SOME PIONEERS OF THE APPLICATIONS OF FRACTIONAL CALCULUS

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Abstract

In the last decades fractional calculus (FC) became an area of intensive research and development. This paper goes back and recalls important pioneers that started to apply FC to scientific and engineering problems during the nineteenth and twentieth centuries. Those we present are, in alphabetical order: Niels Abel, Kenneth and Robert Cole, Andrew Gemant, Andrey N. Gerasimov, Oliver Heaviside, Paul Lévy, Rashid Sh. Nigmatullin, Yuri N. Rabotnov, George Scott Blair.

Key Words and Phrases

Fractional calculus, applications, pioneers, Abel, Cole, Gemant, Gerasimov, Heaviside, Lévy, Nigmatullin, Rabotnov, Scott Blair

1. Introduction

In 1695 Gottfried Leibniz asked Guillaume l'Hôpital if the (integer) order of derivatives and integrals could be extended. Was it possible if the order was some irrational, fractional or complex number? "Dream commands life" and this idea motivated many mathematicians, physicists and engineers to develop the concept of fractional calculus (FC). During four centuries many famous mathematicians contributed to the theoretical development of FC. We can list (in alphabetical order) some important researchers since 1695 (see details at [1, 2, 3], and posters at http://www.math.bas.bg/~fcaa):

- Abel, Niels Henrik (5 August 1802 - 6 April 1829), Norwegian mathematician
- Al-Bassam, M. A. (20th century), mathematician of Iraqi origin
- Cole, Kenneth (1900 - 1984) and Robert (1914 - 1990), American physicists
- Cossar, James (d. 24 July 1998), British mathematician
- Davis, Harold Thayer (5 October 1892 - 14 November 1974), American mathematician
- Djrbashjan, Mkhitar Mkrtichevich (11 September 1918 - 6 May 1994), Armenian (also Soviet Union) mathematician; family name transcripted also as Dzhrbashian, Jerbashian; short CV can be found in *Fract. Calc. Appl. Anal.* **1**, No 4 (1998), 407–414
- Erdélyi, Arthur (2 October 1908 - 12 December 1977), Hungarian-born British mathematician
- Euler, Leonhard (15 April 1707 - 18 September 1783), Swiss mathematician and physicist
- Feller, William (Vilim) (7 July 1906 - 14 January 1970), Croatian-American mathematician
- Fourier, Jean Baptiste Joseph (21 March 1768 - 16 May 1830), French mathematician and physicist
- Gelfand, Israel (Israïl) Moiseevich (2 September 1913 - 5 October 2009), Soviet mathematician (of Jewish origin, born in Russian Empire in a southern Ukrainian town of Okny, moved later to the USA)
- Gemant, Andrew (1895 - 1983), American physicist
- Gerasimov, Andrey Nikolaevich (20th century), Russian (then Soviet) physicist working in the field of mechanics
- Grünwald, Anton K. (1838 - 1920), German mathematician
- Hadamard, Jacques Salomon (8 December 1865 - 17 October 1963), French mathematician
- Hardy, Godfrey Harold (7 February 1877 - 1 December 1947), English mathematician
- Heaviside, Oliver (18 May 1850 - 3 February 1925), English electrical engineer, mathematician, and physicist
- Holmgren, Hjalmar J. (1822 - 1885), Swedish mathematician
- de l'Hôpital, Guillaume François Antoine (1661 - 2 February 1704), French mathematician
- Kilbas, Anatoly Aleksandrovich (20 July 1948 - 28 June 2010), Belarusian mathematician; biographical and memorial notes can be found in *Fract. Calc. Appl. Anal.* **11**, No 4 (2008), an issue dedicated to his 60th anniversary, and in **13**, No 2 (2010), 221–223; a founding editor of this journal with a lot of publications therein

- Kober, Hermann (1888 - 4 October 1973), mathematician born in Poland, studied and worked in Germany, later in Great Britain
- Krug, Anton (19th - 20th century), German mathematician Lacroix,
- Sylvestre François (28 April 1765 - 24 May 1843), French mathematician
- Lagrange, Joseph-Louis (25 January 1736 - 10 April 1813), Italian-French mathematician and astronomer
- Laplace, Pierre-Simon (23 March 1749 - 5 March 1827), French mathematician and astronomer
- Laurent, Paul Matthieu Hermann (2 September 1841 - 19 February 1908), French mathematician
- Leibniz, Gottfried Wilhelm (1 July 1646 - 14 November 1716), German mathematician and philosopher
- Letnikov, Aleksey Vasil'evich (1 January 1837 - 27 February 1888), Russian mathematician; for his contributions, see in [4]
- Lévy, Paul Pierre (15 September 1886 - 15 December 1971), French mathematician
- Liouville, Joseph (24 March 1809 - 8 September 1882), French mathematician
- Littlewood, John Edensor (9 June 1885 - 6 September 1977), British mathematician
- Love, Eric Russel (31 March 1912 - 7 August 2001), British-Australian mathematician
- Marchaud, André (1887 - 1973), French mathematician
- Mikolás, Miklós (20th century), Hungarian mathematician
- Mittag-Leffler, Magnus Gustaf (Gösta) (16 March 1846 - 7 July 1927), Swedish mathematician
- Montel, Paul Antoine Aristide (29 April 1876 - 22 January 1975), French mathematician
- Nagy, Béla Szőkefalvi (29 July 1913 - 22 December 1998), Hungarian mathematician
- Nekrasov, Pavel Alekseevich (13 February 1853 - 20 December 1924), Russian mathematician
- Newton, (Sir) Isaac (4 January 1643 - 31 March 1727), English physicist, mathematician, astronomer, natural philosopher, alchemist, and theologian
- Nigmatullin, Rashid Shakirovich (5 January 1923 - 7 July 1991), Tatarstan, Russian Federation (then Soviet Union) scientist in the field of radio-engineering, radioelectronics, electrochemistry, etc.
- Pincherle, Salvatore (11 March 1853 - 10 July 1936), Italian mathematician

