Supplementary Material for "Approximate ranking from pairwise comparisons"

6 Proofs

In this section, we provide the proofs of our theorems. In order to simplify notation, we assume without loss of generality (re-indexing as needed) that the underlying permutation π equal to the identity, so that $\tau_1 > \tau_2 > \ldots > \tau_n$.

6.1 Proof of Theorem 1

Our analysis uses an argument inspired by the proof of the performance guarantee of the original LUCB algorithm from the bandit literature, presented in [Kal+12]. We begin by showing that the estimate $\widehat{\tau}_i(T_i)$ is guaranteed to be α_i-close to τ_i, for all i, with high probability.

Lemma 1 ([Kau+16, Lem. 19])**.** *For any $\delta \in (0, 0.0005)$, with probability at least $1-\delta$, the event*

$$\mathcal{E}_\alpha := \{|\widehat{\tau}_i(t) - \tau_i| \le \alpha_i, \quad \text{for all } i \in [n] \text{ and for all } t \ge 1\} \tag{12}$$

occurs. The statement continues to hold for any $\delta \in (0,1)$ with $\alpha_i = \alpha(T_i) = \sqrt{\frac{\beta(T_i,\delta')}{2T_i}}$, and $\beta(t,\delta') = 2\log(125\log(1.12t)/\delta')$.

Underlying Lemma 1 is a non-asymptotic version of the law of the iterated logarithm [Kau+16; Jam+14].

We first show that, on the event $\mathcal{E}_\alpha$ defined in equation (12), the Hamming-LUCB algorithm returns sets $\widehat{\mathcal{S}}_1$ and $\widehat{\mathcal{S}}_2$ obeying $\mathrm{D}(\widehat{\mathcal{S}}_\ell, \mathcal{S}_\ell) \le 2h$ for $\ell = 1, 2$, as desired. Indeed, suppose that $\{(1), \ldots, (k-h)\} \subseteq \mathcal{S}_1$. This implies that $\mathcal{S}_1$ and $\widehat{\mathcal{S}}_1$ differ in at most h values, which in turn implies that $\mathcal{S}_2$ and $\widehat{\mathcal{S}}_2$ differ by at most h values. Therefore, $\mathrm{D}(\widehat{\mathcal{S}}_\ell, \mathcal{S}_\ell) \le 2h$ for $\ell = 1, 2$. Next, suppose that $\{(1), \ldots, (k-h)\} \nsubseteq \mathcal{S}_1$. Then, at least one item in $\{(1), \ldots, (k-h)\}$ is in $\mathcal{S}_2$. Thus, on $\mathcal{E}_\alpha$, the termination condition (5) implies that $\{(k+1+h), \ldots, (n)\} \subset \{k+1, \ldots, n\} = \mathcal{S}_2$. Similarly as above, this in turn implies that $\mathrm{D}(\widehat{\mathcal{S}}_\ell, \mathcal{S}_\ell) \le 2h$ for $\ell = 1, 2$.

We next show that on the event $\mathcal{E}_\alpha$, Hamming-LUCB terminates after the desired number of comparisons. Let $\gamma := \frac{\tau_{k-h} + \tau_{k+1+h}}{2}$, and define the event that item i is bad as

$$\mathcal{E}_{\text{bad}}(i) = \begin{cases} \widehat{\tau}_i < \gamma + 3\alpha_i, & i \in \{1, \ldots, k-h\} \\ \widehat{\tau}_i > \gamma - 3\alpha_i, & i \in \{k+1+h, \ldots, n\} \\ \alpha_i > \frac{\tau_{k-h} - \tau_{k+1+h}}{4}, & \text{otherwise.} \end{cases}$$

Lemma 2. *If $\mathcal{E}_\alpha$ occurs and the termination condition (5) is false, then either $\mathcal{E}_{bad}(b_1)$ or $\mathcal{E}_{bad}(b_2)$ occurs.*

Given Lemma 2, we can complete the proof in the following way. For an item i, define

$$\Delta_i = \begin{cases} \tau_i - \tau_{k+1+h}, & i \in \{1, \ldots, k-h\} \\ \tau_{k-h} - \tau_i & i \in \{k+1+h, \ldots, n\} \\ \tau_{k-h} - \tau_{k+1+h}, & \text{otherwise,} \end{cases}$$

and let $\tilde{T}_i$ be the largest integer u satisfying the bound $\alpha(u) \le \Delta_i/4$. A simple calculation (see Section 6.1.1 for the details) yields that

$$\text{On the event } \mathcal{E}_\alpha, \text{ if } T_i \ge \tilde{T}_i \text{ holds, then } \mathcal{E}_{\text{bad}}(i) \text{ is false.} \tag{13}$$

Let $t \ge 1$ be the t-th iteration of the steps in the LUCB algorithm, and let b_1 and b_2 be the two items selected in Step 5 of the algorithm. Note that in each iteration only those two items are compared to other items. By

Lemma 2, we can therefore bound the total number comparisons by

$$
\begin{aligned}
2\sum_{t=1}^{\infty} \mathbb{1}\{\mathcal{E}_{\mathrm{bad}}(b_1) \cup \mathcal{E}_{\mathrm{bad}}(b_2)\} &\leq 2\sum_{t=1}^{\infty}\sum_{i=1}^{n} \mathbb{1}\{(i = b_1 \cup i = b_2) \cap \mathcal{E}_{\mathrm{bad}}(i)\} \\
&\overset{\text{(i)}}{\leq} 2\sum_{t=1}^{\infty}\sum_{i=1}^{n} \mathbb{1}\{(i = b_1 \cup i = b_2) \cap T_i \leq \tilde{T}_i\} \\
&\overset{\text{(ii)}}{\leq} 2\sum_{i=1}^{n} \tilde{T}_i. \qquad (14)
\end{aligned}
$$

For inequality (i), we used the fact (13), and inequality (ii) follows because $T_i(t) \leq \tilde{T}_i$ can only be true for $\tilde{T}_i$ iterations t.

