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DOI:
10.1016/j.ejor.2021.02.041

Citation for published version (Harvard):
Mercadier, M & Strobel, F 2021, 'A one-sided Vysochanskii-Petunin inequality with financial applications', *European Journal of Operational Research*, vol. 295, no. 1, pp. 374-377.
https://doi.org/10.1016/j.ejor.2021.02.041

A one-sided Vysochanskii-Petunin inequality with financial applications*

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February 15, 2021

Abstract

We derive a one-sided Vysochanskii-Petunin inequality, providing probability bounds for random variables analogous to those given by Cantelli's inequality under the additional assumption of unimodality, potentially relevant for applied statistical practice across a wide range of disciplines. As a possible application of this inequality in a financial context, we examine refined bounds for the individual risk measure of Value-at-Risk, providing a potentially useful alternative benchmark with interesting regulatory implications for the Basel multiplier.

Keywords: risk analysis; risk management; finance; OR in banking

*We are grateful to two anonymous reviewers for their thoughtful comments, to P. Armand and J.P. Lardy for their detailed technical advice, as well as to C. Alexander, H. Alexandre, B. Düring, F. Fiordelisi, I. Hasan, C. Hurlin, L. Lepetit, J.S. Oberoi, A. Tarazi, W. Wagner and participants at the LAPE-FINEST Spring Workshop 2019 for further constructive comments; the usual disclaimer applies. Substantial parts of this research were carried out while Mercadier was a doctoral researcher at LAPE, Université de Limoges, France, as well as market finance consultant at JPLC SASU, Limoges, France, with thanks to both their support. Declarations of interest: none.

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1 Introduction

There is a wide range of applied contexts where it is of interest to be able to provide measures that appropriately characterize the worst-case behavior of certain random variables. Probability theory provides useful tools for this in the form of the Chebyshev (1867) inequality, as well as its one-sided equivalent, the Cantelli (1928) inequality; they allow bounds on the probability of a random variable taking on certain ranges of values for the most agnostic of distributional assumptions requiring only the first two moments to exist.

For many practical applications, an additional assumption of unimodality may not be overly restrictive,[1] allowing the use of the Vysochanskii & Petunin (1980) inequality to derive similar, but tighter bounds. However, particularly for many applications in the financial area, one-sided inequalities are most relevant in the sense of "how bad can losses get". As a one-sided version of the Vysochanskii-Petunin inequality is, to the best of our knowledge, not readily available in the literature, our paper first proceeds to the derivation of such a one-sided Vysochanskii-Petunin inequality; it allows tighter bounds than the corresponding two-sided version in contexts focussing on a particular tail of a distribution, and may be relevant for applied statistical practice across a wide range of disciplines.

We then move on to potential applications of this one-sided Vysochanskii-Petunin inequality in a financial context. In particular, extending results by Barrieu & Scandolo (2015) that draw on Cantelli's inequality, we examine refined bounds for the common individual risk measure of Value-at-Risk, both theoretically as well as in an empirical context. These may provide a useful alternative benchmark whose simplicity is nevertheless on a par with that of traditional delta-normal methods, and also has interesting regulatory implications for the Basel multiplier introduced in Basel Committee on Banking Supervision (2009), the minimum capital requirement framework for financial institutions' market risk.

Our derivation of the one-sided Vysochanskii-Petunin inequality is now presented in Section 2; Section 3 illustrates its use with financial applications; and Section 3.3 concludes the paper.

[1] See e.g. Mercadier & Lardy (2019) in the context of credit spread approximation.

2 One-sided Vysochanskii-Petunin inequality: derivation

Let X be a unimodal random variable. According to the Vysochanskii-Petunin inequality (Vysochanskii & Petunin, 1980; Pukelsheim, 1994), for any $\alpha \in \mathbb{R}$, we have

$$\mathbb{P}(|X-\alpha| \geq r) \leq \begin{cases} \frac{4}{9}\frac{\rho^2}{r^2} & \text{for } r \geq \sqrt{\frac{8}{3}}\rho, \\ \frac{4}{3}\frac{\rho^2}{r^2} - \frac{1}{3} & \text{otherwise.} \end{cases} \tag{1}$$

where $\rho^2 = E[(X-\alpha)^2]$.