- Post, Emil Leon (11 February 1897 - 21 April 1954), Polish-American mathematician
- Rabotnov, Yury Nikolaevich (24 February 1914 - 15 May 1985), Russian (then Soviet) scientist in the field of mechanics
- Riemann, Georg Friedrich Bernhard (17 September 1826 - 20 July 1866), German mathematician
- Riesz, Marcel (16 November 1886 - 4 September 1969), Hungarian mathematician
- Ross, Bertram (1917 - 27 October 1993), American mathematician
- Scott Blair, George William (1902 - 1987), British chemist
- Shilov, Georgiy Evgen'evich (3 February 1917 - 17 January 1975), Soviet mathematician
- Sneddon, Ian Naismith (8 December 1919 - 4 November 2000), Scottish mathematician
- Sonine, Nikolay Yakovlevich (22 February 1849 - 27 February 1915), Russian mathematician; for his contributions, see in [4]
- Tardy, Placido (23 October 1816 - 2 November 1914), Italian mathematician
- Wallis, John (23 November 1616 - 28 October 1703), English mathematician
- Weierstrass, Karl Theodor Wilhelm (31 October 1815 - 19 February 1897), German mathematician
- Weyl, Hermann Klaus Hugo (9 November 1885 - 8 December 1955), German mathematician
- Widder, David Vernon (25 March 1898 - 8 July 1990), American mathematician
- Zygmund, Antoni (25 December 1900 - 30 May 1992), Polish-born American mathematician

During the last two decades practical implementations emerged for the FC theory and it is now recognized to be an important tool to describe phenomena that classical integer-order calculus neglects, see e.g. [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17].

Nevertheless, it is important to remember and pay honor to the pioneers initiating the first applications of FC to scientific and engineering problems, some of them found below in alphabetical order. Our paper is not intended to have a detailed biographical description, but solely to highlight some of the main contributions in the area of applicable FC. Also, as these authors worked and published before the Internet era and digitalization of literature, for some of them only a few data could be found, and other names are dropped for the same reason.

2. Niels Abel

Niels Henrik Abel was born in 1802 in Nedstrand, Norway. He became a proficient mathematician when he was a teenager, and died when he was only twenty six, with tuberculosis, in 1829. Much of his important work in Analysis became known only after his death: a paper of his on integration was lost, Gauß did not care to read a letter he sent proving a result on fifth-order equations, Cauchy misplaced a paper of his on elliptic equations, and a treatise he wrote on that subject was only discovered after his death. He got a professorship appointment only two days after he died, so he lived in straitness all his life, [18].

Figure 1. Niels Henrik Abel (1802-1829)

His work in Mathematics, particularly in group theory and elliptic functions, is outstanding, especially considering how early in life he died. His connection to FC comes from an application of fractional integrals he made to the problem in Physics which is finding a tautochrone curve, i.e. a curve $y(x)$ such that a mass m at rest on any of its points will always take the same time T to slide down without friction till reaching the final point $(x, y) = (0, 0)$, irrespective of the initial height h at which it begins to move because of the acceleration of gravity g, [19, 20, 6]. While it is known since Huygen's 1659 work on pendulums that the tautochrone curve is the cycloid (Figure 2), this was found from geometrical arguments; Abel's solution, based upon Mathematical Analysis, leads to a fractional integral. Because of the conservation of mechanical energy (due to the nonexis-tence of friction), the initial energy at rest mgh is conserved when the mass is at an arbitrary height y and has the mechanical energy $mgy + \frac{1}{2}mV^2$, and so its velocity V is given by

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$$V = \sqrt{2g(h-y)} \qquad (2.1)$$

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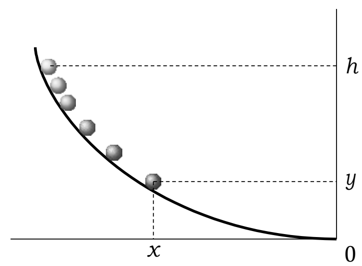

Figure 2. **The cycloid**

$$\ell(y)=\int_0^y\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y, \tag{2.2}$$

and so $V=-\frac{\mathrm{d}R(y)}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}t}$. Equaling this to V above we get

$$\sqrt{2g(h-y)}=-\frac{\mathrm{d}\ell(y)}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}t}\Rightarrow\frac{\mathrm{d}t}{\mathrm{d}y}=-\frac{1}{\sqrt{2g(h-y)}}\frac{\mathrm{d}\ell(y)}{\mathrm{d}y} \tag{2.3}$$

and from here, we arrive at

$$T(h)=\int_0^h\frac{1}{\sqrt{2g(h-y)}}\frac{\mathrm{d}\ell(y)}{\mathrm{d}y}\,\mathrm{d}y. \tag{2.4}$$

