42^{nd} United States of America Mathematical Olympiad

Day I, II 12:30 PM – 5 PM EDT

April 30 - May 1, 2013

USAMO 1. **First Solution:** Assume that ω_B and ω_C intersect again at another point S (other than P). (The degenerate case of ω_B and ω_C being tangent at P can be dealt similarly.) Because $BPSR$ and $CPSQ$ are cyclic, we have $\angle RSP = 180° - \angle PBR$ and $\angle PSQ = 180° - \angle QCP$. Hence, we obtain

$$\angle QSR = 360° - \angle RSP - \angle PSQ = \angle PBR + \angle QCP = \angle CBA + \angle ACB = 180° - \angle BAC;$$

from which it follows that $ARSQ$ is cyclic; that is, $\omega_A, \omega_B, \omega_C$ meet at S. (This is Miquel's theorem.)

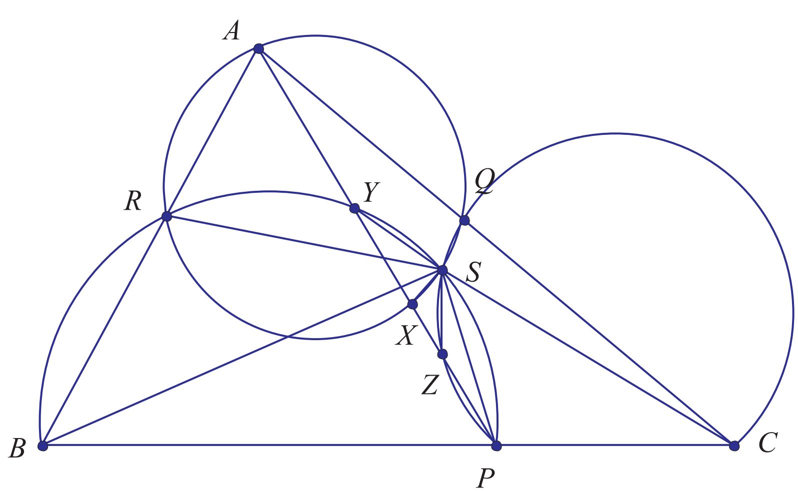
Because $BPSY$ is inscribed in ω_B, $\angle XYS = \angle PYS = \angle PBS$. Because $ARXS$ is inscribed in ω_A, $\angle SXY = \angle SXA = \angle SRA$. Because $BPSR$ is inscribed in ω_B, $\angle SRA = \angle SPB$. Thus, we have $\angle SXY = \angle SRA = \angle SPB$. In triangles SYX and SBP, we have $\angle XYS = \angle PBS$ and $\angle SXY = \angle SPB$. Therefore, triangles SYX and SBP are similar to each other, and, in particular,

$$\frac{YX}{BP} = \frac{SX}{SP}.$$

Similar, we can show that triangles SXZ and SPC are similar to each other and that

$$\frac{SX}{SP} = \frac{XZ}{PC}.$$

Combining the last two equations yields the desired result.



This problem and solution were suggested by Zuming Feng.

Second Solution: Assume that ω_B and ω_C intersect again at another point S (other than P). (The degenerate case of ω_B and ω_C being tangent at P can be dealt with similarly.) Because $BPSR$ and $CPSQ$ are cyclic, we have $\angle RSP = 180^\circ - \angle PBR$ and $\angle PSQ = 180^\circ - \angle QCP$. Hence, we obtain

$$\angle QSR = 360^\circ - \angle RSP - \angle PSQ = \angle PBR + \angle QCP = \angle CBA + \angle ACB = 180^\circ - \angle BAC;$$

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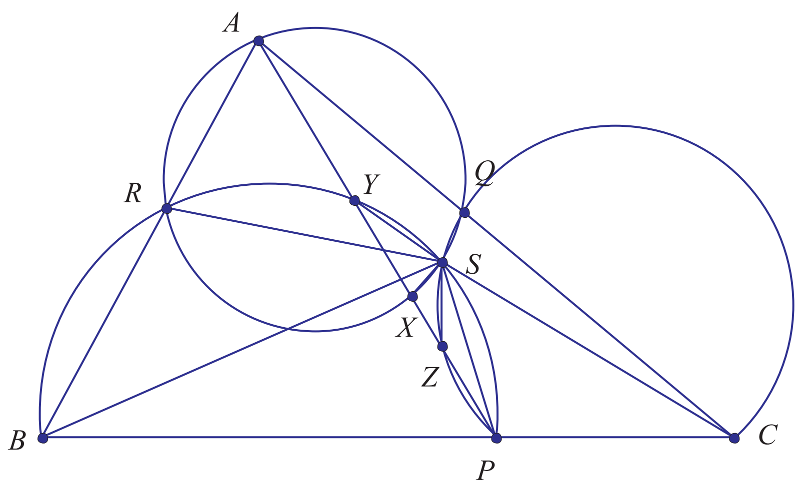


We consider the configuration shown in the above diagram. (We can adjust the proof below easily for other configurations. In particular, our proof is carried with directed angles modulo 180°.)

