A Example of Weighted Vornoi Cells

Fig. 2 shows answers we get from a weak oracle to the query $\mathcal{O}_z(x, y)$ for points z in different regions of a 2-dimensional Euclidean space.

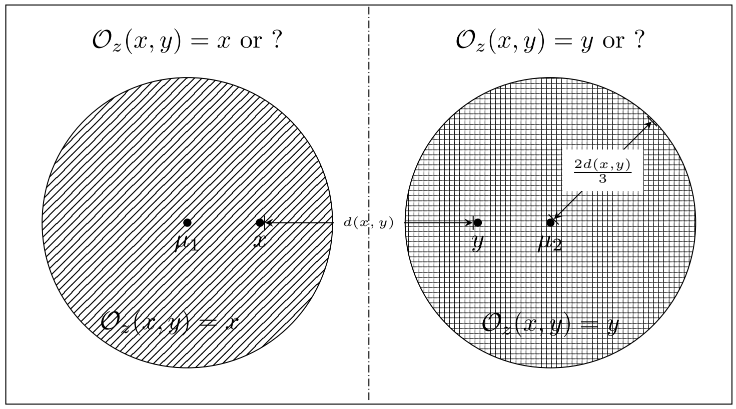

Figure 2: Voronoi cells with a multiplicative distance $\alpha = 2$ for a two dimensional Euclidean space. This figure partition the space into three parts based on the potential answers from the oracle.

B Proof of Lemma 1

Proof. To prove this lemma, we show that any ball $B_x(2R)$ could be covered by at most $k \le c^4$ balls of radius R.

Lemma 3. *An ϵ-net of $\mathcal{A}$ is an ϵ-cover of $\mathcal{A}$.*

Proof. Let $\{x_1, x_2, \cdots, x_k\}$ be an ϵ-net of $\mathcal{A}$. Suppose that there exists $x \in \mathcal{A}$ such that $x \notin \bigcup_{i=1}^k B_{x_i}(\epsilon)$. Then we know that $d(x, x_i) > \epsilon$ for all i. This contradicts the maximality of the ϵ-net. □

Assume $\{x_1, x_2, \cdots, x_k\}$ is an R-net of the ball $B_x(2R)$. We know for all $i \neq j$, $B_{x_i}(R/2) \cap B_{x_j}(R/2) = \varnothing$. Therefore, we have $\mu(\cup_{i=1}^k B_{x_i}(R/2)) = \sum_{i=1}^k \mu(B_{x_i}(R/2))$. Also,

$$\mu(B_{x_i}(R/2)) \ge c^{-3}\mu(B_{x_i}(4R)) \ge c^{-3}\mu(B_x(2R)).$$

Also, $\cup_{i=1}^k B_{x_i}(R/2) \subseteq B_x(4R)$. To sum-up we have

$$\mu(B_x(2R)) \ge c^{-1}\mu(B_x(4R)) \ge c^{-1}\mu(\cup_{i=1}^k B_{x_i}(R/2)) \ge c^{-1}\sum_{i=1}^k \mu(B_{x_i}(R/2)) \ge kc^{-4}\mu(B_x(2R)).$$

□

C Proof of Theorem 1

Proof. To prove this theorem, we show: (a) In each iteration the target t remains in the next version space. Therefore, Worcs-I always find the target t. (b) The μ-mass of the version space shrinks by a factor of at least $1 - c^{-2}$ in each iteration, which results in the total number of $1 + \frac{H(\mu)}{\log(1/(1-c^{-2}))}$ iterations. (c) The number of queries to find the next version space is upper bounded by a polynomial function of the doubling constant c.

(a) Assume c_t is the center of a ball which contains t, i.e., $t \in B_{c_t}(\frac{\Delta}{8(\alpha+1)})$. For all $c_j \neq c_t$ such that $d(c_t, c_j) > \frac{\Delta}{8}$, we have $\mathcal{O}_t(c_t, c_j) = c_t$. This is because $d(c_t, t) \le \frac{\Delta}{8(\alpha+1)}$ and $d(c_j, t) > \frac{\Delta}{8} - \frac{\Delta}{8(\alpha+1)}$, and thus $\alpha d(c_t, t) < d(c_j, t)$.