We want $T(h)$ to be independent of h: thus, the integral in (2.4) can have any value of y as its upper limit of integration. Let us rewrite it as

$$\sqrt{2g}\,T=\int_0^y(y-v)^{-1/2}\frac{\mathrm{d}\ell(v)}{\mathrm{d}v}\,\mathrm{d}v=\Gamma\left(\frac{1}{2}\right){}_0D_y^{-\frac{1}{2}}\frac{\mathrm{d}\ell(y)}{\mathrm{d}y} \tag{2.5}$$

$$\Rightarrow {}_0D_y^{-\frac{1}{2}}\frac{\mathrm{d}\ell(y)}{\mathrm{d}y}=\sqrt{\frac{2gT^2}{\pi}}\Rightarrow\frac{\mathrm{d}\ell(y)}{\mathrm{d}y}=\frac{\sqrt{2g}\,T}{\pi}y^{-\frac{1}{2}}, \tag{2.6}$$

where the last equality is a consequence of the law of exponents. Making $a=\frac{\sqrt{2g}T}{\pi}$ we can get, from (2.2),

$$\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}=ay^{-\frac{1}{2}}\Rightarrow\frac{\mathrm{d}x}{\mathrm{d}y}=\sqrt{\frac{a^2}{y}-1}. \tag{2.7}$$

The parametric equations of a cycloid generated by a circle of radius r, centred on $(0, r)$, rotating an angle θ on $y=2r$, are

$$\begin{cases} x=r\theta+r\sin\theta \\ y=r-r\cos\theta \end{cases}\Rightarrow x=r\arccos\frac{r-y}{r}+r\sin\arccos\frac{r-y}{r}\Rightarrow$$

$$\frac{\mathrm{d}x}{\mathrm{d}y}=\frac{1}{\sqrt{1-\frac{(r-y)^2}{r^2}}}+\frac{r-y}{2ry-y^2}=\frac{2r-y}{y(2r-y)}=\sqrt{\frac{2r}{y}-1} \tag{2.8}$$

which is the same as (2.7). As the curve begins at the origin, we can obtain an integral representation for the curve given by

$$x(y) = \int_0^y \sqrt{\frac{2r}{v} - 1}\, \mathrm{d}v, \quad 2r = \frac{2g\Gamma^2}{\pi^2}. \qquad (2.9)$$

3. Kenneth Cole and Robert Cole

Kenneth Stewart Cole (Figure 3, left) was born on 10 July 1900 in Ithaca, New York, USA, [21, 22, 23]. He graduated from Oberlin College in 1922 and received a PhD in Physics from Cornell University in 1926. He was a pioneer in the application of physical science to biology and received many academic honours. In 1967 he was awarded the National Medal of Science. In 1973 the Membrane Section of the Biophysical Society (USA) established an annual “Cole Award”. Kenneth Cole had a younger brother, Robert, with whom he remained very close throughout his life. They were joint authors of four papers published between 1936 and 1942, namely [24, 25, 26, 27]. Kenneth Cole passed away in April 18, 1984.

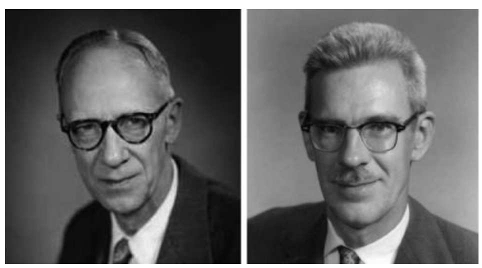

Figure 3. Kenneth Stewart Cole, 1900–1984 (left), and Robert Hugh Cole, 1914–1990 (right)

Robert Hugh Cole (Figure 3, right) was born in 26 October 1914 in Oberlin, Ohio, USA, [35, 36, 37]. After graduating from Oberlin College in 1935, Robert Cole earned his PhD in 1940, and became an Instructor in Physics at Harvard. At Oberlin College, Kenneth S. Cole introduced his brother to research. The collaboration continued during one decade and yielded the famous work on what is now called the Cole-Cole plot, providing a representation of the dielectric properties of a substance. Robert Hugh Cole received also several scientific and academic awards. He died in 17 November 1990.

The electrical model of tissue can be described as an electrical circuit with resistive and capacitive properties. The Cole–Cole model is an empirical description of the measured data that has been successfully applied to a wide variety of tissues, but it does not give any information about the underlying causes of the phenomena being measured. The Cole–Cole equation is a relaxation model given by

$$\epsilon^* = \epsilon_\infty + \frac{\epsilon_s - \epsilon_\infty}{1 + (j\omega\tau_0)^{1-\alpha}}, \qquad (3.1)$$

where E^* is the complex dielectric constant, E_S and E_∞ are the zero (or static) and infinite frequency dielectric constants, ω is the angular frequency, τ_0 is a time constant and $j = \sqrt{-1}$. The parameter α takes a value between 0 and 1, and describes different spectral shapes.

The Cole equation written in terms of electrical impedance Z, instead of a complex permittivity, is given by:

$$Z(\omega) = R_\infty + \frac{R_0 - R_\infty}{1 + \left(j\frac{\omega}{\omega_0}\right)^{1-\alpha}}, \qquad (3.2)$$

where R_∞ and R_0 are the resistances at infinite and zero frequencies, respectively, ω is the angular frequency and ω_0 is a characteristic frequency.