Line RY intersects ω_A again at T_Y (other than R). Because $BPYR$ is cyclic, $\angle T_Y YX = \angle T_Y YP = \angle RBP = \angle ABP$. Because $ARXT_Y$ is cyclic, $\angle XT_Y Y = \angle XAR = \angle PAB$. Hence triangles $T_Y YX$ and ABP are similar to each other. In particular,

$$\angle YXT_Y = \angle BPA \quad \text{and} \quad \frac{YX}{BP} = \frac{XT_Y}{PA}. \tag{1}$$

Likewise, if line QZ intersect ω_A again at T_Z (other than R), we can show that triangles $T_Z ZX$ and ACP are similar to each other and that

$$\angle T_Z XZ = \angle APC \quad \text{and} \quad \frac{XT_Z}{PA} = \frac{XZ}{PC}. \tag{2}$$

In the light of the second equations (on lengths proportions) in (1) and (2), it suffices to show that $T_Z = T_Y$. On the other hand, the first equations (on angles) in (1) and (2) imply that X, T_Y, T_Z lie on a line. But this line can only intersect ω_A twice with X being one of them. Hence we must have $T_Y = T_Z$, completing our proof.

Comment: The result remains to be true if segment AP is replaced by line AP. The current statement is given to simplify the configuration issue. Also, a very common mistake in attempts following the second solution is assuming line RY and QZ meet at a point on ω_A.

This solution was suggested by Zuming Feng.

USAMO 2. **First Solution.** We will show that $a_n = \frac{1}{3}(2^{n+1} + (-1)^n)$. This would be sufficient, since then we would have

$$a_{n-1} + a_n = \tfrac{1}{3}(2^n + (-1)^{n-1}) + \tfrac{1}{3}(2^{n+1} + (-1)^n) = \tfrac{1}{3}(2^n + 2 \cdot 2^n) = 2^n.$$

We will need the fact that for all positive integers n

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} 2^k = \tfrac{1}{3}(2^{n+1} + (-1)^n).$$

This may be established by strong induction. To begin, the cases $n = 1$ and $n = 2$ are quickly verified. Now suppose that $n \geq 3$ is odd, say $n = 2m + 1$. We find that

$$\begin{aligned} \sum_{k=0}^{m} \binom{2m+1-k}{k} 2^k &= 1 + \sum_{k=1}^{m} \binom{2m-k}{k} 2^k + \sum_{k=1}^{m} \binom{2m-k}{k-1} 2^k \\ &= \sum_{k=0}^{m} \binom{2m-k}{k} 2^k + 2 \sum_{k=0}^{m-1} \binom{2m-1-k}{k} 2^k \\ &= \tfrac{1}{3}(2^{2m+1} + 1) + \tfrac{2}{3}(2^{2m} - 1) \\ &= \tfrac{1}{3}(2^{2m+2} - 1), \end{aligned}$$

using the induction hypothesis for $n = 2m$ and $n = 2m - 1$. For even n the computation is similar, so we omit the steps. This proves the claim.

We now determine the number of ways to advance around the circle twice, organizing our count according to the points visited both times around the circle. It is straight-forward to check that no two such points may be adjacent, and that there are exactly two sequences of moves leading from any such point to the next. (These sequences involve only moves of length two except possibly at the endpoints.) Hence given $k \geq 1$ points around the circle, no two adjacent and not including point A, there would appear to be 2^k ways to traverse the circle twice without repeating a move. However, half of these options lead to repeating the same route twice, giving 2^{k-1} ways in actuality. There are $\binom{n-k}{k}$ ways to select k nonadjacent points on the circle not including A (add an extra point behind each of k chosen points), for a total contribution of

$$\sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n-k}{k} 2^{k-1} = \frac{1}{2}\left[-1 + \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} 2^k\right] = \tfrac{1}{6}(2^{n+1} + (-1)^n) - \tfrac{1}{2}.$$

On the other hand, if the $k \geq 1$ nonadjacent points do include point A then there are $\binom{n-k-1}{k-1}$ ways to choose them around the circle. (Select A but not the next point, then add an extra point after each of $k-1$ selected points.) But now there are actually 2^k ways to circle twice, since we can choose either move at A and the subsequent points, then select the other options the second time around. Hence the contribution in this case is

$$\sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n-k-1}{k-1} 2^k = 2 \sum_{k=0}^{\lfloor (n-2)/2 \rfloor} \binom{n-2-k}{k} 2^k = \tfrac{2}{3}(2^{n-1} + (-1)^n).$$

Finally, if n is odd then there is one additional way to circle in which no point is visited twice by using only steps of length two, giving a contribution of $\frac{1}{2}(1 - (-1)^n)$. Therefore the total number of paths is

$$\tfrac{1}{6}(2^{n+1} + (-1)^n) - \tfrac{1}{2} + \tfrac{2}{3}(2^{n-1} + (-1)^n) + \tfrac{1}{2}(1 - (-1)^n),$$

which simplifies to $\frac{1}{3}(2^{n+1} + (-1)^n)$, as desired.