We conclude the proof by noting that the definition of $\alpha(\cdot)$ and some algebra yields (see [Hec+16, Eq. (20)]) that for c_1 sufficiently large

$$
\tilde{T}_i \leq \frac{c_1}{(\Delta_i/4)^2}\log\left(\frac{n}{\delta}\log\left(\frac{2}{(\Delta_i/4)^2}\right)\right) \leq c_2\log\left(\frac{n}{\delta}\right)\frac{\log(2\log(2/\Delta_i))}{\Delta_i^2}.
$$

Applying this inequality to the RHS of equation (14) concludes the proof.

6.1.1 Proof of fact (13)

First, consider an item $i \in \{k+1+h, \ldots, n\}$. We show that if $T_i \geq \tilde{T}_i$, then $\mathcal{E}_{\mathrm{bad}}(i)$ is false. On the event $\mathcal{E}_\alpha$,

$$
\widehat{\tau}_i(\tilde{T}_i) + \alpha(\tilde{T}_i) \leq \tau_i + 2\alpha(\tilde{T}_i) \overset{\text{(i)}}{\leq} \tau_i + \frac{\Delta_i}{2} = \gamma + \frac{\Delta_i}{2} - \frac{\tau_{k-h} - \tau_i + \tau_{k+1+h} - \tau_i}{2} \leq \gamma, \qquad (15)
$$

where inequality (i) follows from $\alpha(T_i) \leq \Delta_i/4$ for $T_i \geq \tilde{T}_i$, by definition of $\tilde{T}_i$, and the last inequality follows from $\Delta_i = \tau_{k-h} - \tau_i$ and $\tau_{k+1+h} - \tau_i \geq 0$. Thus, $\mathcal{E}_{\mathrm{bad}}(i)$ does not occur.

For an item $i \in \{1, \ldots, k-h\}$, $\mathcal{E}_{\mathrm{bad}}(i)$ that is false, the argument is equivalent. For an item in the middle $i \in \{k-h+1, \ldots, k+h\}$, the event $\mathcal{E}_{\mathrm{bad}}(i)$ is false by definition. This concludes the proof.

6.1.2 Proof of Lemma 2

We prove the lemma by considering all different values the indices b_1 and b_2 selected by the LUCB algorithm can take on, and showing that in each case $\mathcal{E}_{\mathrm{bad}}(b_1)$ and $\mathcal{E}_{\mathrm{bad}}(b_2)$ cannot occur simultaneously. For notational convenience, we define the indices

$$
m_1 = \operatorname*{arg\,max}_{i \in \{(k-h+1), \ldots, (k)\}} \alpha_i, \quad \text{and} \quad m_2 = \operatorname*{arg\,max}_{i \in \{(k+1), \ldots, (k+h)\}} \alpha_i,
$$

and note that

$$
b_1 = \operatorname*{arg\,max}_{i \in \{d_1, m_1\}} \alpha_i, \quad \text{and} \quad b_2 = \operatorname*{arg\,max}_{i \in \{d_2, m_2\}} \alpha_i.
$$

1. Suppose that $b_1 \in \{1, \ldots, k-h\}$ and $b_2 \in \{k+1+h, \ldots, n\}$, and that both $\mathcal{E}_{\mathrm{bad}}(b_1)$ and $\mathcal{E}_{\mathrm{bad}}(b_2)$ do not occur. First note that

$$
\widehat{\tau}_{d_1} - \alpha_{d_1} \geq \widehat{\tau}_{b_1} - \alpha_{b_1}. \qquad (16)
$$

In order to establish this claim, note that the inequality holds trivially with equality if $b_1 = d_1$. If $b_1 = m_1$, then it follows from $\widehat{\tau}_{d_1} \geq \widehat{\tau}_{m_1}$ and $\alpha_{d_1} \leq \alpha_{b_1}$. Thus, we obtain

$$
\widehat{\tau}_{d_1} - \alpha_{d_1} \geq \widehat{\tau}_{b_1} - \alpha_{b_1} > \gamma, \qquad (17)
$$

where the last inequality holds by the assumption that $\mathcal{E}_{\mathrm{bad}}(b_1)$ does not occur. An analogous argument yields that

$$
\gamma > \widehat{\tau}_{b_2} + \alpha_{b_2} \geq \widehat{\tau}_{d_2} + \alpha_{d_2}. \qquad (18)
$$

Combining those inequalities yields $\widehat{\tau}_{d_1} - \alpha_{d_1} > \widehat{\tau}_{d_2} + \alpha_{d_2}$, which contradicts that the termination condition (5) is false.

2. Next, suppose that b_1 is an index in the middle and b_2 is in the very bottom, i.e., $b_1 \in \{k-h+1, \ldots, k+h\}$, and $b_2 \in \{k+1+h, \ldots, n\}$, and both $\mathcal{E}_{\mathrm{bad}}(b_1)$ and $\mathcal{E}_{\mathrm{bad}}(b_2)$ do not occur.