Parameter α is often assumed to have values between 0 and 1, but, in fact, there is no theoretical limitation and we can consider real values outside this interval.

Expression (3.1) makes the locus of the complex dielectric constant E^* a circular arc in the complex plane with end points on the real axis.

The Cole brothers were "indirect" FC pioneers, because they have not used explicitly fractional derivatives or integrals. Nevertheless, their model can be easily interpreted in terms of FC. Furthermore, the Mittag-Leffler function is recalled in their paper [27] making reference to the works of Gross [28, 29] and the Davis [30].

The Cole-Cole relaxation reduces to the Debye model [31] when $\alpha = 0$ and, on the other hand, can be generalized leading to the so-called Havriliak-Negami model, [32]:

$$\epsilon^* = \epsilon_\infty + \frac{\epsilon_s - \epsilon_\infty}{[1 + (j\omega\tau)^{1-\alpha}]^\beta}, \qquad (3.3)$$

where $\beta \in \mathbb{R}$. Notice that whereas (3.1) corresponds to an explicit fractional derivative, (3.3) is a fractional model both explicit and implicit; making $\alpha = 0$ we get a purely implicit fractional model of order β, known as the Davidson-Cole model [33].

The Cole-Cole model [34] was one of the first applications of FC and plays nowadays an important role in many different scientific areas.

More information on the Cole brothers and their contributions can be seen at [35, 36, 37].

4. Andrew Gemant

Andrew Gemant was born in 1895 in Nagyvarad, Austro-Hungarian Empire. He studied medicine, and served as a medical student during World War I, but took his PhD in Physics in 1922. He worked in Germany until he lost his professorship in 1933 due to the policies hostile to non-Germans of the National-Socialist government. After some years in England he moved to the United States in 1938, where he continued his research and his teaching until his death in 1983, [38].

Figure 4. Andrew Gemant (1895-1983)

He developed the concept of a complex viscosity [39], that is to say, of a relationship between deformation and stress that is dynamic, so that the deformation depends on the past history of the applied stress. He verified that experimental data for the dynamic viscosity η' of a cellulose ester supported that it varies with frequency ω as

$$\eta' = \frac{\eta_0}{1 + (\omega\tau_0)^{3/4}}, \qquad (4.1)$$

where η_0 is the stationary viscosity [40]. This fractional transfer function, first published in a series of papers in 1941–1943, was the first viscoelastic relationship proposed, see [41]. His models, uncirculated in the Soviet Union, were redeveloped in the 1960's by Shermergor and Meshkov (see the survey [42]).

When Gemant began using a fractional operator in his models, he said that it "only occurs as a useful mathematical symbol, whereas the underlying elementary process, whatever it may be, will probably contain differential quotients of an integral order." This view was denied, as mentioned below, by Scott Blair, [43].

Gemant made a bequest to the American Institute of Physics, thanks to which an Andrew Gemant Award has been attributed since 1987. It "recognizes the accomplishments of a person who has made significant contributions to the cultural, artistic, or humanistic dimension of physics", [44].

$$\rho\frac{\partial^2 u}{\partial t^2} = k\left({}^C D^{\alpha}_{-,t}\left[\frac{\partial^2 u}{\partial x^2}\right]\right)(x,t) \tag{5.5}$$

and

$$\rho x^3\frac{\partial^2 u}{\partial t^2} = k\frac{\partial}{\partial x}\left(x^3\frac{\partial}{\partial x}({}^C D^{\alpha}_{-,t}u)\right)(x,t) \tag{5.6}$$

5. Andrey Gerasimov

Andrey Nikolaevich Gerasimov was a Russian (then Soviet) famous scientist working in the field of Mechanics in the 20th century.

Gerasimov [45] was probably the first to propose, in 1948, a study of models on the anomalous dynamics of the mechanical behavior of viscoelastic materials. He proposed a kernel relation

$$\tau_{ij} = 2\mu_0\int_0^t K(t-\tau)\,e_{ij}(\tau)\,\mathrm{d}\tau, \quad t>0, \tag{5.1}$$

where τ is the Cauchy stress tensor, μ_0 is the viscosity (for the case without dynamics) and e is the strain tensor to describe the behaviour of viscoelastic materials, see [45, 46, 41]. Thus, by using an appropriate kernel K, his generalization of the classical models implied that the tension σ could be proportional to the Riemann-Liouville fractional derivative D^{α}_{+} to the right of the material deformation E, that is,

$$\sigma(t) = E(D^{\alpha}_{+}\epsilon)(t), \quad 0<\alpha<1. \tag{5.2}$$

Since the 1970s, in the studies of one-dimensional differential equations of fractional order, the so-called modified or regularized fractional derivative, known as Caputo (also Caputo-Dzhrbashjan) derivative, was introduced as more adequate for physical interpretations. For $0 < \alpha < 1$ and initial point 0, it takes the form

$${}^C D^{\alpha}_{0+}y(x) = \frac{1}{\Gamma(1-\alpha)}\int_0^x \frac{y'(t)\,\mathrm{d}t}{(x-t)^{\alpha}}, \quad x>0, \tag{5.3}$$

as introduced by Caputo in 1967 in studies of seismic waves.