This problem and solution were suggested by Sam Vandervelde.

Second Solution: We give a bijective proof of the identity

$$a_n = a_{n-1} + 2a_{n-2},$$

which immediately implies that $a_n + a_{n-1} = 2(a_{n-1} + a_{n-2})$. Since trivially $a_0 = a_1 = 1$ (or alternatively $a_1 = 1$, $a_2 = 3$), the desired identity will then follow by induction on n.

To construct the bijection, it is convenient to introduce some alternate representations for the sequences we are counting. Label the points $P_0, \ldots, P_{n-1}$ in order, and define $P_{i+n} = P_i$. One can then represent the sequences to be counted by listing the sequence of vertices $P_{i_0} P_{i_1} \ldots P_{i_m}$ visited by the marker, with the conventions that $i_0 = 0$, $i_m = 2n$, and $i_{j+1} - i_j \in \{1, 2\}$ for $j = 0, \ldots, m-1$. One can represent such sequences of vertices in turn by $2 \times (n+1)$ matrices A by setting

$$A_{ij} = \begin{cases} 1 & P_{ni+j} \text{ is visited} \\ 0 & P_{ni+j} \text{ is not visited} \end{cases} \qquad (i = 0, 1; j = 0, \ldots, n).$$

Such a matrix A corresponds to a valid sequence if and only if $A_{00} = A_{1n} = 1$ (so the sequence of steps starts and ends at P_0), $A_{0n} = A_{n0}$ (so the sequence of steps is well-defined at P_n), and there are no submatrices of any of the forms

$$\begin{pmatrix} 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

(to exclude steps of length greater than 2, duplication of a length 2 step, and duplication of a length 1 step). For example, the valid sequences for $n = 3$ are represented by the matrices

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

Let S_n be the set of valid $2\times(n+1)$ matrices. The correspondence $S_{n-2}\sqcup S_{n-2}\sqcup S_{n-1}\cong S_n$ can then be described by replacing the right end of the matrix in the following fashion, where $\cdots$ represents any row of length $n-2$.

$$
\begin{array}{c|c}
\begin{pmatrix} \cdots & 1 \\ \cdots & 1 \end{pmatrix} & \begin{pmatrix} \cdots & 1 & 1 & 1 \\ \cdots & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \cdots & 1 & 0 & 1 \\ \cdots & 1 & 1 & 1 \end{pmatrix} \\
\begin{pmatrix} \cdots & 0 \\ \cdots & 1 \end{pmatrix} & \begin{pmatrix} \cdots & 0 & 1 & 0 \\ \cdots & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \cdots & 0 & 1 & 0 \\ \cdots & 1 & 1 & 1 \end{pmatrix} \\
\begin{pmatrix} \cdots & 0 & 1 \\ \cdots & 1 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \\
\begin{pmatrix} \cdots & 1 & 1 \\ \cdots & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \\
\begin{pmatrix} \cdots & 1 & 0 \\ \cdots & 1 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\
\begin{pmatrix} \cdots & 1 & 0 \\ \cdots & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}
\end{array}
$$

From this description, it is easy to see that passing from one side to the other preserves the boundary condition and the excluded submatrix conditions (because every submatrix whose entries are not all shown remains unchanged). We thus have the claimed bijection.

This solution was suggested by Kiran Kedlaya.

Third Solution: This solution uses some of the same notation as the second solution.

We first solve a related but simpler counting problem. Let S_n be the set of sequences of steps of lengths 1 or 2 of total length n. For each sequence $s\in S_n$, let $b(s)$ be the number of steps of length 2 in s and define $f_n=\sum_{s\in S_n}2^{b(s)}$. It is clear that $f_0=f_1=1$. For $n\geq 2$, we also have

$$f_n=f_{n-1}+2f_{n-2}$$

by counting sequences of length n according to whether they end in a step of length 1 or 2. Thus

$$f_n+f_{n-1}=2(f_{n-1}+f_{n-2}),$$

from which it follows by induction on n that $f_n+f_{n-1}=2^n$ for $n\geq 1$. By induction on n, we also have

$$f_n=\frac{2^n+(-1)^n}{3}.$$

We now write a_n in terms of f_n. Label the points of the circle as in the previous solution. We may separate sequences of moves into three types.