   First note that from $b_2 \in \{k+1+h, \ldots, n\}$ and $\mathcal{E}_{\mathrm{bad}}(b_2)$ not occurring, we have that

$$\gamma \geq \widehat{\tau}_{b_2} + 3\alpha_{b_2} \overset{\text{(i)}}{\geq} \widehat{\tau}_{d_2} + \alpha_{d_2} + 2\alpha_{b_2} \overset{\text{(ii)}}{\geq} \widehat{\tau}_i + \alpha_i + 2\alpha_{b_2}, \quad \text{for } i \in \{(k+1+h), \ldots, (n)\} .$$

   Here, inequality (i) holds by $\widehat{\tau}_{b_2} \geq \widehat{\tau}_{d_2}$ and $\alpha_{b_2} \geq \alpha_{d_2}$, and inequality (ii) follows by the definition of d_2. On the event $\mathcal{E}_\alpha$, this implies

$$\gamma \geq \tau_i + 2\alpha_{b_2}. \tag{19}$$

   Inequality (19) can only be true for all $i \in \{(k+1+h), \ldots, (n)\}$ if $\gamma - \tau_{k+1+h} \geq 2\alpha_{b_2}$, which is equivalent to

$$\alpha_{b_2} \leq \frac{\Delta}{4}, \quad \Delta := \tau_{k-h} - \tau_{k+1+h}.$$

   Again using that $b_2 \in \{k+1+h, \ldots, n\}$ and $\mathcal{E}_{\mathrm{bad}}(b_2)$ not occurring, we have that

$$\gamma \geq \widehat{\tau}_{d_2} + \alpha_{d_2} \overset{\text{(i)}}{\geq} \widehat{\tau}_{d_1} - \alpha_{d_1} \overset{\text{(ii)}}{\geq} \widehat{\tau}_{d_1} - \frac{\Delta}{4}, \tag{20}$$

   where inequality (i) holds since the termination condition (5) is false, and inequality (ii) follows from $\alpha_{d_1} \leq \alpha_{b_1} \leq \frac{\Delta}{4}$, where the last inequality holds since $\mathcal{E}_{\mathrm{bad}}(b_2)$ does not occur, by assumption.
   From $\widehat{\tau}_{d_1} \geq \widehat{\tau}_i$ for all $i \in \{(k-h+1), \ldots, (n)\}$, it follows that for $i \in \{d_1\} \cup \{(k-h+1), \ldots, (n)\}$,

$$\gamma > \widehat{\tau}_i - \frac{\Delta}{4} \geq \tau_i - \alpha_i - \frac{\Delta}{4}. \tag{21}$$

   Below, we show that

$$\alpha_i \leq \frac{\Delta}{4}, \quad \text{for all } i \in \{d_1\} \cup \{(k-h+1), \ldots, (k+h)\}. \tag{22}$$

   It follows that

$$\gamma > \tau_i - \frac{\Delta}{4} - \frac{\Delta}{4} \Leftrightarrow \tau_{k-h} > \tau_i, \quad \text{for all } i \in \{d_1\} \cup \{(k-h+1), \ldots, (k+h)\}. \tag{23}$$

   Together with equation (19), this yields that $\tau_{k-h} > \tau_i$ for all $i \in \{d_1\} \cup \{(k-h+1), \ldots, (n)\}$, which is a contradiction. This concludes the proof.

   It remains to establish the claim (22). From the bound $\alpha_{b_2} \leq \frac{\Delta}{4}$, as shown above, we have $\frac{\Delta}{4} \geq \alpha_{b_2} \geq \alpha_{m_2} \geq \alpha_i$ for all $i \in \{(k+1), \ldots, (k+h)\}$, by defintion of m_2. Moreover, for $i \in \{(k-h), \ldots, (k)\}$, we have $\alpha_i \leq \alpha_{d_1} \leq \alpha_{b_1} \leq \frac{\Delta}{4}$, where the last inequality holds since b_1 is in the middle and is not bad. This concludes the proof of (22).

3. The case where b_1 lies in the very top and b_2 lies in the middle, i.e., $b_1 \in \{1, \ldots, k-h\}$ and $b_2 \in \{k-h+1, \ldots, k+h\}$, and both $\mathcal{E}_{\mathrm{bad}}(b_1)$ and $\mathcal{E}_{\mathrm{bad}}(b_2)$ do not occur, can be treated analogously as the previous case.

4. Next suppose that both b_1 and b_2 lie in the middle, i.e., $b_1, b_2 \in \{k-h+1, \ldots, k+h\}$ and both $\mathcal{E}_{\mathrm{bad}}(b_1)$ and $\mathcal{E}_{\mathrm{bad}}(b_2)$ do not occur. We show that this leads a contradiction.

   Towards this goal, first note that either

$$\gamma < \widehat{\tau}_{d_2} + \alpha_{d_2}. \tag{24a}$$

   holds true or

$$\gamma > \widehat{\tau}_{d_1} - \alpha_{d_1}, \tag{24b}$$

   holds true, but not both. In order to see this fact, note that if inequality (24a) is violated, then

$$\gamma \geq \widehat{\tau}_{d_2} + \alpha_{d_2} \overset{\text{(i)}}{<} \widehat{\tau}_{d_1} - \alpha_{d_1}, \tag{25}$$

where step (i) follows from the termination condition (5) being false, by assumption. Likewise, if inequality (24b) does not hold, then

$$\gamma \leq \widehat{\tau}_{d_1} - \alpha_{d_1} < \widehat{\tau}_{d_2} + \alpha_{d_2}.$$

Thus, we have shown that either condition (24a) or (24a) holds true, but not both simultaneously; consequently, we may conclude that at least one of these two conditions does *not* hold. Next, we show that this fact leads to a contradiction, which concludes the proof.

First, suppose that inequality (24a) *does not* hold true. Then by the definition of d_2, on $\mathcal{E}_\alpha$,

$$\gamma \geq \widehat{\tau}_{d_2} + \alpha_{d_2} \geq \widehat{\tau}_i + \alpha_i \geq \tau_i, \quad \text{for all } i \in \{(k+1+h), \ldots, (n)\}. \tag{26}$$

Moreover, by inequality (25) together with $\mathcal{E}_{\text{bad}}(b_1)$ and $\mathcal{E}_{\text{bad}}(b_2)$ not occurring, which implies that $\alpha_{b_1}, \alpha_{b_2} \leq \frac{\Delta}{4}$, the following inequality follows by the same argument as inequality (23) follow from inequality (20):

$$\tau_{k-h} > \tau_i, \quad \text{for all } i \in \{d_1\} \cup \{(k-h+1), \ldots, (k+h)\}. \tag{27}$$

Together with (26), this yields a contradiction. The argument for the case in which condition (24a) is true is entirely analogous.