Gerasimov was the first (see [47]) who considered differential equations with partial derivatives of fractional order. In [45] he introduced a partial derivative of similar kind as (5.3) but with respect to t on the whole axis,

$${}^C D^{\alpha}_{-,t}y(x,t) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^t \frac{y_{\tau}(x,\tau)\mathrm{d}\tau}{(t-\tau)^{\alpha}}, \quad t>0,\ x\in\mathbb{R}, \quad 0<\alpha<1. \tag{5.4}$$

In the same paper, he studied two new problems in viscoelasticity describing the motion of a viscous fluid between moving surfaces, and reduced these problems to the following two partial differential equations of fractional order $0 < \alpha < 1$ (see [14, 47, 48]):

$$\rho\frac{\partial^2 u}{\partial t^2} = k\left({}^C D^{\alpha}_{-,t}\left[\frac{\partial^2 u}{\partial x^2}\right]\right)(x,t) \tag{5.5}$$

and

$$\rho x^3\frac{\partial^2 u}{\partial t^2} = k\frac{\partial}{\partial x}\left(x^3\frac{\partial}{\partial x}({}^C D^{\alpha}_{-,t}u)\right)(x,t) \tag{5.6}$$

with unknown $u = u(x,t)$, given constants ρ and k, and (what one may call, [47]) a Gerasimov-Caputo partial fractional derivative (5.4).

6. Oliver Heaviside

Oliver Heaviside was a self-taught scientist that studied electrical circuits, invented mathematical techniques, redesigned Maxwell's equations, predicted the existence of the ionosphere and formulated several important concepts in the area of vector analysis, [49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66]. Heaviside developed independently his operational calculus, a technique by which problems with differential equations are transformed into algebraic equations with a differential operator p. Heaviside defined also fractional powers of p, thus establishing a connection between Operational Calculus and FC.

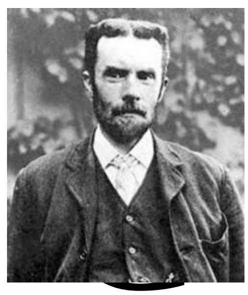

Figure 5. Oliver Heaviside, 1850–1925

Oliver Heaviside (Figure 5) was born in 1850 in London, United Kingdom. He left school at age 16 to study at home the subjects of telegraphy and electromagnetism. When he was 18 years old he took a job as a telegraph operator (the only paid employment he ever had), but in 1874, at age 24, he quitted his job and returned to studying full time on his own. Between 1880 and 1887 Heaviside developed his operational calculus. In 1887 Heaviside proposed that inductors should be added to telephone and telegraph lines to correct the signal's distortion, but the importance of Heaviside's work remained undiscovered for some time after publication of his ideas. AT&T later extended the research and offered Heaviside a financial contribution, but he declined to accept any money unless the company were to give him a proper recognition. In the next years Heaviside studied the electric and magnetic fields using vector calculus. The Maxwell formulation of electromagnetism consisted of 20 equations in 20 variables. Heaviside employed the *curl* and the *div* operators to reformulate the description into four equations and four variables, that is, the form by which they are known nowadays. He developed the transmission line theory and, independently, discovered the Poynting vector. In 1891 the British Royal Society recognized Heaviside's contributions to the modelling of electromagnetism by awarding him as a Fellow of the Royal Society. Moreover, more than fifty pages of the Philosophical Transactions of the Society were devoted to his vector methods and electromagnetic theory. In 1893, before general relativity, Oliver Heaviside addressed the analogies between the equations for electromagnetism and gravitation. In 1902, Heaviside proposed the existence of the Kennelly-Heaviside layer of the ionosphere, by which radio signals are transmitted around the Earth's curvature. (The existence of the ionosphere was confirmed later in 1923.) In 1905 he was given an honorary doctorate by the University of Göttingen, and in 1922, he became the first recipient of the Faraday Medal, which was established that year. Heaviside was not interested in deep mathematical rigour and his techniques seemed to pure mathematicians to be unjustified and outlandish. Heaviside stated his opinion that "Mathematics is an experimental science, and definitions do not come first, but later on". He became bitter as the years went by and, sick of fighting for the recognition, he moved to Torquay, in Southwest England, where he lived as a virtual hermit in his lodgings for 25 years. There he died in 1925. The last, unpublished, volume of his *Electromagnetic Theory* was destroyed by burglars some days after his death. Nevertheless, by the end of 1957, a collection of his papers was found under the floorboards of his room in the house, in Paignton, where he lived. The set consisted of a large number of loose sheets of paper containing his formulæ and notes, many annotated galley proofs, marked page proofs, old letters and postcards, and technical publications. The unearthed papers revealed that Heaviside had extended Maxwell's theory so that gravitation could be fitted in with electromagnetism. It was also found that he had evolved a rigorous justification for his "operation" of extracting the square root of the process of partial differentiation.

In his operational calculus Oliver Heaviside used the symbol p to represent $\frac{d}{dt}$ Since p represented the mathematical operation of differentiation with respect to time, he took $\frac{1}{p}$ to represent the inverse mathematical operation, that is, the integration.

In transmission lines the electrical elements resistance, inductance and capacitance are not discrete, and, in fact, they are distributed continuously along the line length. The model operational equations contained $\sqrt{p}$, which posed new problems in its interpretation. Heaviside paid attention to the problem and wrote: "There is a universe of mathematics lying in between the complete differentiations and integrations." Heaviside was not aware of FC and attacked the problem by trying to generalize his previously established result (where t is time)

$$p^n(t^k) = \frac{k!}{(k-n)!}t^{k-n} \tag{6.1}$$

to fractional values of n.

