On highly composite numbers

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ABSTRACT

Let $d(n)$ be the number of divisors of n. S. Ramanujan has defined n to be highly composite, if, for any $m < n$, we have $d(m) < d(n)$. We shall try to describe the results obtained by Ramanujan about these numbers, and improvements and generalizations of his work. The main problem, which is not completely solved, is to estimate the number of highly composite numbers (or of similar numbers defined with some others arithmetical functions) up to x.

I. **Introduction and notations.**

In 1915, S. Ramanujan published in the Proceedings of the London Mathematical Society a memoir of sixty-three pages entitled "highly composite numbers" and consisting of 52 paragraphs (cf. [54] and [55], no. 15). The purpose of this memoir was to study how large the number of divisors of an integer n can be when n tends to infinity. We shall try to describe the results obtained by Ramanujan, and the improvements and generalizations of his work.

We shall use the following classical notations:

$p_1 = 2,\ p_2 = 3, \ldots, p_k = k^{th}$ prime;

p, P, q will denote prime numbers;

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$$d(n) = \sum_{d|n} 1 = \text{number of divisors of } n;$$

$$\sigma_s(n) = \sum_{d|n} d^s = \text{sum of } s^{th} \text{ power of divisors of } n$$

(observe that $\sigma_{-s}(n) = \sigma_s(n)/n^s$);

$$\sigma(n) = \sigma_1(n) = \sum_{d|n} d;$$

$$\omega(n) = \sum_{p|n} 1 \; ; \quad \varphi(n) = \text{Euler's totient function;}$$

$$d_2(n) = d(n) \; ; \quad d_k(n) = \sum_{d|n} d_{k-1}(d) \quad \text{for} \quad k \geq 3;$$

$$\pi(x) = \sum_{p \leq x} 1 \; ; \quad \theta(x) = \sum_{p \leq x} \log p \quad \text{is the Chebyshev function;}$$

$$\mathrm{Li}\, x = \lim_{\varepsilon \to 0} \left[\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^{x} \frac{dt}{\log t} \right] \quad \text{is the integral logarithm.}$$

The notation $f \ll g$ (or $g \gg f$) will mean $f = O(g)$.

If f is an arithmetical function, we shall define n as an f-champion number if $m < n \Rightarrow f(m) < f(n)$.

For real x, $[x]$ will denote the integral part of x.

The memoir of Ramanujan encompasses 5 parts: Elementary results on the maximal order of $d(n)$, the definition and the structure (that is to say the form of the standard factorization in primes) of the highly composite numbers, the superior highly composite numbers, the maximal order of $d(n)$

with or without the assumption of the Riemann hypothesis, and special forms of N.

In the last part (§46-51), the value of $d(N)$ is studied for various N's: N a perfect power, $N = \text{l.c.m.}\,(1,2,\ldots,n)$, $N = n!$. The smallest integer N with exactly 2^n divisors is also determined, and this is a good contest question.

In the very last paragraph (§52) a few words are said about the iterated d-function: $d^{(1)} = d$, and $d^{(k)}(n) = d(d^{k-1}(n))$. A deeper study of $d^{(k)}$ has been undertaken by Erdös and Kátai (cf. [8]).

II. Elementary results concerning the maximal order of d(n).

It was proved by C. Runge in 1885 (cf. [68]) that for fixed $\varepsilon > 0$,

$$(1) \qquad \lim d(n)/n^{\varepsilon} = 0.$$

S. Wigert proved in 1907 (cf. [74]) that the maximal order of $\log d(n)$ is $\dfrac{\log n \log 2}{\log\log n}$, that is to say that

$$(2) \qquad \overline{\lim} \frac{(\log d(n))(\log\log n)}{(\log n)(\log 2)} = 1.$$

For this result, S. Wigert uses the prime number theorem: $\pi(x) \sim x/\log x$.

S. Ramanujan proved (2) without assuming the prime number theorem, as is mentioned in the notes in Hardy and Wright's book (cf. [19], Chapter 18).

The upper bound of (2) is based on the inequality, valid for all N with $\omega(N) = k$,

$$(3) \qquad d(N) < \frac{((\theta(p_k) + \log N)/k)^k}{(\log p_1)(\log p_2) \cdots (\log p_k)}$$

and on the relation

$$(4) \qquad \pi(x)\log x - \theta(x) = \int_2^x \frac{\pi(t)}{t}\, dt.$$

It is easy to deduce from the prime number theorem that, as $k \longrightarrow \infty$,

$$\theta(p_k) \sim p_k \sim k \log k,$$

but, using (4), it is possible to prove $\theta(p_k) \sim k \log k$ without the prime number theorem (cf. [62]), and this was mainly Ramanumjan's idea.

