 1: Illustration of the example described in the proof of Lemma 1.

For instance, let us define $P = P_0 = UUUUDUDDUUUDDUDDDD$. Performing (1), we obtain $UUUUDDUDUUUDDDDDUD$ and after (2) we have $P_1 = UUUUDDDDUUUDDUDDUD$. The process continues since there is an occurrence of $DUDD$ in P_1. After (1), we have $UUUUDDDDUUUDDDUDUD$ that does not contain any occurrence of zigzag at height at least two. The process finishes with $P' = P_2 = UUUUDDDDUUUDDDUDUD$. See Figure 1 for an illustration of this example.

Now, it suffices to prove that two different Dyck paths P and P' with the same semilength in $\mathcal{A}$ cannot belong to the same class. Let us assume that P and P' belong to the same class. In the case where P does not contain any double rise, we necessarily have $P = P' = (UD)^n$. Whenever P contains at least one double rise, we decompose P and P' as follows:

$$P = \alpha_0 U^{r_1} \alpha_1 U^{r_2} \alpha_2 \dots U^{r_k} \alpha_k \text{ and } P' = \alpha'_0 U^{r_1} \alpha'_1 U^{r_2} \alpha'_2 \dots U^{r_k} \alpha'_k$$

where $k \geq 1$, α_i, α'_i, $0 \leq i \leq k$, do not contain any double rise and where each $r_i \geq 2$, $1 \leq i \leq k$, is maximal.

Obviously, we necessarily have $\alpha_0 = \alpha'_0 = (UD)^{s_0}$ for some $s_0 \geq 0$. Moreover, for $i \geq 0$, α_i (resp. α'_i) contains neither a double rise nor an occurrence of $DUDD$ which means that α_i (resp. α'_i) is necessarily of the form $\alpha_i = D^{t_i}(UD)^{s_i}$ for some $t_i \geq 1$ and $s_i \geq 0$ (resp. $\alpha'_i = D^{t'_i}(UD)^{s'_i}$ for some $t'_i \geq 1$ and $s'_i \geq 0$).

Since P and P' belong to the same class, we have $t_i + 2s_i = t'_i + 2s'_i$ for all $i \geq 0$. For a contradiction, let us assume that there is $j \geq 0$ such that

$s_j \neq s'_j$ (we choose the smallest $j \geq 0$). Without loss of generality, we set $s_j < s'_j$.

So, the difference between the height h of the first point of α_j (which also is the height of the first point of α'_j) and the height of the last point of α_j (resp. α'_j) is t_j (resp. t'_j). The equality $t_j + 2s_j = t'_j + 2s'_j$ induces that $t_j - t'_j \geq 2$ which implies that α'_j has the height of its last point out of the interval $[0, 1]$. Since P and P' do not have any zigzag at height at least two, we deduce that α'_j does not contain any zigzag, that is $s'_j = 0$. A contradiction is obtained, since $s_j < s'_j$. Finally we necessarily have $\alpha_i = \alpha'_i$ for $1 \leq i \leq k$, and then $P = P'$ which completes the proof. □

Before proving Theorem 1, we give a preliminary result in Lemma 2. Let $\mathcal{B}$ be the set of Dyck paths where all zigzags are at height 0, and without zigzag at the end. For instance, the Dyck paths of semilength three in $\mathcal{B}$ are $UDUUDD$ and $UUUDDD$.

Lemma 2 *The generating function of the set $\mathcal{B}$ with respect to the semilength is given by*

$$\frac{2 - x^2 - x + x\sqrt{1 - 2x - 3x^2}}{1 - x + \sqrt{1 - 2x - 3x^2}}.$$

Proof. Let $B(x)$ be the generating function of the set $\mathcal{B}$ with respect to the semilength. Let P be a non-empty Dyck path of $\mathcal{D}$ and $P = U\alpha D\beta$ its first return decomposition where α and β are two Dyck paths. Then, a non-empty Dyck path P belongs to $\mathcal{B}$ if and only if α avoids DUD, and β belongs to $\mathcal{B}$. It is well known (see for instance [11]) that the generating function for the set of Dyck paths avoiding DUD is given by $1 + xM(x)$, where

$$M(x) = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}$$

is the generating function for the classical Motzkin sequence (A001006 in [10]). Therefore, we deduce the functional equation

$$B(x) = 1 + x \cdot (1 + x \cdot M(x)) \cdot (B(x) - x)$$

where $M(x)$ is the generating function for the Motzkin sequence defined above. A simple calculation provides the expected result. □