1. Sequences that visit P_n but not P_{n-1}. Such a sequence starts with some $s\in S_{n-2}$ followed by a step of length 2. The number of complements for s (i.e., the number of ways to complete it to a full sequence) can be seen to be $2^{b(s)}$ as follows. If we decide in order whether to skip each of $P_{n+1},\dots,P_{2n}$, then the choice for P_{n+i} is uniquely forced if $A_{0(i-1)}=1$ and unrestricted if $A_{0(i-1)}=0$. In the notation of the previous solution, we may see this by noting that

$$\begin{pmatrix} A_{0(i-1)} & A_{0i} \\ A_{1(i-1)} & A_{1i} \end{pmatrix} \in \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}.$$

(This logic does not apply to P_{2n}: we have $A_{0(n-1)} = 0$ but must take $A_{1(2n)} = 1$.) We thus get f_{n-2} sequences of this type.

2. Sequences that visit P_{n-1} but not P_n. Such a sequence starts with some $s \in S_{n-1}$ followed by a step of length 2. There are f_{n-1} sequences of this type.
3. Sequences that visit both P_{n-1} and P_n. Such a sequence starts with some $s \in S_{n-1}$ followed by a step of length 1. Here the count is complicated by the constraint that we must skip P_{2n-1}, so the final step of length 2 does not create an option. Therefore, s contributes $2^{b(s)-1}$ complements if $b(s) > 0$. The only case where $b(s) = 0$ is when s consists of only steps of length 1, in which case we get 1 complement if n is even and 0 complements if n is odd.

Putting this together, we get

$$\begin{aligned} a_n &= f_{n-2} + f_{n-1} + \frac{1}{2}(f_{n-1} + (-1)^n) \\ &= \frac{2^{n-2} + (-1)^{n-2}}{3} + \frac{2^{n-1} + (-1)^{n-1}}{3} + \frac{2^{n-1} + (-1)^{n-1}}{6} + \frac{(-1)^n}{2} \\ &= \frac{2^n + (-1)^n}{3} \end{aligned}$$

and so $a_{n-1} + a_n = 2^n$ as desired.

Remark. The sequence a_n is known as the Jacobsthal sequence and has many other combinatorial interpretations. See sequence `A001045` in the Online Encyclopedia of Integer Sequences: `http://oeis.org`.

This solution was suggested by Kiran Kedlaya.

USAMO 3. For $n = 1$ the answer is clearly 1, since there is only one configuration other than the initial one, and that configuration takes 1 step to get to. From now on we will consider $n \geq 2$.

Note that there are $3n$ possible operations in total, since we can select $3n$ lines to perform an operation on (n lines parallel to each side of the triangle.) Performing an operation twice on the same line is equivalent to doing nothing. Hence, we will describe any combination of operations as a triple of n-tuples $((a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n), (c_1, c_2, \ldots, c_n))$, where each element a_i, b_i, c_i is either 0 or 1 (0 means no operation, 1 means the opposite), each tuple of the triple denotes operating on a line parallel to one of the sides, and the indices, i.e. $1, 2, \ldots, n$, denote the number of marks in the row of operation. Let A denote the set of all such $3n$-tuples. Hence $|A| = 2^{3n}$.

Let B denote the set of all admissible configurations. Let $N = \dfrac{n(n+1)}{2}$. We will describe each element of B by an N-tuple $(z_1, z_2, \ldots, z_N)$, where each element is either 0 or 1 (0 means black, 1 means white). (Which element refers to which position is not important.)

For each element $a \in A$, let $b = f(a)$ be the element of B that is the result of applying the operations in a. Then $f(a + a') = f(a) + f(a')$ for all $a, a' \in A$, where addition is considered in modulo 2. Let K be the set of all $a \in A$ such that $f(a)$ is the all-black configuration. The following eight elements are easily seen to be in K.

- $((0,\, 0,\, \ldots,\, 0),\, (0,\, 0,\, \ldots,\, 0),\, (0,\, 0,\, \ldots,\, 0)) = \text{id}$
- $((0,\, 0,\, \ldots,\, 0),\, (1,\, 1,\, \ldots,\, 1),\, (1,\, 1,\, \ldots,\, 1)) = x$
- $((1,\, 1,\, \ldots,\, 1),\, (1,\, 1,\, \ldots,\, 1),\, (0,\, 0,\, \ldots,\, 0)) = y$
- $((1,\, 1,\, \ldots,\, 1),\, (0,\, 0,\, \ldots,\, 0),\, (1,\, 1,\, \ldots,\, 1)) = x + y$
- $((0,\, 1,\, 0,\, 1,\, \ldots),\, (0,\, 1,\, 0,\, 1,\, \ldots),\, (0,\, 1,\, 0,\, 1,\, \ldots)) = z$
- $((0,\, 1,\, 0,\, 1,\, \ldots),\, (1,\, 0,\, 1,\, 0,\, \ldots),\, (1,\, 0,\, 1,\, 0,\, \ldots)) = x + z$
- $((1,\, 0,\, 1,\, 0,\, \ldots),\, (1,\, 0,\, 1,\, 0,\, \ldots),\, (0,\, 1,\, 0,\, 1,\, \ldots)) = y + z$
- $((1,\, 0,\, 1,\, 0,\, \ldots),\, (0,\, 1,\, 0,\, 1,\, \ldots),\, (1,\, 0,\, 1,\, 0,\, \ldots)) = x + y + z$