Let us define $Y = X - \mu$, where $\mu = E[X]$. It follows that $E[Y] = 0$ and $V(Y) = V(X) = \sigma^2$. Let $r \geq 0$. For all $u \geq 0$, we have

$$\mathbb{P}(Y \geq r) = \mathbb{P}(Y + u \geq r + u) \leq \mathbb{P}(|Y+u| \geq r+u).$$

According to inequality (1) applied to the variable $Y+u$, with $\alpha = 0$, for all $u \geq 0$ we then have

$$\mathbb{P}(Y \geq r) \leq \begin{cases} f_r(u) & \text{for } r+u \geq \sqrt{\frac{8}{3}(\sigma^2+u^2)}, \\ g_r(u) & \text{otherwise.} \end{cases}$$

where $f_r(u) = \frac{4}{9}\frac{\sigma^2+u^2}{(r+u)^2}$ and $g_r(u) = \frac{4}{3}\frac{\sigma^2+u^2}{(r+u)^2} - \frac{1}{3}$. Since $f_r(u) \geq g_r(u)$ if and only if $(r+u)^2 \geq \frac{8}{3}(\sigma^2+u^2)$, for all $u \geq 0$ we have

$$\mathbb{P}(Y \geq r) \leq \Phi_r(u) := \max\{f_r(u), g_r(u)\}.$$

By using the fact that the two functions f_r and g_r achieve their minimum at $u^* = \sigma^2/r$, we deduce from the last inequality that

$$\mathbb{P}(Y \geq r) \leq \min_{u \geq 0} \Phi_r(u) \leq \Phi_r(u^*)$$

and

$$\begin{aligned}
\Phi_r(u^*) &= \max\{f_r(u^*), g_r(u^*)\} \\
&= \max\left\{\frac{4}{9}\frac{\sigma^2}{r^2+\sigma^2}, \frac{4}{3}\frac{\sigma^2}{r^2+\sigma^2} - \frac{1}{3}\right\} \\
&= \begin{cases} \frac{4}{9}\frac{\sigma^2}{r^2+\sigma^2} & \text{if } \frac{\sigma^2}{r^2+\sigma^2} \leq \dfrac{3}{8}, \\ \frac{4}{3}\frac{\sigma^2}{r^2+\sigma^2} - \frac{1}{3} & \text{otherwise.} \end{cases}
\end{aligned}$$

We conclude that, for a unimodal random variable X, the one-sided Vysochanskii-Petunin inequality holds as follows:[2]

$$\mathbb{P}(X - E[X] \geq r) \leq \begin{cases} \dfrac{4}{9}\dfrac{V(X)}{r^2+V(X)} & \text{for } r^2 \geq \dfrac{5}{3}V(X), \\ \dfrac{4}{3}\dfrac{V(X)}{r^2+V(X)} - \dfrac{1}{3} & \text{otherwise.} \end{cases} \tag{2}$$

3 Some financial applications

While the one-sided Vysochanskii-Petunin inequality (2) derived in Section 2 is general in nature, it can be particularly useful for applications in finance, where one-sided inequalities are most relevant for risk measurement purposes. As unimodality is often a less controversial assumption than normality in this context,[3] our results allow for alternative, more realistic, risk measurement benchmarks whose simplicity is nevertheless on a par with that of traditional delta-normal methods.

[2] To expand on a commonly used rule of thumb, the one-sided Vysochanskii-Petunin inequality (2) implies $\mathbb{P}(X - E[X] \geq n\sigma) \leq 4/(9+9n^2)$ for $n \geq \sqrt{5/3}$, so we can state a one-sided 3σ rule for the unimodal case as $\mathbb{P}(X - E[X] \geq 3\sigma) \leq 4/90 < 0.0445$.