Let us define the multiplicative function $r(n)$ as follows:

if $p \equiv 1 \bmod 4$, $r(p^k) = d(p^k) = k + 1$;

if $p \equiv 3 \bmod 4$, $r(p^k) = 0$ for k odd, and $r(p^k) = 1$ for k even;

if $p \equiv 2$, $r(2^k) = 1$ for all k.

It is known that the number of ways in which n can be written as a sum of two squares is equal to $4r(n)$ (cf. [19], Chapter 16).

An application of (2) with Ramanujan's proof gives the maximal order of $r(n)$ (cf. [37]):

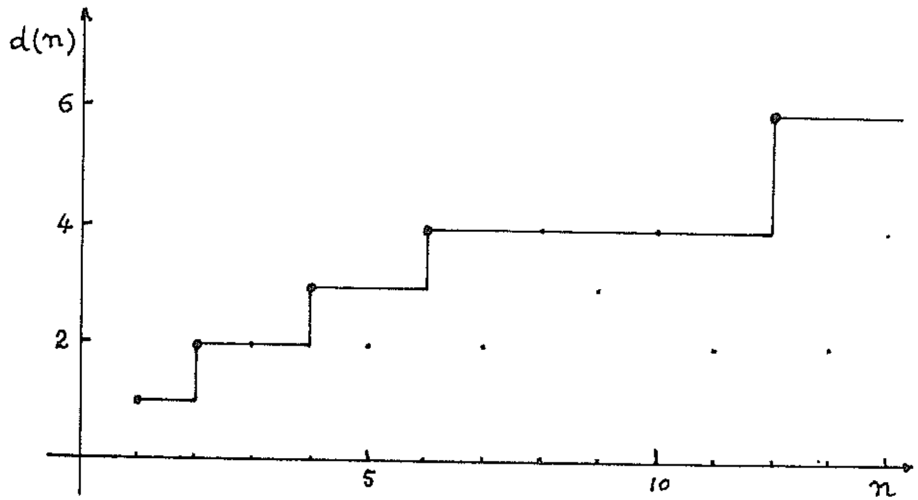
$$(5) \qquad \overline{\lim} \frac{(\log r(n))(\log\log n)}{(\log 2)(\log n)} = 1.$$

Actually $r(n)$ counts (crudely) the divisors of n made up of primes $\equiv 1 \bmod 4$; these primes are about half of all the primes, but the maximal orders of $\log r(n)$ and $\log d(n)$ are the same. This is somewhat surprising and sometimes misleading. For instance, Theorem 338 in [19] gives erroneously 1/2 instead of 1 on the right hand side of (5).

III. **Highly composite numbers.**

S. Ramanujan defined an integer n to be highly composite (we shall write h.c.) if for all $m < n$, we have $d(m) < d(n)$. So, with our definition, h.c. numbers are d-champion numbers.

For every integer $n \geq 1$, let us draw a point with coordinates $(n, d(n))$ and look at the increasing envelope of these points, that is the smallest nondecreasing function lying above all these points. This envelope is a step function, and the vertices of the steps correspond to h.c. numbers.



Now, let us write the standard factorization of an integer n in the form

$$n = \prod_{i=1}^{\infty} p_i^{\alpha_i} \quad \text{with} \quad \alpha_i = \alpha_i(n),$$

where only finitely many α_i are non-zero. Then we have

$$d(n) = \prod_{i=1}^{\infty} (\alpha_i + 1).$$

An integer n is said to be w.n.i.e. (with nonincreasing exponent) if the sequence $(\alpha_i)_i$ is nonincreasing. Clearly, a h.c. number is w.n.i.e..

For a w.n.i.e. integer, we define $q_1 = \max_{p|n} p$ (so that all the primes $\leq q_1$ divide n), and

$$q_j = \max_{p^j|n} p.$$

The nonincreasing sequence $(q_j)_{j\geq 1}$ characterizes n.

IV. Superior highly composite numbers.

Let $\varepsilon > 0$ be fixed. It follows from (1) that $d(n)/n^{\varepsilon}$ is bounded and reaches its maximum in one or several points. Ramanujan has defined an integer N to be superior h.c. (s.h.c.) if there exists $\varepsilon > 0$ such that for all n we have

$$\frac{d(n)}{n^{\varepsilon}} \leq \frac{d(N)}{N^{\varepsilon}}.$$

For $n < N$ we have $d(n) \leq \left(\frac{n}{N}\right)^{\varepsilon} d(N) < d(N)$, which shows that a s.h.c. number is h.c..