5. Finally, if $b_1 \in \{k+1+h, \ldots, n\}$ or if $b_2 \in \{1, \ldots, k-h\}$, and both $\mathcal{E}_{\text{bad}}(b_1)$ and $\mathcal{E}_{\text{bad}}(b_2)$ do not occur, we reach a contradiction using similar arguments as in the previous cases.

6.2 Proof of Theorem 2

We now turn to the proof of the lower bound from Theorem 2.

We first introduce some notation required to state a useful lemma [Kau+16, Lem. 1] from the bandit literature. Let $\nu = \{\nu_j\}_{j=1}^m$ be a collection of m probability distributions, each supported on the real line $\mathbb{R}$. Consider an algorithm $\mathcal{A}$, that, at times $t = 1, 2, \ldots$, selects the index $i_t \in [m]$ and receives an independent draw X_t from the distribution ν_{i_t} in response. Algorithm $\mathcal{A}$ may select i_t only based on past observations, that is, i_t is $\mathcal{F}_{t-1}$ measurable, where $\mathcal{F}_t$ is the σ-algebra generated by $i_1, X_{i_1}, \ldots, i_t, X_{i_t}$. Algorithm $\mathcal{A}$ has a stopping rule ξ that determines the termination of $\mathcal{A}$. We assume that ξ is a stopping time measurable with respect to $\mathcal{F}_t$ and obeying $\mathbb{P}[\xi < \infty] = 1$.

Let $N_i(\xi)$ denote the total number of times index i has been selected by the algorithm $\mathcal{A}$ (until termination). For any pair of distributions ν and ν', we let $\mathrm{KL}(\nu, \nu')$ denote their Kullback-Leibler divergence, and for any $p, q \in [0, 1]$, let $d(p, q) := p \log \frac{p}{q} + (1-p) \log \frac{1-p}{1-q}$ denote the Kullback-Leiber divergence between two binary random variables with success probabilities p, q.

With this notation, the following lemma relates the cumulative number of comparisons to the uncertainty between the actual distribution ν and an alternative distribution ν'.

Lemma 3 ([Kau+16, Lem. 1])**.** *Let ν, ν' be two collections of m probability distributions on $\mathbb{R}$. Then for any $\mathcal{E} \in \mathcal{F}_\xi$ with $\mathbb{P}_\nu[\mathcal{E}] \in (0,1)$, we have*

$$\sum_{i=1}^m \mathbb{E}_\nu[N_i(\xi)] \, \mathrm{KL}(\nu_i, \nu_i') \geq d(\mathbb{P}_\nu[\mathcal{E}], \mathbb{P}_{\nu'}[\mathcal{E}]). \tag{28}$$

Let us now use Lemma 3 to prove Theorem 2.

Define the event

$$\mathcal{E} := \left\{ \mathrm{D}(\widehat{\mathcal{S}}_\ell, \mathcal{S}_\ell) \leq 2h \text{ for } \ell = 1, 2 \right\},$$

corresponding to success of the algorithm $\mathcal{A}$. Recalling that ξ is the stopping rule of algorithm $\mathcal{A}$, we are guaranteed that $\mathcal{E} \in \mathcal{F}_\xi$. Given the linear relations $M_{ij} = 1 - M_{ji}$, the pairwise comparison matrix M is determined by the entries $\{M_{ij}, i = 1, \ldots, n, \ j = i+1, \ldots, n\}$. Let $N_{ij}(\xi)$ be the total number of comparisons between items i and j made by $\mathcal{A}$. For any other pairwise comparison matrix $M' \in \mathcal{C}_0$, Lemma 3 ensures that

$$\sum_{i=1}^n \sum_{j=i+1}^n \mathbb{E}_M[N_{ij}] \, d(M_{ij}, M'_{ij}) \geq d(\mathbb{P}_M[\mathcal{E}], \mathbb{P}_{M'}[\mathcal{E}]). \tag{29}$$

Let $\mathcal{M} := \{m_1, \ldots, m_{2h+1}\}$ be a set of distinct items in $\mathcal{S}_1$. We next construct $M' \in \mathcal{C}_{1/8}$ such that $m_1, \ldots, m_{2h+1} \notin \mathcal{S}_1(M')$ under the distribution M'. Since we assume algorithm $\mathcal{A}$ to be uniformly (h, δ)-Hamming-accurate over $\mathcal{C}_{1/8}$, we have both $\mathbb{P}_M[\mathcal{E}] \geq 1 - \delta$ and $\mathbb{P}_{M'}[\mathcal{E}] \leq \delta$. To see this note that since $\mathcal{S}_1$ and $\mathcal{S}_1(M')$ differ in $2h+1$ elements, there is no set of cardinality k that differs from both $\mathcal{S}_1$ and $\mathcal{S}_1(M')$ in only h elements. It follows that

$$d(\mathbb{P}_M[\mathcal{E}], \mathbb{P}_{M'}[\mathcal{E}]) \geq d(\delta, 1-\delta) = (1-2\delta)\log\frac{1-\delta}{\delta} \geq \log\frac{1}{2\delta}, \tag{30}$$

where the last inequality holds for $\delta \leq 0.15$.