Considering $k = 0$ and $n = \frac{1}{2}$, he obtained

$$\sqrt{p}\,\mathbf{1} = \frac{1}{\sqrt{\pi t}}, \tag{6.2}$$

where **1** is Heaviside's symbol for the function that is zero before $t = 0$ and one for $t > 0$ (nowadays called Heaviside unit step).

Heaviside established a connection of the result with the solution of a heat-flow problem obtained by classical methods. However, the discussion did not appeal to mathematicians, since Heaviside's arguments did not conform to their standards of rigour.

In the middle of Heaviside's calculations, found at Paignton it was verified that he also arrived at the value of $\sqrt{p}$ by another process. Heaviside obtained:

$$\sqrt{p} = \int_0^\infty f(t)\, e^{-pt}\, p\, dt. \tag{6.3}$$

With $f(t) = \frac{1}{\sqrt{\pi t}}$ and replacing t by x^2, he obtained the Gauß integral.

Heaviside, like other genial minds, had the faculty of being able to discover difficult theorems by intuition, without the usual processes of proof. This gift made him supercilious of formal logic and, on the other hand, lead to a considerable number of enemies. Quoting Oliver Heaviside (replying to criticism over use of operators, before justified formally): "Why should I refuse a good dinner simply because I don't understand the digestive processes involved?"

7. Paul Lévy

Paul Lévy was born in 1886 in Paris, France. Following the footsteps of his father and his grandfather, he became a mathematician, with two interruptions during the World Wars — during the first, he served in the artillery; during the second, he lost his professorship during the Vichy regime for being of Jewish descent and had to live in clandestinity, [67]. He was a major figure in the development of probability theory, for which he felt the need of rigourous mathematical treatment — like mathematicians in the Russian school, and unlike Jules Henri Poincaré, a direct influence on Lévy, who however had a different take on probability, [69].

He became a member of the French Academy of Sciences and of the London Mathematical Society, and received the Legion of Honour. Still, he received recognition for his work rather late, first in the United States, and only then in France. Among his few students (he only supervised one PhD student in his entire life, [67]) we can single out Benoît Mandelbrot, who later played such an important role in the study of fractals, also related to fractional calculus. Lévy died in 1971.

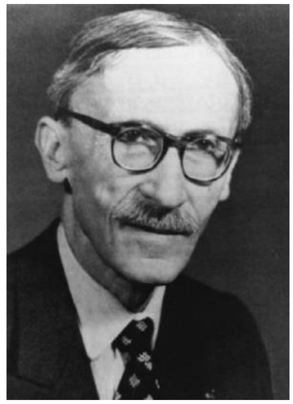

Figure 6. Paul Lévy (1886-1971)

His work is connected to fractional calculus by his study of stochastic processes [70]; indeed, real-valued càdlàg (right-continuous, with left lim- its) stochastic processes $L(t)$ defined for $0 \leq t \leq T$, with (almost surely) $L(0) = 0$, with increments $L(t) - L(s)$ which are independent and sta- tionary (for any $0 \leq s < t \leq T$), and which are stochastically continuous ($\forall \varepsilon > 0 \lim_{s \to t} P(|L(t) - L(s)| > \varepsilon) = 0$, where P denotes probability), are called Lévy processes. Random walks, built from a succession of random steps, are Lévy processes; if the probability distribution that governs the size of the steps is heavy-tailed, the walk is called a Lévy flight. Under mild assumptions, the limit of a random walk as the dimension of the steps tends to zero is a Brownian motion (or rather the Wiener process that models Brownian motion).

Now a continuous-time random walk (CTRW) — when the time at which a random step with probability density function $\lambda(\xi)$ occurs is itself random — may lead to a fractional partial differential equation, if

$$\lambda(\xi) \propto \frac{\sigma^\alpha}{|\xi|^{1+\alpha}}, \; 0 < \alpha < 2, \tag{7.1}$$

$$\langle \xi^2 \rangle = \infty, \tag{7.2}$$

and thus the probability $P(x, t)$ of finding a particle at point x at time t will verify

$$\frac{\partial P(x,t)}{\partial t} = D \frac{\partial^\alpha P(x,t)}{\partial |x|^\alpha}, \tag{7.3}$$

where D is a parameter called diffusion coefficient. This is a variation of the equation in Fick's second law of diffusion, where $\alpha = 2$ and instead of

P we have a concentration of a substance which is undergoing a diffusion process.

Thus the Lévy stable distributions provide solutions for the space-fractional diffusion equation (7.3). The inspiration for P. Lévy to introduce the general class of stable distributions (given his name) was the desire to generalize the celebrated Central Limit Theorem. See more details on the random walks and Lévy-Feller diffusion processes, in [71] and [72].

8. Rashid Nigmatullin

Rashid Shakirovich Nigmatullin was born in 1923 in Tatarstan, Russian Federation (then Soviet Union). He was Rector (1967-1977) of the Kazan Aviation Institute, and it was during his tenure that it adopted, in 1973, the name of Andrey Nikolayevich Tupolev, a famous Russian aircraft designer (who had died the year before). The Institute is currently called Kazan National Research Technical University named after A.N. Tupolev (KNRTU-KAI), [73]. As a publicly active man he was also Chairman (1971-1980) of the Supreme Council of the Tatarstan Autonomous Soviet Socialist Republic (TASSR). For many years he was the head of the Theory of Radio-Engineering and Electronics Department (1954-1988) and had many pupils and followers (including his son Raoul, 4 Dr.Sc., 25 Ph.D., etc.), who further contributed to development of the science and its applications in the related areas. Rashid Nigmatullin died in 1991, leaving a great number of scientific ideas and projects unfinished. In November 2013, KNRTU-KAI organized the International Scientific and Technical Conference called "Nigmatullin's Readings" — a traditional event, this time dedicated to the 90th anniversary of birthday of Prof. R. Sh. Nigmatullin. See details in the Editorial Note in this journal, Vol. **17**, No 1 (2014), [74].