The stucture of these numbers has been completely determined by Ramanujan. Let us define

$$E = \left\{ \frac{\log(1 + 1/k)}{\log p} \ ; \ k \geq 1, \ p \text{ prime} \right\}.$$

If $\varepsilon \notin E$, then the maximum of $d(n)/n^{\varepsilon}$ is attained at only one integer N_{ε}, and we have

$$(6) \qquad N_{\varepsilon} = \prod_{i=1}^{\infty} p_i^{\alpha_i} \quad \text{with} \quad \alpha_i = [1/(p_i^{\varepsilon} - 1)].$$

It was known by Siegel (cf. [1], p. 455) that for real λ and three different primes p, q, r, the numbers $p^{\lambda}, q^{\lambda}, r^{\lambda}$ cannot be all rational, except when λ is an integer (cf. [27] and [28], Chapter 2). This implies that three elements of E cannot be equal. It seems very likely that two elements of E are always distinct, but this is still unproved. If it is true, then, for $\varepsilon \in E$, the maximum of $d(n)/n^{\varepsilon}$ is attained at two integers. If it is false, this maximum is attained at four integers for some ε (cf. [9], p. 71). This question was probably overlooked by Ramanujan; for instance, the result of §44 is false in the latter case.

In both cases, the integer N_{ε} defined by (6) is a s.h.c. number, maximizing $d(n)/n^{\varepsilon}$. If we set

$$(7) \qquad x = 2^{1/\varepsilon}, \; x_k = x^{\log(1+1/k)/\log 2} \quad \text{for} \quad k \geq 1,$$

then we have $x_1 = x$ and $x_2 = x^{\theta}$ with

$$(8) \qquad \theta = \frac{\log (3/2)}{\log 2} = 0.585 \; \ldots,$$

and from (6) we see that

$$p \mid N_\varepsilon \Leftrightarrow p \leq x,$$

$$\alpha_i = k \Leftrightarrow x_{k+1} < p_i \leq x_k,$$

$$\alpha_1 \leq \frac{1}{2^{\varepsilon} - 1} \leq \frac{1}{\varepsilon \log 2} = \frac{\log x}{(\log 2)^2},$$

$$(9) \qquad \log N_\varepsilon = \sum_{1 \leq k \leq \alpha_1} \theta(x_k),$$

$$(10) \qquad \log d(N_\varepsilon) = \sum_{1 \leq k \leq \alpha_1} \pi(x_k) \log(1 + 1/k).$$

It follows from (9) that $\log N_\varepsilon \sim x$.

V. The number of highly composite numbers up to x.

Let us define $Q(X)$ as the number of h.c. numbers $\leq X$. It is easy to see that between X and $2X$ there is always a h.c. number (because $d(2n) > d(n)$), and this implies that $Q(X) \gg \log X$.

It was proved by Ramanujan that $\lim Q(X)/\log X = +\infty$ (cf. [54], §28). Given a h.c. number n, the idea is to construct n' as close as possible to n, and with $d(n') > d(n)$. Then there exists a h.c. number n'' with $n < n'' \leq n'$. Ramanujan chose n' with the same exponents as n for the large primes, modifying only the exponents of the small primes.

The problem of estimating $Q(X)$ was of some interest to Ramanujan. In a joint paper with Hardy (cf. [18] and [55], no. 34), the number of w.n.i.e. (cf. §III) integers up to X is estimated. The introduction mentions: "That class of numbers includes as a subclass the h.c. numbers recently studied by Mr. Ramanujan. The problem of determing the number $Q(X)$ of h.c. numbers not exceeding X appears to be one of extreme difficulty. It is still uncertain whether or not the order of $Q(X)$ is greater than that of any power of $\log X$".

P. Erdös proved in 1944 that for a positive c_1 we have $Q(X) \gg (\log X)^{1+c_1}$. A new tool was Hoheisel's theorem (cf. [21]): for some $0 < \tau < 1$,

$$(11) \qquad \pi(x + x^{\tau}) - \pi(x) \gg x^{\tau}/\log x$$

(the best τ is now $\frac{11}{20} - \frac{1}{384} = 0.547396\ldots$, cf. [32]).

Erdös used it to construct n' by multiplying and dividing n by large primes, and using the diophantine approximations of θ (defined by (8)) given by Dirichlet's theorem (cf. [7]).

In 1971, I proved (cf. [36]) that $Q(X) \ll (\log X)^{c_2}$. I used for this the result of Feldmann (cf. [15]) that there exists κ such that for all integers $u, v \geq 1$ we have:

(12) $$|v\theta - u| \gg v^{-\kappa}.$$

I also used the structure of the h.c. numbers between two consecutive s. h. c. numbers that we shall describe in the next paragraph.