Theorem 1 *The generating function for the set of r-equivalence classes in $\mathcal{D}$ (i.e., modulo the positions of the double rises) with respect to the semilength is given by*

$$\frac{1 - x + \sqrt{1 - 2x - 3x^2}}{1 - 3x + x^2 + x^3 + (1 - x^2)\sqrt{1 - 2x - 3x^2}}.$$

Proof. By Lemma 1, it suffices to provide the generating function $A(x)$ for the the set $\mathcal{A}$, with respect to the semilength. Let P be a non-empty Dyck path of $\mathcal{A}$. It has a unique first return decomposition of the form $P = U\alpha D\beta$ where α belongs to $\mathcal{B}$ and β belongs to $\mathcal{A}$. This induces the functional equation $A(x) = 1 + x \cdot B(x) \cdot A(x)$ where $B(x)$ is the generating function for the set $\mathcal{B}$ (see Lemma 2 for the calculation of $B(x)$), giving the required result. □

3 Equivalence classes modulo peaks

Throughout this section, we study the p-equivalence in $\mathcal{D}$.

Let $\mathcal{E}$ be the set of Dyck paths such that there is no peak UD both on the right of an occurrence of DD and on the left of an occurrence of UU. For instance, $UU\mathbf{DD}U\mathbf{UD}DU\mathbf{UU}DDD$ does not belong to $\mathcal{E}$, while $UUDUUUDDUUDDDD$ belongs to $\mathcal{E}$.

Lemma 3 *There is a bijection between $\mathcal{E}$ and the set of p-equivalence classes of $\mathcal{D}$.*

Proof. Let P be a Dyck path in $\mathcal{D}\backslash\mathcal{E}$. Let us prove that there exists a Dyck path $P' \in \mathcal{E}$ of the same semilength as P such that P and P' belong to the same class.

We define a sequence of Dyck paths $P_0 = P, P_1, P_2, \ldots, P_{k-1}, P_k = P'$, $k \geq 1$.

For any i, $1 \leq i \leq k$, the Dyck path P_i is obtained from P_{i-1} by performing the following process.

We write

$$P_{i-1} = \alpha U D^r \beta U^s D \gamma$$

where $r, s \geq 2$, α (resp. γ) avoids DD (resp. UU) and $\beta = U\delta D$ for some δ. Notice that this decomposition is unique.

We set

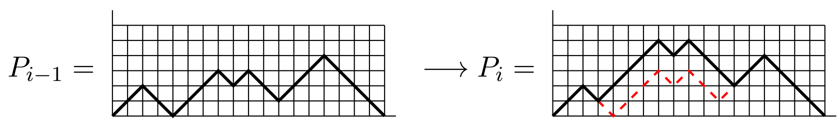
$$P_i = \alpha U D^{r-t} U^t \beta D^t U^{s-t} D \gamma$$

where $t = \min\{r-1, s-1\}$.

Now, let us consider the decomposition of P_i (as above)

$$P_i = \alpha' U D^{r'} \beta' U^{s'} D \gamma'$$

where α', β', γ', r' and s' satisfy the same properties as α, β, γ, r and s. Since at least one of the two values $r-t$ and $s-t$ is equal to one, we can assume $r - t = 1$ without loss of generality. So, we deduce that α' begins with αUDU^tU and γ' ends with γ which implies that the length of $D^{r'}\beta' U^{s'}$ is smaller than the length of $D^r \beta U^s$. Thus, the process finishes whenever either β' becomes empty or there is no occurrence of DD lying on the left of an occurrence of UU, *i.e.*, $P_k \in \mathcal{E}$. Moreover, this construction preserves the positions of peaks which implies that P_{i-1} and P_i belong to the same class. So, at the end of the process P' belongs to $\mathcal{E}$ and to the same p-equivalence class of P. See Figure 2 for an illustration whenever $P_{i-1} = \underline{U}UDD\mathbf{UUUDUDD}UUUD\overline{DDD}$ and $P_i = UUDUUUUDUDDDUUDDDD$ (α, β and γ appear respectively in underlined type, bold face and overlined type in P_{i-1}).



$UUDDUUUDUDDUUUDDDD \rightarrow UUDUUUUDUDDDUUDDDD$

Figure 2: Illustration of the example described in the proof of Lemma 3.

Now, it suffices to prove that two different Dyck paths P and P' in $\mathcal{E}$ cannot belong to the same class. Let us assume that P and P' belong to the same class. Whenever P and P' contain k peaks, $k \geq 1$, we can decompose uniquely P and P' as follows:

$$P = U^{r_1}D^{s_1}U^{r_2}D^{s_2}\dots U^{r_k}D^{s_k} \text{ and } P' = U^{r'_1}D^{s'_1}U^{r'_2}D^{s'_2}\dots U^{r'_k}D^{s'_k}$$

where $r_i, s_i \geq 1$ and $r'_i, s'_i \geq 1$ for $1 \leq i \leq k$.