On the other hand, consider an i such that for all $j \neq i$ with $d(c_i, c_j) > \frac{\Delta}{8}$, we have $\mathcal{O}_t(c_i, c_j) = c_i$. We claim $t \in B_{c_i}(\frac{\Delta(\alpha+2)}{8(\alpha+1)})$. Assume this in not true. Then for c_t we have $d(c_i, c_t) > d(c_i, t) - \frac{\Delta}{8(\alpha+1)} \geq \frac{\Delta}{8}$. Therefore, following the same lines of reasoning as the fist part of the proof, we should have $\mathcal{O}_t(c_i, c_t) = c_t$. This contradicts our assumption.

(b) We have $\mu(\mathcal{V}_{i+1}) \leq (1 - c^{-2})\mu(\mathcal{V}_i)$. To prove this, we first state the following lemma.

Lemma 4. *Assume* $\Delta = \max_{x,y \in \mathcal{V}_i} d(x, y)$. $\forall x \in \mathcal{V}_i$ *we have* $\max_{y \in \mathcal{V}_i} d(x, y) \geq \frac{\Delta}{2}$.

Proof. Assume $x^*, y^* = \arg\max_{x,y \in \mathcal{V}_i} d(x, y)$. By using triangle inequality we have $d(x, x^*) + d(x, y^*) \geq d(x^*, y^*) = \Delta$. We conclude that at least one of $d(x, x^*)$ or $d(x, y^*)$ is larger than or equal to $\frac{\Delta}{2}$. This results in $\max_{y \in \mathcal{V}_i} d(x, y) \geq \max\{d(x, x^*), d(x, y^*)\} \geq \frac{\Delta}{2}$. □

Let's assume c_i is the center of $\mu(\mathcal{V}_{i+1})$ and point c_i^* is the furthest point from c_i. From Lemma 4 we know that $d(c_i, c_i^*) \geq \frac{\Delta}{2}$. Also, it is straightforward to see $B_{c_i}(\frac{\Delta(\alpha+2)}{8(\alpha+1)}) \cap B_{c_i^*}(\frac{\Delta}{4}) = \emptyset$. From the definition of expansion rate we have $\mu(B_{c_i^*}(\frac{\Delta}{4})) \geq c^{-2}\mu(B_{c_i^*}(\Delta)) \geq c^{-2}\mu(\mathcal{V}_i)$.

(c) Let's consider $t \in \operatorname{supp} \mathcal{M}$ as the target. WORCS-I locates the target t, provided $\mu(\mathcal{V}_i) \leq (1-c^{-2})^i \mu(\mathcal{V}_0) \leq \mu(t)$ or equivalently $i \geq 1 + \frac{\log \mu(t)}{\log(1-c^{-2})} \geq \lceil \frac{\log(\mu(t))}{\log(1-c^{-2})} \rceil$. The expected number of iterations is then upper bounded by $\sum_{t \in \operatorname{supp}(\mathcal{M})} \mu(t) \left(1 + \frac{\log \mu(t)}{\log(1-c^{-2})}\right) = 1 + \frac{H(\mu)}{\log(1/(1-c^{-2}))}$. Finally, from Lemma 1, we know that we can cover the version space $\mathcal{V}_i$ with at most $c^{4\lceil \log 8(\alpha+1) \rceil}$ balls of radius $\frac{\Delta}{8(\alpha+1)}$. Note that in the worst case we should query the center of each ball versus centers of all the other balls in each iteration. □

D Proof of Theorem 2

Proof. Let $S_1 \triangleq \mathcal{V}_i \setminus Vor(y, x, \mathcal{V}_i), S_2 \triangleq \mathcal{V}_i \setminus Vor(x, y, \mathcal{V}_i)$ and $S_3 \triangleq \mathcal{V}_i \setminus (Vor(x, y, \mathcal{V}_i) \cup Vor(y, x, \mathcal{V}_i))$. We denote the distance between x and y by $r \triangleq d(x, y)$. Assume Δ is the largest distance between any two pints in $\mathcal{V}_i$, i.e., $\Delta \triangleq \operatorname{diam}(\mathcal{V}_i)$. We have $\beta = \Delta/r$ for $0 \leq \beta \leq 1$. We condition on the target element $t \in \operatorname{supp}(\mathcal{M})$.