Let us define $c(X)$ by

$$Q(X) = (\log X)^{c(X)},$$

and let us assume two very strong conjectures: first, (11) holds for all $\tau > 0$, and secondly, for all $\eta > 0$, there exists a positive constant $B = B(\eta)$ such that for all u, v, w in $\mathbb{Z}$ we have:

$$|u \log 5 + v \log 3 + w \log 2| \geq B((|u| + 1)(|v| + 1))^{-1-\eta}.$$

Then the method of [36] shows that (cf. [73])

$$\lim c(X) = (\log 30)/(\log 16) = 1.227\ldots .$$

More recently (cf. [48]) I used the value $\kappa = 2^{49} \log 3$ given by M. Waldschmidt to show that $\overline{\lim}\, c(X) \leq 3.48$, and a new result of G. Rhin (cf. [57]), namely $\kappa = 7.616$, implies $\overline{\lim}\, c(X) \leq 1.71$. I also show that $\underline{\lim}\, c(X) \leq 1.44$. It was proved in [36] that

$$\underline{\lim}\, c(X) \geq \frac{1}{3} \frac{\log(15/8)}{\log 2} (1 - \tau) = 1.13682\ldots$$

with Mozzochi's value $\tau = 11/20 - 1/384$.

All these results are based on diophantine approximations of $\theta = \frac{\log(3/2)}{\log 2}$, and similar numbers arising from the values of the function d. Actually the 3 is $d(p^2)$ and both 2's occuring in the definition of θ are $d(p)$. Now, suppose we consider a multiplicative function δ, such that $\delta(p^\alpha)$ depends only on α and $\hat{\theta} = \frac{\log(\delta(p^2)/\delta(p))}{\log(\delta(p))}$ is, say, a Liouville number. Then the method of [36] no longer works (cf. [73]). For such a function δ, let $Q_\delta(X)$ be the number of champion numbers up to X. It is an open question whether there exists $c(\delta)$ such that

$$Q_\delta(X) \ll (\log x)^{c(\delta)}.$$

Another open question is the following: Let n_i be the i^{th} h.c. number. Erdös proved in [7] that there exists $c > 0$ such that, for i large enough,

$$n_{i+1}/n_i \leq 1 + (\log n_i)^{-c},$$

and deduced from this that $Q(X) \geq (\log X)^{1+c+o(1)}$. But does there exist c' such that $n_{i+1}/n_i - 1 \gg (\log n_i)^{-c}$? In [48] it is only proved that

$$n_{i+1}/n_i - 1 \gg \exp(-(\log n_i)^{1/4}).$$

VI. The structure of highly composite numbers.

Let n be a h.c. number, and q_1 its largest prime divisor. We write

$$n = \prod_{p \le q_1} p^{a_p}.$$

Ramanujan has proved that $a_{q_1} = 1$ unless $n = 4$ or $n = 36$. Then he divides the primes from 2 to q_1 in five ranges and for each range gives an asymptotic estimate of a_p in terms of q_1. For instance, he has proved that

$$(13) \qquad a_p \log p = \frac{\log q_1}{\log 2} + O(\sqrt{\log q_1 \operatorname{loglog} q_1})$$

holds for $\log p = O(\operatorname{loglog} q_1)$.

The study of the structure of h.c. numbers takes up about half of the whole paper. In the introduction to the "Collected Papers" (cf. [55], p. xxxiv) Hardy has written that this study is "most remarkable, and shows very clearly Ramanujan's extraodinary mastery over algebra of inequalities."

To estimate a_p, the idea of Ramanujan is to write $d(m) < d(n)$ for an appropriate choice of $m < n$. In [1], Alaoglu and Erdös have improved Ramanujan's estimations for a_p, mainly using Hoheisel's result in the construction of a better m.

In [36] and [48] estimates of a_p are obtained with the so-called "benefit" method. Let n be h.c. and N the s.h.c. number just preceding n. Let ε be any parameter such that $N = N_\varepsilon$, and $x = 2^{1/\varepsilon}$. We write (cf. (6))

$$N = \prod_{p \le x} p^{b_p} \quad \text{with} \quad b_p = [1/(p^\varepsilon - 1)].$$

The benefit of n (relative to N and ε) is

$$(14) \qquad \begin{aligned} \operatorname{ben} n &= \log \frac{d(N)}{N^\varepsilon} - \log \frac{d(n)}{n^\varepsilon} \\ &= \sum_{p \le \max(q_1, x)} \left[\log\left(\frac{b_p + 1}{a_p + 1}\right) - \varepsilon(b_p - a_p)\log p\right]. \end{aligned}$$

From the definition of the s.h.c. numbers, each term in the above sum in nonnegative. In [36] I proved that there exists C, such that, when n is h.c.,

$$\operatorname{ben} n \le C\, x^{-\gamma}$$

for $\gamma = \theta(1 - \tau)(\kappa + 1) = 0.0307\ldots$ with the Mozzochi and Rhin values of τ and κ. Using this I showed:

$$(15) \quad \begin{cases} \text{if } \varepsilon \log p - \log\left[1 + \frac{1}{b_p + 1}\right] \le Cx^{-\gamma}, & \text{then } a_p = b_p \text{ or } a_p = b_p + 1; \\ \text{if } \log\left[1 + \frac{1}{b_p + 1}\right] - \varepsilon \log p \le Cx^{-\gamma}, & \text{then } a_p = b_p \text{ or } a_p = b_p - 1; \\ \text{in other cases,} & a_p = b_p. \end{cases}$$