It remains to specify the alternative matrix $M' \in \mathcal{C}_0$. The alternative matrix M' is defined as

$$M'_{ij} = \begin{cases} M_{mj} - \frac{n-1}{n-1-2h}(\tau_m - \tau_{k+1+2h}), & \text{if } i = m \text{ for } m \in \mathcal{M}, j \in [n] \setminus \mathcal{M} \\ M_{im} + \frac{n-1}{n-1-2h}(\tau_m - \tau_{k+1+2h}), & \text{if } j = m \text{ for } m \in \mathcal{M}, i \in [n] \setminus \mathcal{M} \\ M_{ij} & \text{otherwise.} \end{cases} \tag{31}$$

It follows that, for $m \in \mathcal{M}$,

$$\begin{aligned} \tau'_m &= \frac{1}{n-1}\sum_{j\in[n]\setminus\{m\}} M'_{mj} \\ &= \frac{1}{n-1}\sum_{j\in[n]\setminus\{m\}} M_{mj} - \frac{1}{n-1}\sum_{j\in[n]\setminus\mathcal{M}} \frac{n-1}{n-1-2h}(\tau_m - \tau_{k+1+2h}) \\ &= \tau_{k+1+2h}. \end{aligned}$$

Similarly, all other scores τ'_i are larger than τ_i by a common constant, that is, for $i \in [n] \setminus \mathcal{M}$,

$$\tau'_i = \tau_i + \frac{1}{n-1-2h}\sum_{m\in\mathcal{M}}(\tau_m - \tau_{k+1+2h}).$$

It follows that, under the distribution M' the items in the set $\mathcal{M}$ are not among the k highest scoring items, which ensures that $\mathcal{M} \cap \mathcal{S}_1(M) = \emptyset$. Moreover, $M' \in \mathcal{C}_{1/8}$. This follows from the assumption $M \in \mathcal{C}_{3/8}$, which implies

$$M'_{mj} \leq \frac{5}{8} + \left(\frac{5}{8} - \frac{3}{8}\right) \leq \frac{7}{8},$$

and similarity $M'_{mj} \geq \frac{1}{8}$.

Next consider the total number of comparisons of item m with all others items, that is, $N_m = \sum_{j\in[n]\setminus\{m\}} N_{mj}$. By linearity of expectation, we have

$$\begin{aligned} \sum_{m\in\mathcal{M}} \max_{j\in[n]\setminus\{m\}} d(M_{mj}, M'_{mj})\mathbb{E}_M[N_m] &= \sum_{m\in\mathcal{M}} \max_{j\in[n]\setminus\{m\}} d(M_{mj}, M'_{mj}) \sum_{j'\in[n]\setminus\{m\}} \mathbb{E}_M[N_{mj'}] \\ &\overset{\text{(i)}}{\geq} \sum_{m\in\mathcal{M}}\sum_{j\in[n]\setminus\{m\}} \mathbb{E}_M[N_{mj}]\, d(M_{mj}, M'_{mj}) \\ &\overset{\text{(ii)}}{=} \sum_{i=1}^{n}\sum_{j=i+1}^{n} \mathbb{E}_M[N_{ij}]\, d(M_{ij}, M'_{ij}) \\ &\overset{\text{(iii)}}{\geq} d(\mathbb{P}_M[\mathcal{E}], \mathbb{P}_{M'}[\mathcal{E}]) \\ &\geq \log\frac{1}{2\delta}. \end{aligned} \tag{32}$$

Here steps (i) and (ii) follows from the fact that $d(M_{ij}, M'_{ij}) = 0$ for all (i,j) not in $\{(m,j) \mid m \in \mathcal{M}, j \in [n]\setminus\mathcal{M}\}$ and not in $\{(i,m) \mid m \in \mathcal{M}, i \in [n] \setminus \mathcal{M}\}$, by definition of the M'_{ij} (see equation (31)), and step (iii) follows from inequality (29) (that is, from Lemma 3). Finally, inequality (32) follows from inequality (30).

We next upper bound the KL divergence on the left hand side of inequality (32). Using the inequality $\log x \leq x-1$ valid for $x > 0$, we have that

$$d(M_{mj}, M'_{mj}) \leq \frac{(M_{mj} - M'_{mj})^2}{M'_{mj}(1 - M'_{mj})} \leq d_m, \quad d_m := 16(\tau_m - \tau_{k+1+2h})^2. \tag{33}$$

Here, the last inequality follows from the definition of M' in equation (31), for $j \in [n] \setminus \{m\}$, and from $\frac{1}{8} \leq M'_{mj} \leq \frac{7}{8}$, which implies $\frac{1}{M'_{mj}(1-M'_{mj})} \leq 16$. Applying inequality (33) to the left hand side of inequality (32) yields

$$\sum_{m \in \mathcal{M}} d_m \mathbb{E}_M\left[N_m\right] \geq \log \frac{1}{2\delta}, \quad \text{valid for each subset } \mathcal{M} \subseteq \mathcal{S}_1 \text{ of cardinality } 2h+1. \tag{34}$$

We can therefore obtain a lower bound on $\sum_{i \in \mathcal{S}_1} \mathbb{E}_M\left[N_m\right]$ by solving the minimization problem:

$$\underset{e_m \geq 0}{\text{minimize}} \sum_{m \in \mathcal{S}_1} e_m \quad \text{subject to} \quad \sum_{m \in \mathcal{M}} d_m e_m \geq \log \frac{1}{2\delta} \quad \text{for each subset } \mathcal{M} \subseteq \mathcal{S}_1 \text{ of cardinality } 2h+1. \tag{35}$$

Since the d_m are decreasing in m, the solution to this optimization problem is $e_{k-2h}, \ldots, e_k = 0$ and $e_m = \log(1/2\delta)/d_m$.

Using an analogous line of arguments for items in the set $\mathcal{S}_2$, we arrive at the following lower bound

$$\log \frac{1}{2\delta} \left(\sum_{i=1}^{k-2h} \frac{1}{8(\tau_i - \tau_{k+1+2h})} + \sum_{i=k+1+2h}^{n} \frac{1}{8(\tau_{k-2h} - \tau_i)} \right)$$

on the number of comparisons. This concludes the proof.

6.3 Alternative lower bound

In this section, we state a second lower bound on the number of comparisons, which shows that to obtain an (h, δ)-Hamming accurate ranking, an algorithm has to compare *each* item a certain number of times. The proof of this lower bound also forms the foundation for the proof of Theorem 3.