We will show that they are the only elements of K.

Suppose $L = ((a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n), (c_1, c_2, \ldots, c_n))$ is in K. Then $a_i + b_j + c_k = 0$ whenever $i + j + k = 2n + 1$ (why this is is left as an exercise for the reader.) By adding x and/or y if necessary, we will assume that $b_n = c_n = 0$. Since $a_2 + b_{n-1} + c_n = a_2 + b_n + c_{n-1} = 0$, we have that $b_{n-1} = c_{n-1}$. There are two cases:

(a) $b_{n-1} = c_{n-1} = 0$. Then from $a_3 + b_{n-2} + c_n = a_3 + b_{n-1} + c_{n-1} = a_3 + b_n + c_{n-2}$, we have that $b_{n-2} = c_{n-2} = 0$. Continuing in this manner (considering equalities with $a_4, a_5, \ldots$), we find that all the b_i's and c_i's are 0, from which we deduce that $L = \text{id}$.

(b) $b_{n-1} = c_{n-1} = 1$. Then from $a_3 + b_{n-2} + c_n = a_3 + b_{n-1} + c_{n-1} = a_3 + b_n + c_{n-2}$, we have that $b_{n-2} = c_{n-2} = 0$. Continuing in this manner (considering equalities with $a_4, a_5, \ldots$), we find that $(b_1, b_2, \ldots, b_n) = (c_1, c_2, \ldots, c_n) = (\ldots, 1, 0, 1, 0)$, from which we deduce that either $L = z$ or $L = x + z$.

Hence L is one of the eight elements listed above. It follows that the 2^{3n} elements of A form 2^{3n-3} sets, each set corresponding to an element of B. For each element $a \in A$, let x_1 be the number of $a_1, a_3, \ldots$ that are 1, and let x_2 be the number of $a_2, a_4, \ldots$ that are 1. Define y_1, y_2, z_1, and z_2 similarly with the b_i's and c_i's. We want to find the element in the set containing a that has the smallest value of $T = x_1 + x_2 + y_1 + y_2 + z_1 + z_2$. The maximum of this value over all the sets is the desired answer.

We observe that an element $a \in A$ has the minimal value of T in its set if and only if it satisfies the following inequalities:

(a) $x_1 + x_2 + y_1 + y_2 \leq n$

(b) $x_1 + x_2 + z_1 + z_2 \leq n$

(c) $y_1 + y_2 + z_1 + z_2 \leq n$

(d) $x_2 + y_2 + z_2 \leq \left\lfloor \dfrac{3\lfloor n/2 \rfloor}{2} \right\rfloor = V$

(e) $x_1 + y_1 + z_2 \leq \left\lfloor \dfrac{2\lceil n/2 \rceil + \lfloor n/2 \rfloor}{2} \right\rfloor = W$

(f) $x_2 + y_1 + z_1 \leq \left\lfloor \dfrac{2\lceil n/2 \rceil + \lfloor n/2 \rfloor}{2} \right\rfloor = W$

(g) $x_1 + y_2 + z_1 \le \left\lfloor \frac{2\lceil n/2 \rceil + \lfloor n/2 \rfloor}{2} \right\rfloor = W$

We wish to find the maximal value of T that an element satisfying all these inequalities can have. Adding the last four inequalities and dividing by 4, we obtain $T \le \left\lfloor \frac{V + 3W}{2} \right\rfloor$. We consider four cases:

(a) $n = 4k$. $V = W = 3k$, and so $T \le 6k$. We can choose $x_1 = x_2 = y_1 = y_2 = z_1 = z_2 = k$ to attain the bound.

(b) $n = 4k + 1$. $V = 3k$ and $W = 3k + 1$, and so $T \le 6k + 1$. We can choose $x_1 = x_2 = y_1 = y_2 = z_2 = k$ and $z_1 = k + 1$ to attain the bound.