Formulas (15) show that $|a_p - b_p|$ is at most 1. If q_k is the largest prime dividing n with exponent $\ge k$, and x_k is defined by (7), then we have for $k \le k'$, $k' = [1/(e^{\gamma \log 2} - 1)] = 46$, (cf. [48])

$$(16) \qquad |\pi(x_k) - \pi(q_k)| < \sqrt{x_k x^{-\gamma}}\,,$$

and for $k' < k \leq a_2$

$$|\pi(x_k) - \pi(q_k)| \leq 1.$$

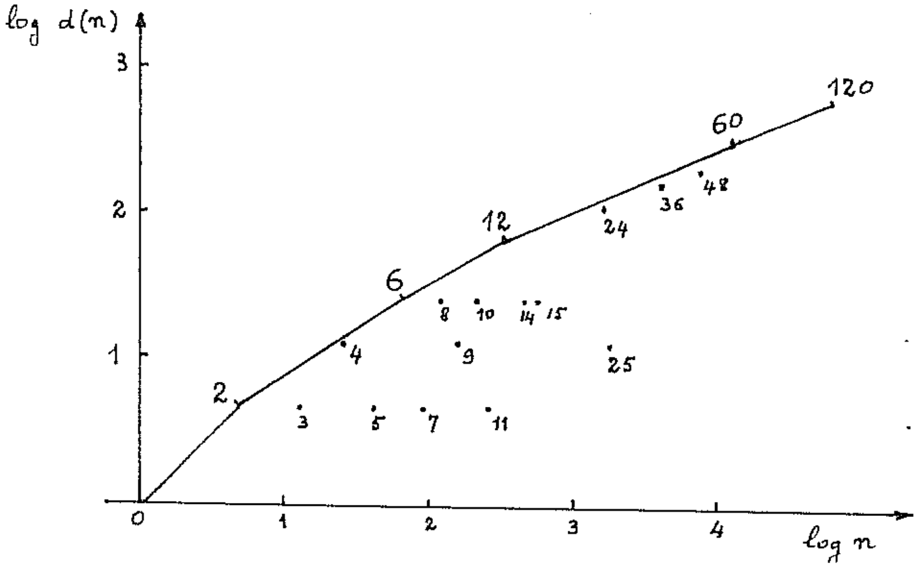
With $k = 1$, we get, by Hoheisel's theorem

(17) $$q_1 = x + O(x^\tau) \sim \log N \sim \log n.$$

From this we can see that in (13) the error term is in fact $O(1)$. Moreover, from (15) and (17), when q_1 is given, we can calculate a_p with an error of at most 1, for all p's between 2 and q_1.

VII. Effective upper bounds.

For each integer $n \geq 1$, let us draw a point with coordinates $\log n$ and $\log d(n)$, and then consider the convex envelope of all these points.



The first s.h.c. numbers are 2,6,12,60. Observe that h.c. numbers which are not superior such that 4,24,36,48 are close to the convex envelope. The

Now consider all the straight lines with fixed slope ε and going through the points $(\log n, \log d(n))$. These lines cut the y-axis in a point whose ordinate is $\log d(n) - \varepsilon \log n$, and so, from the definition of s.h.c. numbers, the highest possible such line, is going through $(\log N, \log d(N))$ where $N = N_\varepsilon$. Thus s.h.c. numbers are characterized by the vertices of the preceding convex envelope.

It follows from (2) that there exists an absolute constant A such that

(18) $$\frac{\log d(n)}{\log 2} \leq A \frac{\log n}{\log\log n} \quad \text{for all} \quad n \geq 3$$

($n = 2$ is not possible because $\log\log 2$ is negative).

We now observe that the function $x \longmapsto \frac{Ax}{\log x}$ is concave for $x \geq e^2$ and that 2520 is the smallest s.h.c. number bigger than $\exp(e^2)$. So to prove (18) for a certain A, it is sufficient to prove it for all $n \leq 2520$, and then for all s.h.c. number bigger than 2520, because, if the curve $y = Ax/\log x$ is above all the vertices of the convex envelope, it will be above all the points $(\log n, \log d(n))$.