[3] We tested for both unimodality and normality in stock returns for a large sample of 1748 firms in 44 mainly developed countries, for the period 1991Q1 to 2020Q1. We applied Hartigans' dip test for unimodality (Hartigan & Hartigan, 1985) and the Shapiro-Wilk test for normality (Shapiro & Wilk, 1965), with their standard null hypotheses of unimodality and normality, respectively; resulting p-values were compared to a common threshold of $\alpha = 0.05$. Our results showed that for conditional (i.e. GARCH(1,1) filtered) firm returns, the hypothesis of unimodality (normality) was not rejected in 96% (54%) of all cases at the quarterly level (analogous results were obtained for unconditional firm returns). Further details are available from the authors on request; see also the supplementary material in the online appendix.

3.1 Bounds on Value-at-Risk

One of the most influential risk measures developed by market practitioners is Value-at-Risk (VaR), defined as "the worst expected loss over a given horizon under normal market conditions at a given level of confidence" (Jorion 2001).[4] In the context of a short time-horizon, in which losses X are generally assumed as $E[X] = 0$ and $V(X) = \sigma^2$, we can write:

$$\mathbb{P}(X \geq \mathrm{VaR}_\alpha(X)) = \alpha$$

as an implicit definition of the VaR of X at confidence level α.

Barrieu & Scandolo (2015) provide an upper bound of the VaR of X using Cantelli's inequality:

$$\mathrm{VaR}_\alpha(X) \leq \sigma\sqrt{\frac{1}{\alpha} - 1} := \mathrm{VaR}_\alpha^{cant}(X) \tag{3}$$

If losses X are unimodal, we can refine this upper bound using the one-sided Vysochanskii-Petunin inequality (2), which implies

$$\alpha \leq \begin{cases} \frac{4}{9}\frac{\sigma^2}{\mathrm{VaR}_\alpha(X)^2 + \sigma^2} & \text{for } \mathrm{VaR}_\alpha(X) \geq \sqrt{\frac{5}{3}}\sigma, \\ \frac{4}{3}\frac{\sigma^2}{\mathrm{VaR}_\alpha(X)^2 + \sigma^2} - \frac{1}{3} & \text{otherwise.} \end{cases}$$

and hence gives

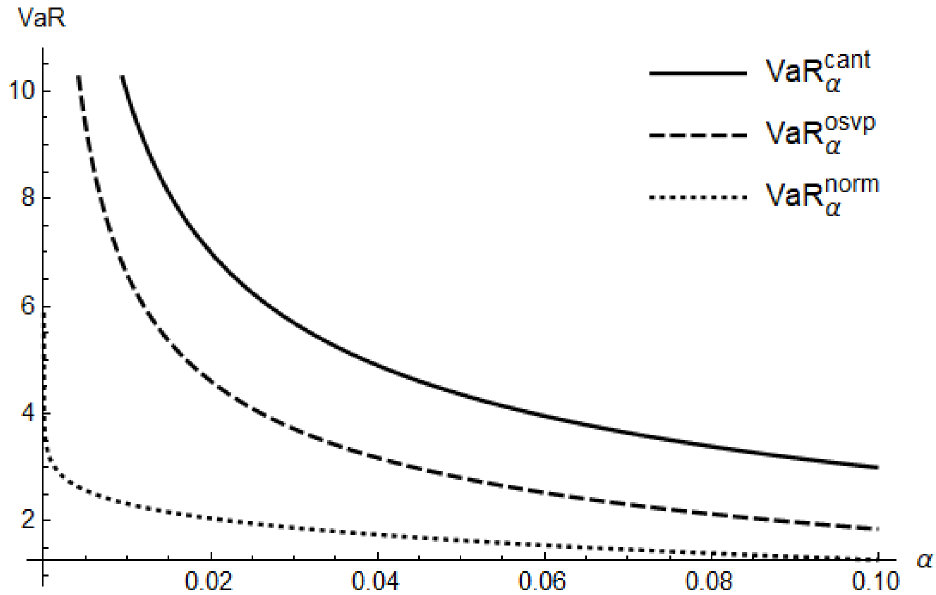
$$\mathrm{VaR}_\alpha(X) \leq \begin{cases} \sigma\sqrt{\frac{4}{9\alpha} - 1} & \text{for } \alpha \leq \frac{1}{6}, \\ \sigma\sqrt{\frac{3(1-\alpha)}{3\alpha + 1}} & \text{otherwise.} \end{cases} \tag{4}$$