Since the positions of peaks in P and P' are the same, we have $r_1 = r'_1$, $s_k = s'_k$ and $s_i + r_{i+1} = s'_i + r'_{i+1}$ for $1 \leq i \leq k-1$.

For a contradiction, let us assume that there is $j \geq 1$ such that $s_j \neq s'_j$ (we choose the smallest $j \geq 1$). Without loss of generality, we consider $s_j < s'_j$ and thus $s'_j \geq 2$. Since P' belongs to $\mathcal{E}$, we necessarily have $r'_\ell = 1$ for all $\ell \geq j+2$. As $s_j < s'_j$ the height of the $(j+1)$-th peak of P is greater than the height of the $(j+1)$-th peak of P'. Since $r'_\ell (= 1)$ is minimal for any $\ell \geq j+2$, the height of the ℓ-th peak of P is greater than the height of the ℓ-th peak of P' for all $\ell \geq j+1$. We obtain a contradiction with $s_k = s'_k$ which means that the k-th peak of P and P' are located at the same height. Thus, we necessarily have $P = P'$ which completes the proof. □

Theorem 2 *The generating function for the set of p-equivalence classes of $\mathcal{D}$ (i.e., modulo the positions of peaks) with respect to the semilength is given by*

$$\frac{1-6x+12x^2-8x^3+x^4}{(1-2x)^2(1-3x+x^2)}.$$

Proof. By Lemma 3, it suffices to provide the generating function $E(x)$ of the set $\mathcal{E}$, with respect to the semilength. Every nonempty path $P \in \mathcal{E}$ is decomposed as $P = U\alpha D\beta$, where either α is empty and $\beta \in \mathcal{E}$, or α is a nonempty element of $\mathcal{E}$ that contains DDU and $\beta = (UD)^\lambda$, $\lambda \geq 0$, or α avoids DDU and β avoids DUU. It is known that the generating function of Dyck paths avoiding DDU (equivalently DUU) is equal to $\frac{1-x}{1-2x}$ (see [1]). The above decomposition gives that

$$E(x) = 1+xE(x)+x\left(E(x)-1-\frac{1-x}{1-2x}\right)\frac{1}{1-x}+x\left(\frac{1-x}{1-2x}-1\right)\frac{1-x}{1-2x},$$

which implies the results. □

4 Equivalence classes modulo valleys

Throughout this section, we study the v-equivalence in $\mathcal{D}$.

Let $\mathcal{H}$ be the set of Dyck paths such that the height of any valley DU is at most one. For instance, $UUDDUUDUDD$ belongs to $\mathcal{H}$, while $UUU\mathbf{DU}DDUDD$ does not belong to $\mathcal{H}$ since its first valley DU is at height 2.

Lemma 4 *There is a bijection between $\mathcal{H}$ and the set of v-equivalence classes of $\mathcal{D}$.*

Proof. Let P be a Dyck path in $\mathcal{D}$. Let us prove that there exists a Dyck path $P' \in \mathcal{H}$ of the same semilength as P such that P and P' belong to the same class.

Before describing the construction of P', it is easy but worth to notice the following fact.

If a Dyck path P contains a valley DU of height at least one, then there exists an occurrence of UU on its left and an occurrence of DD on its right.

Let DU be a valley of height $h \geq 2$. We call *left-abutment* (resp. *right-abutment*) of this valley the rightmost occurrence of UU located on the left of the valley (resp. the leftmost occurrence of DD located on the right of the valley). Of course, the right- and left-abutments are at height at least $h-1$.

For instance, if $P = UU\underline{UU}DUD\mathbf{DU}UU\overline{DD}UDDDD$ then the second valley DU (bold face) has its left-abutment at position 3 (underlined) and its right-abutment at position 12 (overlined).

Now, we define a sequence of Dyck paths $P_0 = P, P_1, P_2, \ldots, P_k = P'$, $k \geq 1$.

For any i, $1 \leq i \leq k$, the Dyck path P_i is obtained from P_{i-1} by performing the following process.

We consider the leftmost valley of height at least two and we replace its left-abutment UU with UD and its right-abutment DD with UD.

The process finishes because at each step the leftmost valley of P_{i-1} at height $h \geq 2$ is moved into a valley in P_i at height $h-2$. At the end of the process, the Dyck path P' belongs to $\mathcal{H}$ since it does not contain any valley of height at least two. Moreover, all Dyck paths $P_0, P_1, \ldots, P_k$ belong to the same equivalence class.

