A Supplementary Material

A.1 Relaxation on Local Polytope

The relaxation of (1) over the *local polytope* is given by:

$$\begin{aligned}
\min_{\mu} \quad & \sum_{u\in V}\sum_{i\in L}\mu_u(i)c(u,i) + \sum_{e=(u,v)}\sum_{i,j}\mu_e(ij)\theta_{(u,v)}(i,j) \\
\text{s.t.} \quad & \sum_i \mu_u(i) = 1, && \forall i \in L. \\
& \sum_j \mu_e(ij) = \mu_u(i), && \forall e=(u,v)\in E, i \in L. \\
& \sum_i \mu_e(ij) = \mu_v(j), && \forall e=(u,v)\in E, j \in L. \\
& \mu_u(i) \geq 0, && \forall u \in V,\ i \in L. \\
& \mu_e(ij) \geq 0, && \forall e \in E,\ i,j \in L.
\end{aligned}$$

For a Ferromagnetic Potts Model, the objective becomes:

$$\min_{\mu}\sum_{u\in V}\sum_{i\in L}\mu_u(i)c(u,i)+\sum_{e=(u,v)}w(u,v)\sum_{i,j}\mu_e(ij)\mathbb{1}(i\neq j)$$

Fix the values $\mu_u(i)$. We want to minimize

$$\sum_{e=(u,v)}w(u,v)\sum_{i,j}\mu_e(ij)\mathbb{1}(i\neq j)$$

subject to the constraints

$$\begin{aligned}
& \sum_j \mu_e(ij) = \mu_u(i), && \forall e=(u,v)\in E, i \in L. \\
& \sum_i \mu_e(ij) = \mu_v(j), && \forall e=(u,v)\in E, j \in L. \\
& \mu_e(ij) \geq 0, && \forall e \in E,\ i,j \in L.
\end{aligned}$$

Because $w(u,v) \geq 0$ and $\mu_e(ij) \geq 0$, we want to put as much mass on $\mu_e(ii)$ as possible without violating a constraint, since those terms do not appear in the objective. To that end, we set $\mu_e(ii) = \min(\mu_u(i), \mu_v(i))$. Then using the first constraint, the objective becomes:

$$\begin{aligned}
&\sum_{e=(u,v)}w(u,v)\sum_i \mu_u(i) - \min(\mu_u(i),\mu_v(i)) \\
&= \sum_{e=(u,v)}w(u,v)\left(1-\frac{1}{2}\sum_i \mu_u(i)+\mu_v(i)\right. \\
&\left. + \sum_i |\mu_u(i)-\mu_v(i)|\right) \\
&= \sum_{e=(u,v)}w(u,v)\sum_i |\mu_u(i)-\mu_v(i)| \\
&= \sum_{e=(u,v)}w(u,v)\frac{|\mu_u-\mu_v|}{2},
\end{aligned}$$

where we use multiple times that $\sum_i \mu_u(i) = 1$. The LP objective is thus:

$$\min_{\mu}\sum_{u\in V}\sum_{i\in L}\mu_u(i)c(u,i) + \sum_{e=(u,v)}w(u,v)\frac{|\mu_u-\mu_v|}{2}$$

Identifying μ_u with $\bar{u}$ and μ_v with $\bar{v}$, we obtain the LP (3).

A.2 Proof of Lemma 1

Proof. This argument is similar to the one in Angelidakis et al. (2017). First, we verify the last two conditions in Lemma 1. Let $\alpha = \frac{2}{k\theta} = \frac{5}{3}$ and $\beta = k\theta = \frac{6}{5}$. The algorithm clearly returns a feasible solution (i.e. a valid labeling). Consider any two vertices u and v, and let $\Delta = d(u,v)$. There are two cases: $j(u) = j(v)$ and $j(u) \neq j(v)$. In the first case, let $j = j(u) = j(v)$. We have $P(u) \neq P(v)$ exactly when $r \in (\min(\bar{u}_i,\bar{v}_i), \max(\bar{u}_i,\bar{v}_i))]$ and $i \neq j$. r is uniformly distributed in $(0,\theta)$, so the probability of this occurring is

$$\mathbb{P}[P(u)\neq P(v)] = \frac{1}{k}\sum_{i:i\neq j}\frac{|\bar{u}_i-\bar{v}_i|}{\theta} \leq \frac{2}{k\theta}d(u,v) = \alpha\Delta.$$