Considering that usual values of confidence levels, such as 1 or 5 percent, comfortably satisfy $\alpha \leq 1/6$, this conveniently simplifies to

$$\mathrm{VaR}_\alpha(X) \leq \sigma\sqrt{\frac{4}{9\alpha} - 1} := \mathrm{VaR}_\alpha^{osvp}(X) \tag{5}$$

[4] For some recent contributions, see e.g. Leung et al. (2021), Staino & Russo (2020), Leippold & Vasiljevic (2020), Meng & Taylor (2020), Taylor (2019), Babat et al. (2018), Berkowitz et al. (2011).

Figure 1: (Theoretical) Cantelli/OSVP VaR bounds vs Gaussian baseline (for $\sigma = 1$ & range of confidence levels α)



as the (improved) upper bound of the VaR of X using the one-sided Vysochanskii-Petunin (OSVP) inequality.

The improvement from using the one-sided Vysochanskii-Petunin inequality instead of Cantelli's inequality to provide an upper bound of the VaR of X is substantial, with

$$\frac{\text{VaR}_\alpha^{osvp}(X)}{\text{VaR}_\alpha^{cant}(X)} = \sqrt{\frac{4-9\alpha}{9-9\alpha}}.$$

Note this represents a factor of improvement between 0.662 and 0.644 for common values of confidence levels between 1 and 5 percent. For illustration, Figure 1 plots the (theoretical) Cantelli/OSVP VaR bounds, for a range of confidence levels α, as compared to the Gaussian baseline (for $\sigma = 1$).

Empirical relevance We now examine whether the upper bound of the VaR using the one-sided Vysochanskii-Petunin inequality is consistent with the data. For this, we use a large dataset of stock returns for 1748 firms, representing 44 (mainly developed) countries and components of each country's main index. The stock return data is daily, over 42 years from 1978-01-04 to 2020-04-02, and is extracted from the Bloomberg L.P. database.[5]

The comparison analysis of empirical VaR vs the OSVP VaR bound (5) is carried out on a rolling daily basis; for this, we compute the 1% quantile (for the empirical VaR) and standard deviation (for the OSVP VaR bound) over the previous 60 as well as 260 days. Significantly, over a total of 10,964,478 such comparisons for 60-day windows, and 10,614,878 comparisons for 260-day ones, we record zero instances where the empirical VaR exceeds the OSVP VaR bound at a 99% confidence level, both for conditional, i.e. GARCH(1,1) filtered, as well as unconditional firm returns. The upper bound of the VaR using the one-sided Vysochanskii-Petunin inequality could thus represent a reliable (conservative) educated guess of the VaR for financial market practitioners, computable using easily accessible volatility measures[6] (e.g. for $\alpha = 0.01$, $\mathrm{VaR}_{\alpha}^{osvp}(X) \approx 6.5912\sigma$).

3.2 Implications for the Basel multiplier

The Basel multiplier features in the Basel regulatory framework's allowance for financial institutions to use internal risk models to assess the capital requirement related to market risk. According to the Basel Committee on Banking Supervision (2009) framework, capital requirements can be calculated according to the formula

$$c = \max\left\{\mathrm{VaR}_{t-1}; m_c \cdot \mathrm{VaR}_{avg}\right\} + \max\left\{\mathrm{sVaR}_{t-1}; m_s \cdot \mathrm{sVaR}_{avg}\right\},$$

where sVaR is a stressed VaR measure (using historical data from a 12-month period of significant financial stress), VaR_{avg} and sVaR_{avg} are averages over the preceding sixty business days, and the "multipliers" m_c, m_s are set by individual supervisory authorities (subject to absolute minima of 3 for each).