Calculations can be carried out easily for two reasons: first, s.h.c. numbers are rare, and secondly, their factorization into primes is known by (6) or (9), and effective estimates of Rosser and Schoenfeld can be used (cf. [66], [67], [70]). The result is that

$$\frac{\log d(n)}{\log 2} \leq 1.5379 \ldots \frac{\log n}{\log\log n}, \quad n \geq 3$$

with equality for $n = 6983776800$ cf ([43]). By the same method more accurate estimates can be given (cf. [61]):

$$\frac{\log d(n)}{\log 2} \leq \frac{\log n}{\log\log n}\left[1 + \frac{1.9349\ldots}{\log\log n}\right], \quad n \geq 3;$$

$$\frac{\log d(n)}{\log 2} \leq \frac{\log n}{\log\log n}\left[1 + \frac{1}{\log\log n} + \frac{4.7624\ldots}{(\log\log n)^2}\right], \quad n \geq 3;$$

$$\frac{\log d(n)}{\log 2} \leq \frac{\log n}{\log\log n - 1.39177\ldots}, \quad n \geq 56.$$

Under the assumption of the Riemann hypothesis, it follows from the upper bound obtained by Ramanujan (cf. [54], §43, and (19) below), that there exists c such that

$$\frac{\log d(n)}{\log 2} \leq \mathrm{Li}(\log n) + c(\log n)^{\theta}$$

with θ defined by (8). The above method enables one to find the best possible c, but the calculations have not yet been done.

VIII. The maximal order of d(n).

Using the definition of s.h.c. numbers, Ramanujan has defined the maximal order of $d(n)$ as a certain function D (cf. [54], §38). Consider the piecewise linear function $u \longmapsto \Delta(u)$ such that for all s.h.c. numbers N, $\Delta(\log N) = \log d(N)$, that is the convex envelope of the set of points $(\log n, \log d(n))$ considered in the preceding paragraph. Then Ramanujan's D-function is equal to

$$D(t) = \exp(\Delta(\log t))$$

and satisfies $d(n) \leq D(n)$ for all n, with equality when n is s.h.c.

The reasons why Ramanujan chose D as the maximal order of $d(n)$ are not clear to me. $F(t) = \max_{n \leq t} d(n)$ might be a better choice. Anyway D and F are very close (cf. [41], p. 13-15, where more about this notion of maximal order can be found).

However, it was a great idea of Ramanujan to use s.h.c. numbers to get a good estimate of the maximal order, that is to find an analytic function as close as possible to the maximal order. His estimation for D, under the assumption of the Riemann hypothesis,

$$(19) \qquad \frac{\log D(n)}{\log 2} = \mathrm{Li}(\log n) + \theta\,\mathrm{Li}((\log n)^{\theta}) - \frac{(\log n)^{\theta}}{\log\log n} - R(\log n) + O\left[\frac{\sqrt{\log n}}{(\log\log n)^3}\right]$$

with

$$R(x) = \left[2\sqrt{x} + \sum_{\rho} \frac{x^{\rho}}{\rho^2}\right]/(\log x)^2$$

where the sum is over all the nonreal zeroes of the Rieman ζ-function, is certainly very nice.

Let $\lambda > 1$, and let

$$f_{\lambda}(n) = \frac{\log d(n)}{\log 2} - \lambda\,\frac{\log n}{\log\log n}.$$

It follows from (2) that $\lim_{n\to\infty} f_{\lambda}(n) = -\infty$, and therefore $f_{\lambda}(n)$ has an absolute maximum attained for at least one integer $\tilde{N}_{\lambda}$ (cf. [64]). Little is known about these integers $\tilde{N}_{\lambda}$, but they are still closer to the maximal order of $d(n)$ than the s.h.c. numbers themselves.

IX. Tables.

In his memoir, Ramanujan has included a table of the first one hundred h.c. numbers, and of the first fifty s.h.c. numbers. It is worth while mentioning that the table on pp. 2 and 3 in the notebooks (cf. [56], Vol. 2) is a table of the h.c. numbers.

In [60], Robin has calculated the first 5000 h.c. numbers, and independently the same calculation was carried out by te Riele. They used a method of dynamical programming. Let us define S_k as the set of integers made up of primes $p_1, \dots p_k$. We say that n is k – h.c. if $n \in S_k$ and if

$$m \in S_k \text{ and } m < n \Rightarrow d(m) < d(n).$$

These k – h.c. numbers are easily determined by induction, and small k – h.c. numbers are actually h.c.

A theoretical study of 2 – h.c. numbers (of the form $2^\alpha 3^\beta$) has been undertaken in [2], using the continued fraction expansion of $\log 3/\log 2$.

A more powerful algorithm is also given in [60]. It allows one to calculate h.c. numbers between two consecutive s.h.c. numbers. This algorithm uses the "benefit" method mentioned in §VI. First you guess a positive real number B which should be the maximal value of the benefit of a h.c. number. Then you calculate all integers n in the considered range, the benefit of which is smaller than B. From these n's you calculate an exact upper bound B' for the maximal benefit of a h.c. number. If $B' \le B$, h. c. numbers are included in the calculated n's. If $B' > B$, you start again with B' instead of B.

Robin has used this algorithm to determine the smallest number which has more than 10^{1000} divisors. It is an integer of 13198 decimal digits, the largest prime factor of which is 30113.