Note that we used $u_i \leq \varepsilon < \theta$ for $i \neq j$ and for all u. Now consider the case where $j(u) \neq j(v)$. Here $d(u,v) \geq d(e_{j(u)}, e_{j(v)}) - d(u, e_{j(u)}) - d(v, e_{j(v)})$ by the triangle inequality (e_i is the ith standard basis vector in $\mathbb{R}^k$). So $d(u,v) \geq 1-2\varepsilon \geq 1-2/30$ for $k \geq 3$. So $d(u,v) \geq 14/15$, and $\alpha = 5/3$ so $\alpha\Delta > 1$ and the bound trivially applies.

Next we verify the "co-appoximation" condition. First consider the case where $j(u) = j(v) = j$. Then $d(u,v) \leq d(u,e_j) + d(e_j,v) \leq 2\varepsilon \leq 1/15$. As we showed, $\mathbb{P}[P(u)\neq P(v)] \leq \alpha\Delta$. So $\mathbb{P}[P(u) = P(v)] \geq 1-\alpha\Delta \geq \beta^{-1}(1-\Delta)$, where the last inequality is because $\frac{1-\beta^{-1}}{\alpha-\beta^{-1}} = \frac{1/6}{5/3-5/6} = \frac{1}{5} \geq \Delta$. Now assume $j(u) \neq j(v)$. Note that if $\bar{u}_i \geq r$ and $\bar{v}_i \geq r$, u and v are both added to P_i. So

$$\begin{aligned}
\mathbb{P}[P(u) = P(v)] &\geq \mathbb{P}[u_i \geq r, v_i \geq r] \\
&= \frac{1}{k}\sum_{i=1}^{k}\frac{\min(\bar{u}_i,\bar{v}_i)}{\theta}.
\end{aligned}$$

Here we used that for all i, $\min(\bar{u}_i,\bar{v}_i) \leq \varepsilon < \theta$ since $j(u) \neq j(v)$. Then

$$\begin{aligned}
\mathbb{P}[P(u) = P(v)] &\geq \frac{1}{k}\sum_{i=1}^{k}\frac{\bar{u}_i+\bar{v}_i-|\bar{u}_i-\bar{v}_i|}{2\theta} \\
&= \frac{1}{k\theta}(1-d(u,v)) = \beta^{-1}(1-d(u,v)).
\end{aligned}$$

The approximation conditions hold.

Finally, we check the first two conditions of Lemma 1. First consider $\mathbb{P}[P(u) = i, i \neq j(u)]$. This can only occur when i is selected and u is assigned to P_i. So

$$\mathbb{P}[P(u) = i, i \neq j(u)] = \frac{1}{k}\mathbb{P}[\bar{u}_i \geq r] = \frac{1}{k}\frac{\bar{u}_i}{\theta} = \frac{5}{6}\bar{u}_i.$$

Now we compute $\mathbb{P}[P(u) \neq j(u)]$. A vertex u clearly can only be assigned a label $i \neq j(u)$ if such an i is selected and u is assigned to it; namely,

$$\begin{aligned}\mathbb{P}[P(u) \neq j(u)] &= \frac{1}{k}\sum_{i:i\neq j(u)}\frac{\bar{u}_i}{\theta} = \frac{1}{k\theta}(1 - \bar{u}_{j(u)})\\ &= \frac{5}{6}(1 - \bar{u}_{j(u)}).\end{aligned}$$

This concludes the proof. □

A.3 Full Proof of Theorem 1

Here we reproduce the proof of Theorem 1 in more detail.

Theorem. *On a (2,1)-stable instance of* Uniform Metric Labeling *with optimal integer solution g, the LP relaxation* (3) *is tight.*

Proof. Assume for a contradiction that the optimal LP solution $\{\bar{u}^{LP}\}$ of (3) is fractional. To construct a stability-violating labeling, we will run Algorithm 2 on a fractional labeling $\{\bar{u}\}$ constructed from $\{\bar{u}^{LP}\}$ and the optimal integer solution g. We then use Lemma 1 to show that in expectation, the output of $\mathcal{R}(\{\bar{u}\})$ must be better than the optimal integer solution in a particular $(2,1)$-perturbation, which contradicts $(2,1)$-stability.