Barrieu & Scandolo (2015), expanding on previous results by Stahl (1997), highlight that these multipliers can be related to probabilistic bounds giving some upper limit to common risk measures such as the VaR in the context of properly allowing for model risk.

In particular, it proves interesting to compare an upper bound of the VaR of losses X using Cantelli's inequality, stated as inequality (3) above, with the VaR obtained by using the commonly used delta-normal method, which assumes that X is normally distributed

[5] The firms cover the following major sectors: basic material, communications, consumer cyclical, consumer non-cyclical, energy, financial, industrial, utilities, technology and diversified sectors. Further details are available from the authors on request; see also the supplementary material in the online appendix.

[6] Such as e.g. VOLATILITY_260D from Bloomberg L.P.

and gives:

$$\text{VaR}_{\alpha}^{norm}(X) = \sigma \cdot |z_{\alpha}|$$

where $z_{\alpha} = \Phi^{-1}(\alpha)$ is the quantile of a standard normal, for $\alpha < 0.5$. Writing the ratio:

$$\lambda_{VaR}^{cant} = \frac{\text{VaR}_{\alpha}^{cant}(X)}{\text{VaR}_{\alpha}^{norm}(X)} = \frac{1}{|z_{\alpha}|}\sqrt{\frac{1}{\alpha} - 1},$$

we observe that for common values of confidence levels between 1 and 5 percent, the ratio λ_{VaR}^{cant} ranges between 4.277 and 2.650. As this corresponds roughly to the range of [3, 4] for the Basel multipliers m_c, m_s in Basel Committee on Banking Supervision (2009, Section V.4(l)), Barrieu & Scandolo (2015) argue that the Basel multiplier could be motivated such that "if the VaR computed under normal assumptions is multiplied by λ, we obtain an upper bound for the worst possible VaR compatible with partial information (mean and variance)".

If losses X can be assumed to be unimodal, we can refine the upper bound of the VaR using the one-sided Vysochanskii-Petunin inequality, obtained as inequality (5) above (for $\alpha \leq 1/6$), and write the ratio:

$$\lambda_{VaR}^{osvp} = \frac{\text{VaR}_{\alpha}^{osvp}(X)}{\text{VaR}_{\alpha}^{norm}(X)} = \frac{1}{|z_{\alpha}|}\sqrt{\frac{4}{9\alpha} - 1}; \tag{6}$$

in this case, we note that for common values of confidence levels between 1 and 5 percent, the ratio λ_{VaR}^{osvp} would range between 2.833 and 1.708. Clearly, with the assumption of unimodality the ratio is much smaller (by a factor of $\sqrt{(4 - 9\alpha) / (9 - 9\alpha)}$), so that, following similar logic, one could argue that the Basel Multiplier for VaR could be in the range [2, 3], instead of [3, 4], in this case, i.e. be set at considerably less conservative levels.

3.3 Conclusion

We derived a one-sided Vysochanskii-Petunin inequality, giving bounds on the probability of a random variable analogous to those provided by Cantelli's inequality under the additional assumption of unimodality, which may be relevant for applied statistical practice across a wide range of disciplines. We then pursued potential applications of this one-sided Vysochanskii-Petunin inequality in a financial context. In particular, we examined refined bounds for the individual risk measure of Value-at-Risk, both theoretically as well as in an empirical context. These may provide a useful alternative benchmark whose simplicity is nevertheless on a par with that of traditional delta-normal methods, and also has interesting regulatory implications for the Basel multiplier featuring in the minimum capital requirement framework for financial institutions' market risk.

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