Figure 7. Rashid Shakirovich Nigmatullin (1923-1991)

The basic directions of his scientific activity were: molecular electronics, analysis and synthesis of electric circuits, mathematical and electric modelling of the charge transmission on the interphase boundary electrode/electrolyte. He was working and creating new trends in the fields of radio-engineering, radioelectronics, electrochemistry and other related applicable sciences. Yet in the 1950s Prof. Nigmatullin introduced new concepts and ideas to build miniature electronic devices and detectors by using the properties of the systems at the boundary electrode/fluid, and established that in these models the fractional differential and integration operations are realized. As an electrical analogue of diffusion resistance, he suggested a semi-infinite resistor-capacity RC-cable, in which the process of distribution of the potential is similar to the diffusion process. Nigmatullin established that the input impedance of the system electrode/electrolyte is proportional to $p^{-1/2}$, where p is the Laplace operator. This was a technical realization of the mathematical operations of fractional order (semi-order) integro-differentiation, that can be found in his 1964 papers [75, 76]. He proposed a polarography method for construction of fractional-differentiated polarograms, by means of which it was possible to find, with enough speed and exactness (1-2%), the form of the polarogram after a fractional differentiation.

Another proposal for FC applications came from analysis of the integral equation given by Nigmatullin to relate the surface concentration and the density of the substance flow through the electrode. He showed that differentiating by $\mathrm{d}^{1/2}/\mathrm{d}t^{1/2}$ the time variance of the surface concentration $C(0, t)$ it is possible to find directly the gradient of concentration or density, thus avoiding to solve the boundary value problem for the diffusion equation. This property became the base of electrical modelling of cells (see details in [77]).

In the mid-1980s, FC was related with the so-called constant phase elements (CPE) and also with the objects of the fractal geometry, thus finding new horizons for applications. Nigmatullin expected such a close relationship to exist, since any arbitrary mathematical operation has, at its physical realization, a definite geometry or topology. He thought about the representation of the diffusion impedance operator via a cascade-model of involved RC-elements. And should the frequency for preserving the equivalence of the diffusion impedance increase, it would be necessary to increase as well the number of cascades by a geometrical division of the initial RC-element into smaller RC-elements, self-similar in a topological sense. These ideas and relationships were later developed and mathematically proved by his son Raoul, see e.g. [78] and many earlier his papers.

One of the last projects of Rashid Nigmatullin in developing the trend of use of FC operators was the transition from problems in analysis to problems in synthesis of electrical chains with an arbitrary fractional order Laplace operator and system with a fixed phase. For such a synthesis, he introduced a new type of elements in the theory of electrical chains, combining resistor-condensator and resistor-induction properties, in 1982. These new elements he called respectively "recond" and "reind" and planned further investigations on the subject, that however he was not able to finish; see details in [77].

In this way, Nigmatullin was the first in Russia, and one of the pioneers in the world, to think about practical applications of the operators of fractional differentiation, in the 1960s, long before the first specialized conferences on FC and round table discussions dedicated to the "open question" of the existence of any physical or geometrical interpretation of fractional order derivatives. He suggested and realized for the first time an electrochemical device that performs the fractional integration/differentiation operation of order one-half, and a reference to his 1964 pioneering paper [75] can be found in [6], recently also in [48].

R. Sh. Nigmatullin published a total of 178 papers and had 39 inventions and patents. But this publication activity was not well-known in the West because his papers were published in Russian and in that time (the Cold War period) were related with military subject matter of the USSR. In the next journal's issue (No 3) an English translation of his paper [75] will be republished.

9. Yuri Rabotnov

Yuri Nikolaevich Rabotnov was born in 1914 in Nizhny Novgorod, Russian Empire. He obtained his PhD in 1940 and was a member of the Soviet Academy of Sciences from 1958 on. He worked in Moscow State University and was closely associated with the Academy Town of Novosibirsk in Siberia, see e.g. [79]. Yuri Rabotnov died in 1985, [80]. The 100th anniversary of this Russian Academician, who made important contributions to the theory of viscoelasticity, especially related to the algebra of FC operators based on fractional derivatives and their applications to mechanics of solids, is celebrated by an international conference "Hereditary Mechanics of Deformation and Destruction of Solids – Scientific Heritage of Yu. N. Rabotnov", February 24-26, 2014 in Moscow.

Along with the pure FC approach to linear viscoelasticity, and the power law approach, there is also another approach involving integrals of convolution type — as an implementation of the Volterra idea, developed and extensively presented mainly by Rabotnov, in his 1948 paper [81], as well as in his books (see e.g. his 1966 book [82]).

Figure 8. Yuri Nikolaevich Rabotnov (1914-1985)

Rabotnov worked in mechanics of materials, studying the mathematical modelling of elastic and plastic behaviour of materials, as discussed and analyzed in [42], [41], [48]. In the confluence of these topics, he was led to a fractional model of viscoelasticity [4], which he found more accurate to describe as hereditary elasticity, [82].