Let $\{\bar{u}^g\}$ be the solution to (3) corresponding to g, and define the following ε-close solution $\{\bar{u}\}$: for every u and every i, set $\bar{u}_i = (1-\varepsilon)\bar{u}_i^g + \varepsilon\bar{u}_i^{LP}$. Note that $\{\bar{u}\}$ is fractional and $j(u) = g(u)$ for all u.

Recall that E_g is the set of edges cut by the optimal solution g. Define the following $(2,1)$-perturbation w' of the weights w:

$$w'(u,v) = \begin{cases} w(u,v) & (u,v) \in E_g \\ \frac{1}{2}w(u,v) & (u,v) \in E \setminus E_g. \end{cases}$$

We refer to the objective with modified weights w' as Q' (that is, Q' is the objective in the instance with weights w' and costs c).

Now let $h = \mathcal{R}(\{\bar{u}\})$. To compare g and h, we will compute $\mathbb{E}[Q'(g) - Q'(h)]$, where the expectation is over the randomness of the rounding algorithm. By definition,

$$\begin{aligned}\mathbb{E}\left[Q'(g) - Q'(h)\right] &= \mathbb{E}[Q'(g) - Q'(h) | h = g]\Pr(h = g)\\ &\quad + \mathbb{E}[Q'(g) - Q'(h) | h \neq g]\Pr(h \neq g).\end{aligned}$$

The first term of the sum above is clearly zero. Further, as $\{\bar{u}\}$ is fractional, the guarantees in Lemma 1 imply that $\Pr(h \neq g) > 0$. By $(2,1)$-stability of the instance, any labeling $h \neq g$ must satisfy $Q'(h) > Q'(g)$. So stability and fractionality of the LP imply $\mathbb{E}[Q'(g) - Q'(h)] < 0$.

If we compute $\mathbb{E}[Q'(g) - Q'(h)]$ and simplify using Lemma 1 and the definition of w' (see the appendix for a full derivation), we obtain:

$$\begin{aligned}Q'(g) - Q'(h) &= \sum_{u\in V_\Delta} c(u, g(u)) + \sum_{(u,v)\in E_g\setminus E_h} w'(u,v)\\ &\quad - \sum_{u\in V_\Delta} c(u, h(u)) - \sum_{(u,v)\in E_h\setminus E_g} w'(u,v).\end{aligned}$$

Taking the expectation, we obtain:

$$\begin{aligned}\mathbb{E}[Q'(g) - Q'(h)] &= \sum_{u\in V} c(u, g(u))\Pr(h(u) \neq g(u))\\ &\quad + \sum_{(u,v)\in E_g} w'(u,v)\Pr((u,v) \text{ not cut})\\ &\quad - \sum_{u\in V}\sum_{i\neq g(u)} c(u,i)\Pr(h(u) = i)\\ &\quad - \sum_{(u,v)\in E\setminus E_g} w'(u,v)\Pr((u,v) \text{ cut}).\end{aligned}$$

Applying Lemma 1 with $j(u) = g(u)$,

$$\begin{aligned}\mathbb{E}[Q'(g) - Q'(h)] \geq \frac{5}{6}\Bigg(&\sum_{u\in V} c(u, g(u))(1 - \bar{u}_{g(u)})\\ &+ \sum_{(u,v)\in E_g} w'(u,v)(1 - d(u,v))\\ &- \sum_{u\in V}\sum_{i\neq g(u)} c(u,i)\bar{u}_i\\ &- \sum_{(u,v)\in E\setminus E_g} 2w'(u,v)d(u,v)\Bigg)\end{aligned}$$

Using the definition of w',

$$\begin{aligned}\mathbb{E}[Q'(g) - Q'(h)] \geq \frac{5}{6}\Bigg(&\sum_{u\in V} c(u, g(u)) + \sum_{(u,v)\in E_g} w(u,v)\\ &- \sum_{u\in V}\sum_{i\in L} c(u,i)\bar{u}_i - \sum_{(u,v)\in E} w(u,v)d(u,v)\Bigg)\end{aligned}$$

The first two terms are simply $Q(g)$, and the last two are the objective $Q(\{\bar{u}\})$ of the LP solution $\bar{u}$. Since $\bar{u} = (1-\varepsilon)\bar{u}^g + \varepsilon\bar{u}^{LP}$ and $Q(\{\bar{u}^{LP}\}) \leq Q(\{\bar{u}^g\})$, the convexity of the LP objective implies $Q(\{\bar{u}\}) \leq Q(\{\bar{u}^g\}) = Q(g)$. So $\mathbb{E}[Q'(g) - Q'(h)] \geq 0$. But stability of the instance and fractionality of the LP solution implied $\mathbb{E}[Q'(g) - Q'(h)] < 0$. □

A.4 Generating Counterexamples

The following procedure can be used to find (β, γ)-stable instances.