We first prove that $\mu(\mathcal{V}_i) \leq (1-c_{\text{strong}}^{-l})^i \mu(\mathcal{V}_0) = (1-c_{\text{strong}}^{-l})^i$. Note that we have $2^l \cdot \frac{r}{\alpha+1} \geq \frac{\alpha+1}{\beta} \cdot \frac{r}{\alpha+1} \geq \Delta$. The first step is to show that $Vor(x, y, \mathcal{V}_i) \supseteq B_x(\frac{r}{\alpha+1})$. For any element $v \in B_x(\frac{r}{\alpha+1})$, we have $d(x, v) \leq \frac{r}{\alpha+1}$. Therefore, $\alpha d(x, v) \leq \frac{\alpha r}{\alpha+1} \leq r - d(x, v) = d(x, y) - d(x, v) \leq d(y, v)$, which yields immediately that $v \in Vor(x, y, \mathcal{V}_i)$. As a result,

$$\mu(Vor(x, y, \mathcal{V}_i)) \geq \mu(B_x(\frac{r}{\alpha+1})) \geq c_{\text{strong}}^{-l}\mu(B_x(2^l \cdot \frac{r}{\alpha+1})) \geq c_{\text{strong}}^{-l}\mu(B_x(D)) \geq c_{\text{strong}}^{-l}\mu(\mathcal{V}_i).$$

We deduce that $\mu(S_2) = \mu(\mathcal{V}_i) - \mu(Vor(x, y, \mathcal{V}_i)) \leq (1 - c_{\text{strong}}^{-l})\mu(\mathcal{V}_i)$. Similarly, we have $\mu(Vor(y, x, \mathcal{V}_i)) \geq c_{\text{strong}}^{-l}\mu(\mathcal{V}_i)$ and $\mu(S_1) \leq (1 - c_{\text{strong}}^{-l})\mu(\mathcal{V}_i)$. In addition, $\mu(S_3) \leq (1 - c_{\text{strong}}^{-l})\mu(\mathcal{V}_i)$. To sum up, we have $\max_{1 \leq j \leq 3} \mu(S_j) \leq (1 - c_{\text{strong}}^{-l})\mu(\mathcal{V}_i)$.

The search process ends after at most i iterations provided $\mu(\mathcal{V}_i) \leq (1 - c_{\text{strong}}^{-l})^i \mathcal{V}_\prime \leq \mu(t)$, or equivalently $i \geq 1 + \frac{\log \mu(t)}{\log(1-c_{\text{strong}}^{-l})}$. The average number of iterations is then bounded from above by

$$\sum_{t \in \operatorname{supp}(\mathcal{M})} \mu(t) \left(1 + \frac{\log \mu(t)}{\log(1 - c_{\text{strong}}^{-l})}\right) \leq 1 + \frac{H(\mu)}{\log(1/(1 - c_{\text{strong}}^{-l}))}.$$

Also, in each iteration we need to query only one pair of objects. □

E Proof of Lemma 2

Proof. We first prove that we can always find at least one point with this property. Define $x^* = \arg\max_{z \in \mathcal{V}_i} d(x, z)$. From Lemma 4, we know that $d(x, x^*) \geq \frac{\Delta}{2}$. We claim there is no $z \neq x^*$ such that $x \in Vor(x^*, z, \mathcal{V}_i)$. Assume there is a z. This means $\alpha d(x, x^*) \leq d(x, z)$, where it contradicts with the choice of x^*. This means that the set of points with this property is not empty. If $y = z^*$ then we are done with the proof, because $d(x, x^*) \geq \frac{\Delta}{2} \geq \frac{\Delta}{2\alpha}$. Next, we prove that for any $y \neq x^*$ with this property, we have $d(x, y) \geq \frac{\Delta}{2\alpha}$. Assume $y \neq x^*$. We know $x \notin Vor(y, x^*, \mathcal{V}_i)$. Therefore, we have $\alpha d(x, y) \geq d(x, x^*) \geq \frac{\Delta}{2}$ and $d(x, y) \geq \frac{\Delta}{2\alpha}$. □