Aloaglu and Erdös conjectured in [1] that if n is h.c., then there exist two primes p and q such that Np and N/q are h. c.. G. Robin has found a counterexample to this conjecture (cf. [60]).

X. Optimization problems in integers.

Calculation of the largest h.c. numbers $\le A$ is equivalent to solving

$$(20)\quad \begin{cases} \max \sum_{k=1}^{\infty} \log(x_k + 1), \\ \sum_{k=1}^{\infty} x_k \log p_k \le a = \log A,\ x_k \in \mathbb{N}. \end{cases}$$

As $p_k \le A$, the number of variables is finite, and solving (20) with x_k real is easy using Lagrange multipliers.

In fact, it is also possible to use Lagrange multipliers to solve (20) when the x_k's are integers. Suppose that f and g are two real-valued functions defined on a subset Ω of $\mathbb{R}^n$, and that g is nonnegative. We want to solve

$$(21)\quad \begin{cases} \max_{x \in \Omega} f(x) \\ g(x) \le C \end{cases}$$

for different values of C. Suppose that for $\lambda \ge 0$ there exists x_0 such that $f - \lambda g$ is maximal at x_0, that is to say

$$\forall x \in \Omega,\ f(x) - \lambda g(x) \le f(x_0) - \lambda g(x_0).$$

Then x_0 is a solution of (21) for $C = g(x_0)$. Indeed, we have for $x \in \Omega$, with $g(x) \leq C = g(x_0)$:

$$f(x) \leq f(x_0) + \lambda(g(x) - g(x_0)) \leq f(x_0).$$

Such C's which can be written in the form $g(x_0)$ are called Lagrange bounds for the problem (21). Lagrange bounds of (20) are logarithms of s.h.c. numbers.

In general, not all possible values of C in (21) are Lagrange bounds, and to solve (20) when C is not a Lagrange bound, we can use Everett's method (cf. [14]) which is about the same as the benefit method I used in §VI.

A few bridges have been built between h.c. numbers and optimization problems in integers. (cf. [38], [39], [40], [58], [60]). Probably it is worthwhile working in that area. In my opinion, optimization theory sheds an interesting light on h.c. numbers, and from this point of view it can no longer be said that h.c. numbers are in a backwater of mathematics.

XI. Other champion numbers.

Ramanujan's work on h.c. numbers has been first extended to the sum of the divisors of n by Alaoglu and Erdös (cf. [1], [51] and [69]). They define a highly abundant (h.a.) number as a champion number for the function $n \longmapsto \sigma(n)$, and a superabundant (s.a.) number as a champion number for the function $n \longmapsto \sigma(n)/n$. Furthermore they say that n is colossally abundant (c.a.) if there exists $\varepsilon > 0$ such that for all m,

$$\frac{\sigma(m)}{m^{1+\varepsilon}} \leq \frac{\sigma(n)}{n^{1+\varepsilon}}.$$

It is easy to see that c.a. $\Rightarrow$ s.a. $\Rightarrow$ h.a.

Let n be h.a. and P its largest prime divisor. It is not knwon whether $P \sim \log n$, or if P^2 divides n for infinitely many n's. Let $Q_h(X)$ and $Q_s(X)$ be the number of h.a. and s.a. numbers up to X. It has been proved in [9] that $Q_s(X) > (\log X)^{1+\delta}$. We don't know whether $Q_h(X)$, or even $Q_s(X)$, is smaller than $(\log X)^c$.

More recently, Masser and Shiu (cf. [31] and [4]) have studied sparsely totient numbers, that is to say integers n such that $m > n \Rightarrow \varphi(m) > \varphi(n)$. In this case the superior numbers are easy: they are the product of the first k primes (cf. [49], Chapter 1), but that does not make the study of sparsely totient numbers really easier.

Landau has defined $g(n)$ as the maximal order of an element in the symmetric group of n elements. Let ℓ be the additive function defined by $\ell(p^\alpha) = p^\alpha$. One can prove that

$$g(n) = \max_{\ell(M) \leq n} M$$

and

$$N \in g(\mathbb{N}) \Leftrightarrow M > N \Rightarrow \ell(M) > \ell(N).$$

So the values of $g(n)$ appear as a generalization of h.c. numbers (cf. [34], [35], [29], [30]).

Let us define n to be largely composite (l.c.) if

$$m \leq n \Rightarrow d(m) \leq d(n).$$

These numbers are not necessarily w.n.i.e. (cf. §III), and they are much more numerous than h.c. numbers (cf. [42]). An open question is whether between two consecutive h.c. numbers that are large enough there is always a l.c. number.