(c) $n = 4k + 2$. $V = 3k + 1$ and $W = 3k + 1$, and so $T \le 6k + 2$. We can choose $x_1 = x_2 = y_1 = y_2 = k$ and $z_1 = z_2 = k + 1$ to attain the bound.

(d) $n = 4k + 3$. $V = 3k + 1$ and $W = 3k + 2$, and so $T \le 6k + 3$. We can choose $x_1 = x_2 = y_2 = k$ and $y_1 = z_1 = z_2 = k + 1$ to attain the bound.

This concludes our proof.

This problem and solution were suggested by Warut Suksompong.

USAMO 4. **First Solution:** Let a, b, c be nonnegative real numbers such that $x = 1 + a^2$, $y = 1 + b^2$ and $z = 1 + c^2$. We may assume that $c \le a, b$, so that the condition of the problem becomes

$$(1 + c^2)(1 + (1 + a^2)(1 + b^2)) = (a + b + c)^2.$$

The Cauchy-Schwarz inequality yields

$$(a + b + c)^2 \le (1 + (a + b)^2)(c^2 + 1).$$

Combined with the previous relation, this shows that

$$(1 + a^2)(1 + b^2) \le (a + b)^2,$$

which can also be written $(ab - 1)^2 \le 0$. Hence $ab = 1$ and the Cauchy-Schwarz inequality must be an equality, that is, $c(a + b) = 1$. Conversely, if $ab = 1$ and $c(a + b) = 1$, then the relation in the statement of the problem holds, since $c = \frac{1}{a+b} < \frac{1}{b} = a$ and similarly $c < b$.

Thus the solutions of the problem are

$$x = 1 + a^2, \quad y = 1 + \frac{1}{a^2}, \quad z = 1 + \left(\frac{a}{a^2 + 1}\right)^2$$

for some $a > 0$, as well as permutations of this. (Note that we can actually assume $a \ge 1$ by switching x and y if necessary.)

This problem and solution were suggested by Titu Andreescu.

Second Solution: We maintain the notations in the first solution and again consider the equation

$$(a+b+c)^2 = 1 + c^2 + (1+a^2)(1+b^2)(1+c^2).$$

Expanding both sides of the equation yields

$$a^2+b^2+c^2+2ab+2bc+2ca = 1+c^2+1+a^2+b^2+c^2+a^2b^2+b^2c^2+c^2a^2+a^2b^2c^2$$

or

$$a^2b^2c^2+a^2b^2+b^2c^2+c^2a^2-2ab-2bc-2ca+c^2+2 = 2(ab+bc+ca).$$

Setting $(u,v,w)=(ab,bc,ca)$, we can write the above equation as

$$uvw+u^2+v^2+w^2-2u-2v-2w+\frac{vw}{u}+2 = 2(u+v+w).$$

which is the equality case of the sum of the following three special cases of the AM-GM inequality:

$$uvw+\frac{vw}{u} \ge 2vw, v^2+w^2+2vw+1 = 2(v+w) \ge 0, \quad u^2+1 \ge 2u.$$

Hence we must have the equality cases these AM-GM inequalities; that is, $ab=u=1$ and $a(b+c)=v+w=1$. We can then complete our solution as we did in the first solution.

This solution was suggested by Zuming Feng.

USAMO 5. For a given positive integer k, write $10^k m - n = 2^r 5^s t$, where $\gcd(t,10)=1$. For large enough values of k the number of times 2 and 5 divide the left-hand side is at most the number of times they divide n, hence by choosing k large we can make t arbitrarily large. Choose k so that t is larger than either m or n.

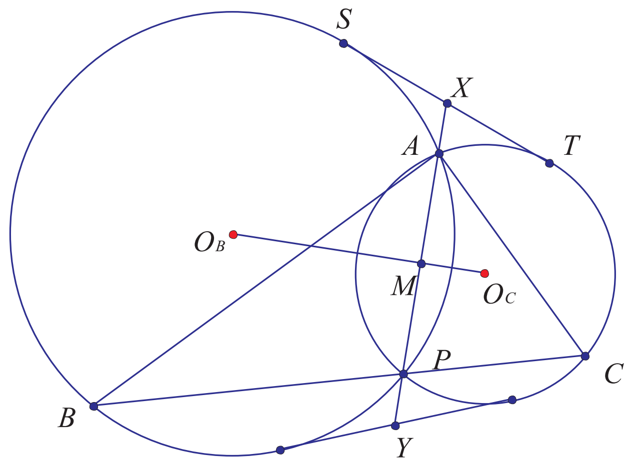
Since t is relatively prime to 10 there is a smallest exponent b for which $t \mid (10^b-1)$. Thus b is the number of digits in the repeating portion of the decimal expansion for $\frac{1}{t}$. More precisely, if we write $tc=(10^b-1)$, then the repeating block is the b-digit decimal representation of c, obtained by prepending extra initial zeros to c as necessary. Since t is larger than m or n, the decimal expansions of $\frac{m}{t}$ and $\frac{n}{t}$ will consist of repeated b-digit representations of cm and cn, respectively. Rewriting the identity in the first line as