Theorem 4. *Let $\mathcal{A}$ be a symmetric algorithm, i.e., its distribution of comparisons commutes with permutations of the items, that is uniformly (h, δ)-Hamming accurate over $\mathcal{C}$, with $\delta \leq \frac{1}{2} \min(\frac{1}{k}, \frac{1}{n-k})$. Choose an integer $q \geq 1$. Then, for any item $a \in [n]$, when applied to a given pairwise comparison model $M \in \mathcal{C}$, the algorithm $\mathcal{A}$ must make at least*

$$\frac{2}{3} \left(\frac{2q-1}{2h+q} \right)^2 \Big/ \left(\max_{b \in \{k-2(h+q), \ldots, k+1+2(h+q)\}} \max \left(\max_{j \neq \{a,b\}} d(M_{aj}, M_{bj}), d(M_{ab}, 1/2) \right) \right)$$

comparisons on average.

In the remainder of this section, we provide a proof of Theorem 4. For a given item a, we divide our proof into two cases, corresponding to whether or not $\mathbb{P}\left[a \notin \widehat{\mathcal{S}}_1\right] > c_1 + \eta$, where we define the scalar $\eta := \frac{1-c_1-c_2}{2}$.

Case 1: First, suppose that $\mathbb{P}\left[a \notin \widehat{\mathcal{S}}_1\right] > c_1 + \eta$. Pick some other item b in $\{k-h', \ldots, k\}$ that obeys $\mathbb{P}\left[b \notin \widehat{\mathcal{S}}_1\right] \leq c_1$. The following lemma guarantees that such an item exists:

Lemma 4. *Let $\mathcal{A}$ be an algorithm that is (h, δ)-Hamming accurate, with $\delta \leq \frac{1}{2} \min(\frac{1}{k}, \frac{1}{n-k})$. Let $\widehat{\mathcal{S}}_1$ and $\widehat{\mathcal{S}}_2$ be $\mathcal{A}$'s estimate of the top k items $\mathcal{S}_1$ and the bottom $n-k$ items $\mathcal{S}_2$, respectively. Choose constants c_1, c_2 and h' such that $h + \frac{1}{2} \leq c_1 h'$, $h + \frac{1}{2} \leq c_2 h'$, and $c_1 + c_2 < 1$. Then*

i) there exists an item $b \in \{k-h', \ldots, k\}$ such that $\mathbb{P}\left[b \notin \widehat{\mathcal{S}}_1\right] \leq c_1$, and *ii) there exists an item* $b' \in \{k+1, \ldots, k+1+h'\}$ *such that* $\mathbb{P}\left[b' \in \widehat{\mathcal{S}}_1\right] \leq c_2$.

We use Lemma 3 from Kaufmann et al., which relates the expected number of comparisons to the uncertainty between the actual distribution M and an alternative distribution M' about the events $\mathcal{E}_a := \{a \notin \widehat{\mathcal{S}}_1\}$ and $\mathcal{E}_b = \{b \notin \widehat{\mathcal{S}}_1\}$. Concretely, define the alternative matrix M' as

$$M'_{ij} = \begin{cases} M_{bj}, & i = a, j \in [n] \setminus \{a,b\} \\ M_{ib}, & j = a, i \in [n] \setminus \{a,b\} \\ 1/2, & i = a \text{ and } j = b, \text{ or } i = b \text{ and } j = a \\ M_{i,j}, & \text{otherwise.} \end{cases} \tag{36}$$

Since the algorithm $\mathcal{A}$ is invariant to permutations of the labels, by assumption, we have that $\mathbb{P}_{M'}[\mathcal{E}_a] = \mathbb{P}_{M'}[\mathcal{E}_b]$, since a and b have the same distribution under the distribution specified by M', and we assume $\mathcal{A}$ to be symmetric. Moreover, by construction of M', we have

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} \mathbb{E}_\nu\left[N_{ij}(\xi)\right] \mathrm{KL}(M_{ij}, M'_{ij}) = \sum_{j \notin \{a,b\}} \mathbb{E}_M\left[N_{aj}(\xi)\right] \mathrm{KL}(M_{aj}, M_{bj}) + \mathbb{E}_M\left[N_{ab}(\xi)\right] \mathrm{KL}(M_{aj}, 1/2).$$

Applying Lemma 3 from Kaufmann et al. (see equation (29)) then yields

$$\begin{aligned} \sum_{j \notin \{a,b\}} & \mathbb{E}_M\left[N_{aj}(\xi)\right] \mathrm{KL}(M_{aj}, M_{bj}) + \mathbb{E}_M\left[N_{ab}(\xi)\right] \mathrm{KL}(M_{aj}, 1/2) \\ & \geq \max\{d(\mathbb{P}_\nu[\mathcal{E}_a], \mathbb{P}_{\nu'}[\mathcal{E}_a]), d(\mathbb{P}_\nu[\mathcal{E}_b], \mathbb{P}_{\nu'}[\mathcal{E}_b])\} \\ & = \max\{d(\mathbb{P}_\nu[\mathcal{E}_a], \mathbb{P}_{\nu'}[\mathcal{E}_a]), d(\mathbb{P}_\nu[\mathcal{E}_b], \mathbb{P}_{\nu'}[\mathcal{E}_a])\} \\ & \geq \min_{p \in [0,1]} \max\{d(\mathbb{P}_\nu[\mathcal{E}_a], p), d(\mathbb{P}_\nu[\mathcal{E}_b], p)\} \\ & \geq \frac{2}{3}\eta^2, \end{aligned} \tag{37}$$

where the last inequality follows from Lemma 5 stated below, together with $\mathbb{P}\left[a \notin \widehat{\mathcal{S}}_1\right] - \mathbb{P}\left[b \notin \widehat{\mathcal{S}}_1\right] \geq \eta$, which follows from $\mathbb{P}\left[a \notin \widehat{\mathcal{S}}_1\right] > c_1 + \eta$ and $\mathbb{P}\left[b \notin \widehat{\mathcal{S}}_1\right] \leq c_1$.