Champion numbers are considered in [52] for $d_k(n)$ (cf. §1), in [3] for the function $f(n)$, defined as the number of unordered factorizations of n into factors > 1, in [48] for the function $n \longmapsto d(n) + d(n+1)$, in [10] for the function

$$F(n) = \max_t \Big[\sum_{\substack{d|n \\ t/2 \leq d < t}} 1 \Big].$$

in [13] for the function $f(n) = \sum_{i=1}^{k-1} q_i/q_{i+1}$ where $n = q_1^{\alpha_1} \cdot \ldots \cdot q_k^{\alpha_k}$ with $q_1 < q_2 < \ldots < q_k$, and for the function $\omega - f$, and in [11] for the function f, where $f(n)$ is the largest integer k for which there exists m such that n divides the product $\prod_{1 \leq i \leq k} (m+i)$, but does not divide this product if any of its factors is omitted.

Champion numbers for ω are the products of the first primes. Integers n such that

$$m \leq n \Rightarrow \omega(m) \leq \omega(n)$$

have been studied in [12].

XII. Maximal order of various functions.

It has been proved by Landau (cf. [26]) that

$$\overline{\lim} \frac{n}{\varphi(n) \operatorname{loglog} n} = e^{\gamma},$$

where γ is Euler's constant. In [45] and [46] it is proved that for infinitely many n's,

$$n > e^{\gamma} \varphi(n) \operatorname{loglog} n$$

holds.

The maximal order of $\sigma(n)$ was first obtained by Gronwall (cf. [16]) who showed

$$\overline{\lim} \frac{\sigma(n)}{n \operatorname{loglog} n} = e^{\gamma}$$

Robin has proved in [63] that the property

$$\forall n \geq 5041, \ \sigma(n) < e^{\gamma} n \operatorname{loglog} n$$

is equivalent to the Riemann hypothesis (cf. also [59] and [65]).

Let $a(n)$ be the number of abelian groups of order n. This is a multiplicative function, and $a(p^{\alpha})$ is equal to the number of partitions of α. The maximal order of $a(n)$ is a little more difficult to study than that of $d(n)$. The reason is that the "superior" numbers are more complicated. Schwarz and Wirsing have proved in [71] that the maximal order of $a(n)$ is

$$(\log 5)\mathrm{Li}\left[\frac{\log n}{4}\right] + O(\log n \exp(-c\sqrt{\log\log n})),$$

improving the results of [22] and [25]. In [41] the maximal order of $a(n)$ under the assumption of the Riemann hypothesis is given.

When dealing with an arithmetical function, it is now a classical problem to study its maximal order. This has been done for the coefficients of some modular forms and especially Ramanujan's function τ (cf. [5] and [33]), and some other functions (cf. [17], [23] and [24]).

A more general study has been undertaken for those multiplicative functions $f(n)$ for which $f(p^\alpha)$ does depend on α, but not on p. (cf. [6], [53], [20], [72], and [44]).

Explicit upper bounds for $d_3(n)$ and $r(n)$ defined in §II can be found in [61].

XIII. The unpublished manuscript.

In the notes on the memoir "Highly composite numbers" at the end of the "Collected Papers" of S. Ramanujan (cf. [55], p. 339), it is stated: "The paper, as long as it is, is not complete. The London Math. Soc. was in some financial difficulty at the time, and Ramanujan suppressed part of what he had written, in order to save expense."

During the Ramanujan centenary conference at Urbana, many documents were displayed, and among them, I have found about 20 pages, handwritten by Ramanujan, that belong to this suppressed part. This unpublished part deals with the maximal order of some arithmetical functions under the assumption of the Riemann hypothesis, and generalizes the results of §§ 39–43. One type of these arithmetical functions is the number of representations of n by a sum of 2, 4,, 6, 8 squares, or by some other simple quadratic forms. Large values of $d_k(n)$ are also studied.

The more interesting part of this manuscript pertains to the maximal order of

$$\sigma_{-s}(n) = \sum_{d|n} d^{-s}$$

where $s > 0$. We have

$$\sigma_s(n) = n^s\sigma_{-s}(n)$$

and Ramanujan studied in detail those functions $\sigma_s(n)$ which occur in Eisenstein series. To study the maximal order of $\sigma_{-s}(n)$, generalized superior h.c. numbers are introdcued. In the case $s = 1$, these numbers were rediscovered by Alaoglu and Erdös who call them colossally abundant numbers.

Three cases are to be considered: $0 < s < 1/2$, $s = 1/2$ and $s > 1/2$. When $s = 1$, Ramanujan gives the formula

$$\overline{\lim}\, (\sigma_{-1}(n) - e^\gamma \log\log n)(\sqrt{\log n}) \le e^\gamma(4 - 2\sqrt{2} + \gamma - \log 4\pi)$$

$$= 1.39\ldots,$$

which was rediscovered by Robin (cf. [63], p. 194). In fact, Ramanujan has estimations for every s.

I shall try to get this manuscript of Ramanujan published elsewhere.

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