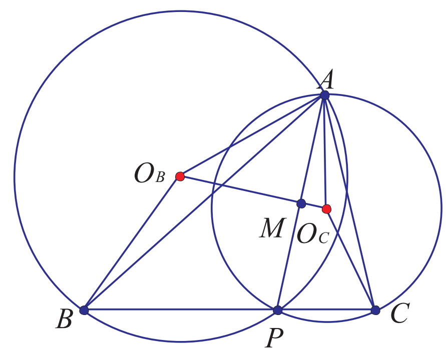
$$10^k\left(\frac{m}{t}\right) = 2^r5^s + \frac{n}{t},$$

we see that the decimal expansion of $\frac{n}{t}$ is obtained from that of $\frac{m}{t}$ by shifting the decimal to the right k places and removing the integer part. Thus the b-digit representations of cm and cn are cyclic shifts of one another. In particular, they have the same number of occurrences of each nonzero digit. (Because they may have different numbers of leading zeros as b-digit numbers, the number of zeros in their decimal expansions may differ.)

This problem and solution were suggested by Richard Stong.

USAMO 6. We consider the left-hand side configuration shown below. Let O_B and ω_B (O_C and ω_C) denote the circumcenter and circumcircle of triangle ABP (ACP) respectively. Line ST, with S on ω_B and T on ω_C, is one of the common tangent lines of the two circumcircles. Point X lies on segment ST. Point Y lies on the other common tangent line.





We will start with the following simple and well known geometry facts.

Let M be the intersection of segments XY and O_BO_C. By symmetry, M is the midpoint of both segments AP and XY, and line O_BO_C is the perpendicular bisector of segments XY and AP. By the power-of-a-point theorem,

$$XS^2 = XA \cdot XP = XT^2 \quad \text{and} \quad X \text{ is the midpoint of segment } ST. \tag{3}$$

Triangles ABC and AO_BO_C are similar to each other, which is the so called *Salmon theorem*. Indeed, $\angle ABC = \angle MO_BA = \angle O_CO_BA$, because each angle is equal to half of the angular size of arc $\overset{\frown}{AP}$ of ω_B. Likewise, $\angle O_BO_CA = \angle C$. In particular, we have

$$\frac{AB}{AO_B} = \frac{BC}{O_BO_C} = \frac{CA}{O_CA} \tag{4}$$

Set $AB = c$, $BC = a$, and $CA = b$. We will establish the following key fact in two approaches.

$$1 - \left(\frac{PA}{XY}\right)^2 = \frac{BC^2}{(AB+AC)^2} = \frac{a^2}{(b+c)^2}. \tag{5}$$

With this fact, the given condition in the problem becomes

$$\frac{PB \cdot PC}{AB \cdot AC} = \frac{a^2}{(b+c)^2} \quad \text{or} \quad PB \cdot PC = \frac{a^2bc}{(b+c)^2}. \tag{6}$$

There are precisely two points P_1 and P_2 (on segment BC) satisfying (6): AP_1 is the bisector of $\angle BAC$ and P_2 is the reflection of P_1 across the midpoint of segment BC. Indeed, by the angle-bisector theorem, $P_2C = P_1B = \frac{ac}{b+c}$ and $P_2B = P_1C = \frac{ab}{b+c}$, from which (6) follows.

In order to settle the question, it remains to show that we can't have more than two points satisfying (6). We just write (6) as

$$\frac{a^2bc}{(b+c)^2} = PB \cdot PC = PB \cdot (a - PB).$$

This a quadratic equation in PB, which can have at most two solutions.

Solution 1. Rays O_BX and O_CT meet in W. Because of (3) and $O_BS \parallel O_CT$, triangles O_BSX and WTX are congruent to each other. Hence $O_BX = XW$ and triangles O_BXO_C and WXO_C have the same area. Note that XM and XT are altitudes in triangles O_BXO_C and WXO_C respectively. Hence

$$\frac{XY \cdot O_BO_C}{4} = \frac{XM \cdot O_BO_C}{2} = \frac{XT \cdot O_CW}{2} = \frac{ST \cdot (O_CT + TW)}{4} = \frac{ST \cdot (O_CT + O_BS)}{4}.$$

By (4), we can write the above equation as

$$\frac{XY}{ST} = \frac{O_CT + O_BS}{O_BO_C} = \frac{O_CA + O_BA}{O_BO_C} = \frac{AB + AC}{BC} \quad \text{or} \quad \frac{XY^2}{ST^2} = \frac{(b+c)^2}{a^2}. \tag{7}$$