Lemma 5. *For scalar* $p_a, p_b \in [0,1]$*, let* $\bar{p}$ *denote their average* $\frac{p_a+p_b}{2}$*. Then,*

$$d(p_a, \bar{p}) + d(p_b, \bar{p}) \geq \min_{p \in [0,1]} \max\{d(p_a, p), d(p_b, p)\} \geq \frac{1}{2}(d(p_a, \bar{p}) + d(p_b, \bar{p})). \tag{38}$$

Moreover, if $p_b - p_a \geq \eta$*, then*

$$\frac{1}{2}(d(p_a, \bar{p}) + d(p_b, \bar{p})) \geq \frac{2}{3}\eta^2. \tag{39}$$

Case 2: Turning to the other case, suppose that $\mathbb{P}\left[a \notin \widehat{\mathcal{S}}_1\right] \leq c_1 + \eta$. Pick some other item b' in $\{k+1, \ldots, k+1+h'\}$ obeying $\mathbb{P}\left[b' \notin \widehat{\mathcal{S}}_1\right] > 1 - c_2$, and note that Lemma 4 ensures that such an item exits. Using a line of argument analogous to that above, we find that

$$\sum_{j \notin \{a,b\}} \mathbb{E}_M\left[N_{aj}(\xi)\right] \mathrm{KL}(M_{aj}, M_{b'j}) + \mathbb{E}_M\left[N_{ab}(\xi)\right] \mathrm{KL}(M_{aj}, 1/2) \geq \frac{2}{3}\eta^2. \tag{40}$$

Here we used that Lemma 5 together with the lower bound $\mathbb{P}\left[a \in \widehat{\mathcal{S}}_1\right] - \mathbb{P}\left[b' \in \widehat{\mathcal{S}}_1\right] \geq (1 - c_1 - \eta) - c_2 = \eta$, which in turn follows from the relations $\mathbb{P}\left[a \notin \widehat{\mathcal{S}}_1\right] \leq c_1 + \eta$, $\mathbb{P}\left[b' \notin \widehat{\mathcal{S}}_1\right] > 1 - c_2$, and $1 - c_1 - c_2 = 2\eta$.

Combining inequalities (37) and (40) yields

$$\max_{b \in \{k-h', \ldots, k+1+h'\}} \left\{ \sum_{j \notin \{a,b\}} \mathbb{E}_M\left[N_{aj}(\xi)\right] \mathrm{KL}(M_{aj}, M_{bj}) + \mathbb{E}_M\left[N_{ab}(\xi)\right] \mathrm{KL}(M_{aj}, 1/2) \right\} \geq \frac{2}{3}\eta^2. \tag{41}$$

Choosing $h' = 2(h+q)$ and $c_1 = c_2 = 1/2 - \eta$ concludes the proof.

6.3.1 Proof of Lemma 4

Since $\mathcal{A}$ is (h,δ)-accurate, we have

$$\sum_{i\in\mathcal{S}_1}\mathbb{P}\left[i\notin\widehat{\mathcal{S}}\right]\le h+\delta k\le h+\frac{1}{2}\le c_1 h',$$

where the last inequality holds by assumption. Thus, there are at most h' many $i\in\mathcal{S}_1$ with $\mathbb{P}\left[i\notin\widehat{\mathcal{S}}\right]\ge c_1$, which implies that for at least $k-h'$ many items $i\in\mathcal{S}_1$, we have that $\mathbb{P}\left[i\notin\widehat{\mathcal{S}}\right]\le c_1$. This in turn implies that there is at least one item $b\in\{k-h',\dots,k\}$ obeying $\mathbb{P}\left[b\notin\widehat{\mathcal{S}}\right]\le c_1$.

Likewise, assuming that $\mathcal{A}$ is (h,δ)-accurate, we have

$$\sum_{i\in\mathcal{S}_2}\mathbb{P}\left[i\in\widehat{\mathcal{S}}\right]\le h+\delta(n-k)\le h+\frac{1}{2}\le c_2 h'.$$

Then there exists at least one arm $b'\in\{k+1,\dots,k+1+h'\}$ such that $\mathbb{P}\left[b'\in\widehat{\mathcal{S}}_1\right]\le c_2$.

6.3.2 Proof of Lemma 5

We start with proving inequality (38). Observe that, since $d(x,y)\ge 0$, we have

$$\min_{p\in[0,1]}(d(p_a,p)+d(p_b,p)\ge\min_{p\in[0,1]}\max\{d(p_a,p),d(p_b,p)\}\ge\min_{p\in[0,1]}\frac{1}{2}(d(p_a,p)+d(p_b,p)).$$

Hence, it suffices to show that $\min_{p\in[0,1]}\frac{1}{2}(d(p_a,p)+d(p_b,p))=\frac{1}{2}(d(p_a,\bar{p})+d(p_b,\bar{p}))$. To this end, define the binary entropy $H(q):=-q\log q-(1-q)\log(1-q)$. We then have

$$\begin{aligned}\frac{1}{2}(d(p_a,p)+d(p_b,p))&=-\frac{1}{2}(H(p_a)+H(p_b))+\bar{p}\log\frac{1}{p}+(1-\bar{p})\log\frac{1}{1-p}\\&=-\frac{1}{2}(H(p_a)+H(p_b))+H(\bar{p})+\bar{p}\log\frac{\bar{p}}{p}+(1-\bar{p})\log\left(\frac{1-\bar{p}}{1-p}\right)\\&=-\frac{1}{2}(H(p_a)+H(p_b))+H(\bar{p})+d(\bar{p},p),\end{aligned}$$

which is minimized by taking $p=\bar{p}$, for which $d(\bar{p},\bar{p})=0$. We can then expand

$$\begin{aligned}-H(\bar{p})&=\frac{p_a+p_b}{2}\log(\bar{p})+\left(1-\frac{p_a+p_b}{2}\right)\log\left(1-\bar{p}\right)\\&=\frac{p_a}{2}\log(\bar{p})+\frac{1-p_a}{2}\log(1-\bar{p})+\frac{p_b}{2}\log(\bar{p})+\frac{1-p_b}{2}\log(1-\bar{p}).\end{aligned}$$

Thus

$$\begin{aligned}-\frac{1}{2}(H(p_a)+H(p_b))+H(\bar{p})&=\frac{1}{2}(-H(p_a)-p_a\log(\bar{p})-(1-p_a)\log(1-\bar{p}))\\&\quad+\frac{1}{2}(-H(p_b)-p_b\log(\bar{p})-(1-p_b)\log(1-\bar{p}))\\&=\frac{1}{2}\{d(p_a,\bar{p})+d(p_b,\bar{p})),\end{aligned}$$

as needed.