For instance, if we perform the above process on $P = UUUUDUDDUDDUUDDDUD$, then we obtain $P_1 = UUUDDUUDUDDUUDDDUD$ and $P_2 = P'$ is given by $UUUDDUDDUUDUUDDDUD$ (see Figure 3 for an illustration of this example).

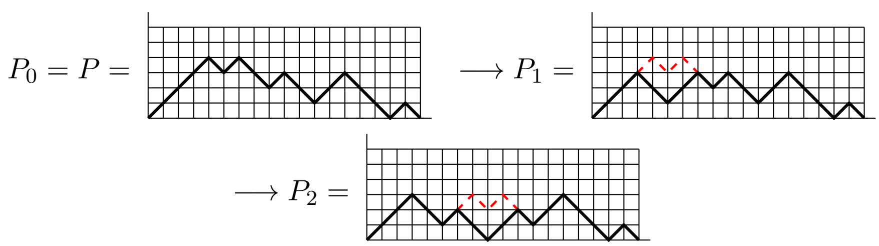


Figure 3: Illustration of the example described in the proof of Lemma 4.

Now, it suffices to prove that two different Dyck paths P and P' of the same semilength in $\mathcal{H}$ cannot belong to the same class. Let us assume that P and P' belong to the same class. Whenever P and P' contain k valleys ($k \geq 0$), we can uniquely decompose P and P' as follows:

$$P = U^{r_1}D^{s_1}U^{r_2}D^{s_2}\ldots U^{r_{k+1}}D^{s_{k+1}} \text{ and } P' = U^{r'_1}D^{s'_1}U^{r'_2}D^{s'_2}\ldots U^{r'_{k+1}}D^{s'_{k+1}}$$

where $r_i, s_i \geq 1$ and $r'_i, s'_i \geq 1$ for $1 \leq i \leq k+1$.

Since the positions of valleys are the same in P and P', we have $s_i + r_i = s'_i + r'_i$ for $1 \leq i \leq k+1$. For a contradiction, let us assume that there is

$j \geq 1$ such that $s_j \neq s'_j$ (we choose the smallest $j \geq 1$). Without loss of generality, we assume $s'_j < s_j$. Considering $s_j + r_j = s'_j + r'_j$ we deduce from $s'_j \leq s_j - 1$ the inequality $r'_j - s'_j \geq r_j - s_j + 2$.

Notice that the height of the j-th valley in P (resp. P') is given by $\sum_{i=1}^{j}(r_i - s_i)$ (resp. $\sum_{i=1}^{j}(r'_i - s'_i)$). Therefore, as P and P' belong to $\mathcal{H}$, we necessarily have $0 \leq \sum_{i=1}^{j}(r_i - s_i) \leq 1$ and $0 \leq \sum_{i=1}^{j}(r'_i - s'_i) \leq 1$.

Thus, we have:

$$\begin{aligned}\sum_{i=1}^{j}(r'_i - s'_i) &= \sum_{i=1}^{j-1}(r'_i - s'_i) + r'_j - s'_j \\ &= \sum_{i=1}^{j-1}(r_i - s_i) + r'_j - s'_j \\ &\geq \sum_{i=1}^{j-1}(r_i - s_i) + r_j - s_j + 2 = 2 + \sum_{i=1}^{j}(r_i - s_i) \geq 2\end{aligned}$$

This means that P' has its j-th valley of height greater than or equal two, which is a contradiction. Thus, we necessarily have $P = P'$, which completes the proof. □

Theorem 3 *The generating function for the set of v-equivalence classes of $\mathcal{D}$ (i.e., modulo the positions of valleys) with respect to the semilength is given by the generating function for the Fibonacci sequence restricted to the odd ranks (see A001519 in [10])*

$$\frac{1-2x}{1-3x+x^2}.$$

Proof. By Lemma 4, it suffices to provide the generating function $H(x)$ for the set $\mathcal{H}$ with respect to the semilength. Let P be a non-empty Dyck path in $\mathcal{H}$ and $P = U\alpha D\beta$ its first return decomposition where α and β are some Dyck paths.

So, $P = U\alpha D\beta$ does not contain any valley at height at least two if and only if α has all its valleys at height 0, and β belongs to $\mathcal{H}$. This means that $\alpha = U^j D^j \alpha'$ for some j, $1 \leq j \leq n$, and where α' has all its valleys at height 0. The generating function for the number of such α is given by $\frac{1-x}{1-2x}$. Finally we have the functional equation $H(x) = 1 + x \cdot \frac{1-x}{1-2x} \cdot H(x)$ which gives the result. □

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