Note that O_BSTO_C is a right trapezoid. Let U be the foot of the perpendicular from O_C on O_BS. We have

$$ST^2 = UO_C^2 = O_BO_C^2 - O_SU^2 = O_BO_C^2 - (O_BS - O_CT)^2 = O_BO_C^2 - (O_BA - O_CA)^2.$$

By (4), we can write the above equation as

$$ST^2 = \frac{O_BO_C^2}{BC^2}(BC^2 - (BA - CA)^2) = \frac{O_BO_C^2}{BC^2}(a^2 - (b-c)^2) = \frac{O_BO_C^2}{BC^2}(a+b-c)(a-b+c). \tag{8}$$

Multiplying (7) and (8) together gives

$$XY^2 = \frac{O_BO_C^2}{BC^2} \cdot \frac{(a+b-c)(a-b+c)(b+c)^2}{a^2}. \tag{9}$$

Let h_a denote length of the altitude from A to side BC in triangle ABC. Then h_a and AM are corresponding parts in similar triangles ABC and AO_BO_C, and so

$$\frac{O_BO_C^2}{BC^2} = \frac{AM^2}{h_a^2} = \frac{AM^2}{4h_a^2}. \tag{10}$$

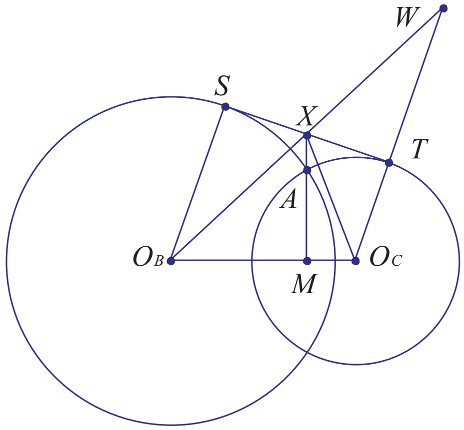
Multiplying (9) and (10) together gives

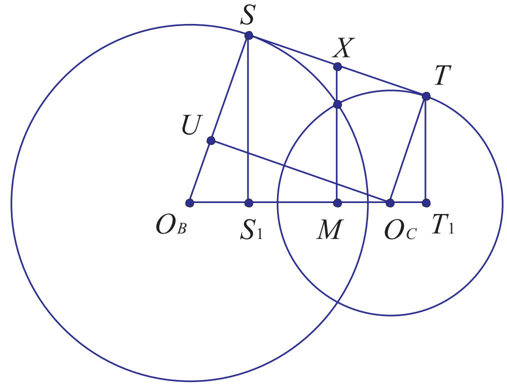
$$XY^2 = \frac{AP^2}{4h_a^2} \cdot \frac{(a+b-c)(a-b+c)(b+c)^2}{a^2}$$

By Heron's formula, we have

$$\frac{AP^2}{XY^2} = \frac{4h_a^2a^2}{(a+b-c)(a-b+c)(b+c)^2} = \frac{(a+b+c)(b+c-a)}{(b+c)^2} = \frac{(b+c)^2 - a^2}{(b+c)^2} = 1 - \frac{a^2}{(b+c)^2},$$

from which (5) follows.





Solution 2. By the power-of-a-point theorem, we have $XA \cdot XP = XS^2$. Therefore,

$$1 - \left(\frac{PA}{XY}\right)^2 = \frac{XY^2 - PA^2}{XY^2} = \frac{(XY + PA)(XY - PA)}{XY^2} = \frac{4XA \cdot XP}{XY^2} = \frac{4XS^2}{XY^2}\frac{ST^2}{XY^2}. \tag{11}$$

Let S_1 and T_1 be the feet of the perpendiculars from S and T to line O_BO_C. It is easy to see that right triangles $O_BSS_1, O_CTT_1, O_SO_CU$ are similar to each other. Note also that XM is the midline of right trapezoid S_1STT_1 (because of (3)). Therefore, we have

$$\frac{ST}{O_BO_C} = \frac{UO_C}{O_BO_C} = \frac{S_1S}{O_BS} = \frac{T_1T}{O_CT} = \frac{S_1S + T_1T}{O_BS + O_CT} = \frac{2XM}{O_BS + O_CT} = \frac{XY}{O_BS + O_CT},$$

or, by (4),

$$\frac{ST}{XY} = \frac{O_BO_C}{O_BS + O_CT} = \frac{O_BO_C}{O_BA + O_CA} = \frac{BC}{BA + CA} = \frac{a}{b + c}. \tag{12}$$

It is clear that (5) follows from (11) and (12).

This problem and Solution 1 were suggested by Titu Andreescu and Cosmin Pohoata. Solution 2 was suggested by Zuming Feng.