We next prove inequality (39). We have

$$\begin{aligned}d(p_a,\bar{p})&=d\left(p_a,\frac{p_b-p_a}{2}+p_a\right)\\&\ge\min_{p\in[0,1]}d(p,\eta/2+p)=d(1/2+\eta/4,1/2-\eta/4)=\frac{\eta}{2}\log\left(\frac{1/2+\eta/4}{1/2-\eta/4}\right)\\&\ge\frac{\eta}{2}\left(1-\frac{1/2-\eta/4}{1/2+\eta/4}\right)\ge\frac{2}{3}\eta^2,\end{aligned}$$

where the second to last, and the last inequality follow from $\log x \geq 1 - 1/x$ and $\eta \in [0,1]$, respectively. This concludes the proof of inequality (39).

6.4 Proof of Theorem 3

The proof is analogous to that of the proof of Theorem 4 in Section 6.3, and only requires minor changes. Specifically, we only need to show that for a given pairwise comparison matrix $M \in \mathcal{C}_{\mathrm{PAR}(\Phi)} \cap \mathcal{C}_{M_{\min}}$, we can construct an alternative matrix obeying equality (36), that lies in $\mathcal{C}_{\mathrm{PAR}(\Phi)} \cap \mathcal{C}_{M_{\min}}$ as well.

Consider any parametric pairwise comparison matrix $M \in \mathcal{C}_{\mathrm{PAR}(\Phi)} \cap \mathcal{C}_{M_{\min}}$. Then there exists a parameter vector $w \in \mathbb{R}^n$ such that $M_{ij} = \Phi(w_i - w_j)$. For the items $a, b \in [n]$, in the proof of Theorem 4, define a set of alternative parameters as

$$w_i' := \begin{cases} w_b & \text{if } i = a \ , \\ w_i & \text{otherwise.} \end{cases}$$

Now let M' be the matrix with pairwise comparison probabilities $M'_{ij} = \Phi(w'_i - w'_j)$. Note that $M' \in \mathcal{C}_{\mathrm{PAR}(\Phi)} \cap \mathcal{C}_{M_{\min}}$, and observe that it obeys equality (36), as desired.

Thus, the proof of Theorem 4 yields that for any item a, when applied to a given pairwise comparison model $M \in \mathcal{C}_{\mathrm{PAR}(\Phi)} \cap \mathcal{C}_{M_{\min}}$, the algorithm $\mathcal{A}$ must make at least

$$\begin{aligned} &\frac{2}{3}\left(\frac{2q-1}{2h+q}\right)^2 \Big/ \left(\max_{b \in \{k-2(h+q),\ldots,k+1+2(h+q)\}} \max\left(\max_{j \neq \{a,b\}} d(M_{aj}, M_{bj}), d(M_{ab}, 1/2)\right)\right) \\ &\leq \frac{2}{3}\left(\frac{2q-1}{2h+q}\right)^2 \Big/ \left(\frac{2\phi_{\max}^2}{M_{\min}\phi_{\min}^2}(\tau_a - \tau_b)^2\right) \end{aligned}$$

comparisons on average. Here, the last inequality follows from [Hec+16, Eq. (31)], which holds for any $i \in [n]$:

$$d(M_{ia}, M'_{ib}) \leq \frac{2\phi_{\max}^2}{M_{\min}\phi_{\min}^2}(\tau_a - \tau_b)^2. \tag{42}$$

Moreover, we used that

$$d(M_{ab}, 1/2) \leq \frac{2\phi_{\max}^2}{M_{\min}\phi_{\min}^2}(\tau_a - \tau_b)^2, \tag{43}$$

which follows along the lines as [Hec+16, Eq. (31)]. This concludes the proof.

A Proof of equation (9)

Equation (9) follows by upper bounding the terms in

$$N_h^{\mathrm{up}}(M) = \widetilde{O}\left(\sum_{i=1}^{k-3h} \Delta_{i,k+1+3h}^{-2} + \sum_{i=k+1+3h}^{n} \Delta_{k-3h,i}^{-2} + 2(3h)\Delta_{k-3h,k+1+3h}^{-2}\right).$$

Specifically, if $i_1 < i_2$ and $j_2 > j_1$, then $\Delta_{i_1,j_1} \leq \Delta_{i_1,j_1}$. Therefore, the terms above can be upper bounded by

$$\sum_{i=1}^{k-3h} \Delta_{i,k+1+3h}^{-2} \leq \sum_{i=1}^{k-3h} \Delta_{i,k+1+2h}^{-2}, \qquad \sum_{i=k+1+3h}^{n} \Delta_{k-3h,i}^{-2} \leq \sum_{i=k+1+3h}^{n} \Delta_{k-2h,i}^{-2},$$

$$\text{and } 2h\Delta_{k-3h,k+1+3h}^{-2} \leq \sum_{i=k-3h+1}^{k-2h} \Delta_{i,k-2h+1}^{-2} + \sum_{i=k+2h+1}^{k-3h} \Delta_{k-2h,i}^{-2}.$$