Let us present in brief the ideological inheritance of Rabotnov devoted to FC model involving more than two different fractional parameters, following for instance the surveys [83, 84]. As a matter of fact, a rheological equation of the type

$$\sum_{i=0}^{n} a_i D^{i\alpha}\varepsilon = \sum_{j=0}^{n} b_j D^{j\alpha}\sigma, \qquad (9.1)$$

(ε and σ are the strain and stress, resp.) interpolating in some sense the classical equations of viscoelasticity (with differentiations of integer orders) and the fractional rheological equation proposed later by Bagley in 1979 (with differentiations of different fractional orders), was introduced by Rabotnov [85] in 1966 (see also [82]) in the equivalent form (see also [82])

$$\varepsilon = J_\infty \left[\sigma + \sum_{i=1}^{n} g_i \ni_\alpha^* (-\tau_i^{-\alpha})\sigma\right] \qquad (9.2)$$

(where $J^{-1} := E^\infty$ is the relaxed (prolonged) modulus of elasticity; g_i and τ_i are constants, the latter to extend his earlier model with τ_ε and τ_σ). The operator in (9.2) is defined as

$$\ni_\alpha^* (-\tau_i^{-\alpha})\sigma = \int_0^t \ni_\alpha (-1, t'/\tau_i)\, \sigma(t-t')dt'. \qquad (9.3)$$

For the equivalence of the models (9.1) and (9.2) one can see Rossikhin and Shitikova [83].

Note that Rabotnov's approach was essentially based on the use of a special case of the Mittag-Leffler function (we use below the denotations by Podlubny [12])

$$\ni_\alpha(\beta, t) = t^\alpha \sum_{k=0}^{\infty} \frac{\beta^k t^{k(\alpha+1)}}{\Gamma((k+1)(1+\alpha))} = t^\alpha E_{\alpha+1,\alpha+1}(\beta t^{\alpha+1}), \qquad (9.4)$$

as involved with $\beta = -1$ in (9.3), and called by him in [85] as the fractional exponential function (because for $\alpha = 1$ it reduces to the exponential function). In the literature (e.g. [12, 17], etc.) it is known now as the Rabotnov function. In his further papers, as well as in his books [86], [82], one can find tabulations of the function (9.4) and of integrals of it,

$$\int_0^t \ni_\alpha(\beta, \tau) d\tau, \quad \int_0^t \frac{\partial \ni_\alpha(\beta, \tau)}{\partial \beta} d\tau,$$

as well as discussions on its relation to the Mittag-Leffler function.

This means that Rabotnov's theory was related also to the FC approach, implicitly involving fractional integrals and derivatives. The question is whether Rabotnov could have written his rheological model in terms of fractional derivatives. As mentioned in [83], he could. In his classical papers of 1948, such as [81], he wrote about this matter. However, Rabotnov might have considered the FC operators as some mathematical abstraction without a physical meaning, and thus suggested the use of hereditary mechanics models instead of the fractional derivatives.

Similarly as mentioned for other Russian/Soviet scientists in this survey, the contributions of Rabotnov remained not well known to the authors on the West, as they were published in Russian. On occasion of his centennial jubilee, in the next issue (No 3) of the FCAA journal, the publishing of an English translation of Rabotnov's paper [81] is planned, accompanied by a survey by Rossikhin and Shitikova on his pioneering role in the application of fractional calculus operators in mechanics of solids.

10. Scott Blair

George William Scott Blair was born in 1902 in Weybridge, United Kingdom. He gave up studying Physics in Cambridge to take up Chemistry in Oxford [87], and was one of the creators of Rheology, the science of the deformation and flow of matter [88] (a definition he attributes, as well as the name rheology, to the chemist Eugene Bingham [43]). He applied this science to problems such as the quality of flour for panification, the study of soils, the mechanical behaviour of butter and cheese, the mucus from the uterine cervix of cows (so as to detect pregnancies earlier; this latter would be applied to human gynaecology as well), or the coagulation of blood.

Figure 9. George William Scott Blair (1902-1987)

Scott Blair received the Herbert Freundlich medal of the German Society of Rheology in 1954, the Jean-Leonard-Marie Poiseuille Award in 1969, and the Founders Gold Medal of the British Society of Rheology in 1970 [89, 90, 91]. He died in 1987.

Scott Blair proposed that just as a fluid and a solid verify, respectively, the equations

$$\frac{S}{\frac{d\sigma}{dt}} = \eta, \qquad (10.1)$$

$$\frac{S}{\sigma} = n, \qquad (10.2)$$

where S is a stress, σ is a strain, and η and n are constants, there are "intermediate bodies" verifying [88, 43]

$$\frac{S}{\frac{d^\mu \sigma}{dt^\mu}} = \chi, \qquad (10.3)$$

where μ is the fractional order and x is again a constant. His stance on the use of fractional operators was rather qualified: on the one hand he says they are "an over-simplification" but disagreed with Gemant's opinion quoted above according to which they are but "a useful mathematical symbol", having justified their use on physical grounds.

For more details on the pioneering contributions of Scott Blair to FC models in rheology, the reader is referred to the recent survey by Mainardi [41] and to the forthcoming one, by Rogosin and Mainardi [92].

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