1. Given a fixed number of nodes n and labels k, randomly generate a graph G as follows:
   (a) Connect any two nodes (u, v) with an edge with probability `connectProb`.
   (b) When connecting two nodes, choose the edge weight $w(u, v)$ uniformly at random from $\mathbb{Z} \cap [0, \texttt{weightMax}]$.
2. For each node u, choose an index i uniformly at random from $\{1 \ldots k\}$. Draw $c(u, i)$ uniformly at random from $\mathbb{Z} \cap [0, \texttt{costMax}]$. Set $c(u, j) = 0$ for $j \neq i$.
3. Find the optimal solution g to the instance (G, w, c, L).
4. Let E_g be the set of edges cut by g, and consider the following adversarial perturbation w' of w:
$$w'(u,v) = \begin{cases} \frac{1}{\beta} w(u,v) & (u,v) \in E \setminus E_g \\ \gamma w(u,v) & (u,v) \in E_g \end{cases}$$
   Let Q' be the objective with these modified weights.
5. Enumerate the $k^n - 1$ possible labelings not equal to g. If any of them have $Q'(h) \leq Q'(g)$, return to step 1. Otherwise, print V, E, w, c.

Following this procedure, we can also enforce additional properties of the instance in step 5 before printing it out. For instance, we can enforce that the LP must be fractional on the instance, or that α-expansion must not find the optimal solution. If these additional conditions fail to hold, we return to step 1.

The examples in Section 6 were found with `connectProb` $= 0.5$, `weightMax` $= 4$, `costMax` $= 20$, and then modified for simplicity. Steps 3-5 were repeated for each modification to ensure the resulting instances satisfied the correct stability conditions. In Section 6, $\beta = 1$ and $\gamma = 2$; in Section 6, $\beta = 2$ and $\gamma = 1$.

The following lemma proves that steps 3-5 are sufficient to verify stability.

Lemma A.1. *Let w^* be an arbitrary (β, γ)-perturbation of the weights w, and let w' be the adversarial perturbation for the optimal solution g. Then for any labeling h, $Q^*(h) \leq Q^*(g)$ implies $Q'(h) \leq Q'(g)$. In other words, if a labeling h violates stability in any perturbation, it violates stability in the adversarial perturbation w'.*

Proof. We show that $Q^*(g) - Q^*(h) \leq Q'(g) - Q'(h)$. Let $V_\Delta = \{u \in V \mid g(u) \neq h(u)\}$. Recall that E_g and E_h are the sets of edges cut by g and h, respectively. We compute

$$\begin{aligned} Q'(g) - Q'(h) = &\sum_{u \in V_\Delta} c(u, g(u)) + \sum_{(u,v) \in E_g \setminus E_h} w'(u,v) \\ &- \sum_{u \in V_\Delta} c(u, h(u)) - \sum_{(u,v) \in E_h \setminus E_g} w'(u,v). \end{aligned}$$

Using the definition of w',

$$\begin{aligned} Q'(g) - Q'(h) = &\sum_{u \in V_\Delta} c(u, g(u)) + \sum_{(u,v) \in E_g \setminus E_h} \gamma w(u,v) \\ &- \sum_{u \in V_\Delta} c(u, h(u)) - \sum_{(u,v) \in E_h \setminus E_g} \frac{w(u,v)}{\beta}. \end{aligned}$$

Since w^* is a valid (β, γ)-perturbation, $\frac{1}{\beta} w(u,v) \leq w^*(u,v) \leq \gamma w(u,v)$. Then since all the c's and w's are nonnegative,

$$\begin{aligned} Q'(g) - Q'(h) \geq &\sum_{u \in V_\Delta} c(u, g(u)) + \sum_{(u,v) \in E_g \setminus E_h} w^*(u,v) \\ &- \sum_{u \in V_\Delta} c(u, h(u)) - \sum_{(u,v) \in E_h \setminus E_g} w^*(u,v) \\ = &\; Q^*(g) - Q^*(h). \end{aligned}$$

□