Cut-Free Modal Theory of Definite Descriptions

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Abstract

We present a standard sequent calculus for first-order modal logic with definite descriptions. It is equivalent to Garson's system which is a generalization and simplification of the approach originally introduced in Q3 system of Thomason. This particular theory of definite descriptions is based on free logic with identity and existence predicate where both rigid and nonrigid terms are present. We show that, despite of the complexities unavoidable for any characterization of definite descriptions, it is possible to provide a structural proof theoretic analysis of such theory. In particular, cut elimination theorem for this sequent calculus is proved in a constructive manner. We briefly consider some possible extensions of this calculus. Finally, some other approaches to modal description theories, due to Goldblatt, and to Fitting and Mendelsohn, are also discussed from the standpoint of structural proof theory.

Keywords: first-order modal logic, free logic, definite descriptions, sequent calculus, cut elimination.

1 Introduction

The aim of this paper is to present a cut-free sequent calculus for first-order modal logic with definite descriptions. It seems that a satisfactory structural proof theory for such logics is not yet developed. We mean by that a formalization provided in terms of sequent calculi enabling analysis of proofs. Roughly speaking, in order to allow such an analysis, suitable sequent calculus must be defined in terms of rules having analytical character and admitting cut elimination. These requirements will be discussed in more detail in section 4.

The first problem requiring a clarification is which logic should be taken into account. There is a variety of first-order modal logics based on different assumptions concerning such questions as existence or denotation [2] and we can hardly say that some of them are treated as commonly acceptable. The same

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[2] For a survey see e.g. Garson [10] or Fitting and Mendelsohn [7].

may be said about theories of definite descriptions. Since the publication of B. Russell's famous paper "On Denoting" [28] several theories were formulated but neither can be claimed to be a definitive solution to the problem of descriptions. In particular, a treatment of improper descriptions which fail to designate a unique object leads to significant differences between several approaches. Many researchers dealing with the problem of definite descriptions follow the Russellian route and eliminate them in favour of ordinary first-order logic with identity but such reductionist approach has serious disadvantages. However, there is an older tradition, starting with Frege [8], [9], in which definite descriptions are treated as genuine terms and a fixed denotation is assigned to all improper descriptions. This account was formally developed by Kalish and Montague [21] but it has also some disadvantages. It seems that a detailed treatment of definite descriptions requires richer resources beyond those offered by classical logic. In fact, a construction of a satisfactory theory of definite descriptions was one of the aims of developing free logics, as reported by Bencivenga [2]. Lambert [24] shows also that free logics offer an useful setting for comparison of Russellian and Fregean approaches to definite descriptions.

Modal logics and semantics of possible worlds provide even better framework for construction of such a theory. A good witness to this claim is a detailed study of first-order modal logics with complex terms of different kinds, including definite descriptions, developed by Fitting and Mendelsohn [7]. It is probably the most subtle theory of definite descriptions which is rich enough for expressing differences between terms that designate existent and nonexistent object, and terms that do not (and even cannot) designate. As such it certainly deserves attention but it is difficult to provide a suitable sequent formalization of it. We will comment on these problems in the last section.

Another formalization of modal logic with definite descriptions, also discussed in the last section, is due to Goldblatt [12]. His approach does not require introduction of some extra machinery beyond standard apparatus. It is in fact not difficult to provide adequate sequent calculus for it but the problem of proving cut elimination is open.

It seems that for the aims of proof theoretic analysis, the approach presented by Garson [11] is a better option. He provided elegant, relatively simple, yet well justified treatment of definite descriptions on the basis of some variant of free logic. It is a slightly strenghtened version of Lambert's system [23] of minimal free description theory MFD in the language with modalities. A strenghtening is due not only to the addition of modalities but also to the addition of a rule specifying the relationship between rigid and nonrigid terms. The first version of such logic was developed by Thomason [30] under the name Q3 but with some unecessary complications. Garson provides much simpler formalization of this system in terms of natural deduction, and this formulation will be our basic point of reference. In what follows we present a sequent calculus formalization of Garson's system and prove cut elimination theorem for it. Hence "Free" in the title is purposely ambigous in the sense of being cut-free formalization of free modal logic.

2 Garson's System !S

The system is formulated in the standard predicate language with identity and existence predicate and with iota-operator forming definite descriptions from formulae of the language. More precisely, we will use the following categories of expressions denoted by the following symbols:

- denumerably infinite set of bound variables $VAR = \{x, y, z, ...\}$
- denumerably infinite set of free variables (rigid names) $CON = \{a, b, c, ...\}$
- denumerably infinite set of predicate symbols $PRED = \{A, B, C, ...\}$
- connectives: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \Box$
- predicates of identity and existence: $=$, E
- (free) quantifiers: $\forall, \exists$
- iota-operator: $\imath$

In general we will use the same symbols in the metalanguage but with aditional metavariables φ, ψ, χ used for any formulae and $\Gamma, \Delta, \Pi, \Sigma$ for their multisets. A definition of a term and formula is standard; note however, that we do not admit formulae containing $x, y, ...$ not bound by quantifiers or iota-operator. Accordingly, the category of terms covers free variables and descriptions which will be written as $\imath x\varphi$ where φ is a formula in the scope of iota-operator. Metavariables $t, t_1, ...$ will be applied for any terms, including descriptions. Moreover, we will use a metavariable d for denoting any definite description if its structure is not essential. $\varphi[x/t]$ is officially used for the operation of correct substitution of a term t for x. However, to simplify matters, we will be also using freely in proof schemata a notation $\varphi(x), \varphi(a), \varphi(t)$. In particular, $\varphi(x)$ will be used to denote that φ (being a scope of some operator which binds x) contains at least one occurrence of free x, whereas $\varphi(a)$ or $\varphi(t)$ will denote the result of substitution.

Note that to simplify matters, and following Garson's policy, we do not introduce function symbols and we regard only elements of CON as rigid, and definite descriptions as nonrigid terms. It is possible to divide all terms into rigid and nonrigid, then to subdivide both classes into simple (names) and complex terms and the latter into descriptions and functional terms. It seems that such syntactic extensions require only additional notational complications (two-sorted language) and no substantial changes into presented systems are necessary. However, we will see that admitting rigid definite descriptions or universal instantiation on nonrigid terms may lead to troubles in proving cut elimination for sequent calculus formulation. We will comment on this problem in the last section while discussing Goldblatt's approach.

Garson presents his system as Jaśkowski-style natural deduction. We omit propositional details of his system and briefly recall only his rules for quantifiers, identity and descriptions:

($\forall E$) $\forall x\varphi \ \vdash \ Ea \rightarrow \varphi[x/a]$, where a is any (rigid) constant.
($\forall I$) $Ea \rightarrow \varphi[x/a] \ \vdash \ \forall x\varphi$, where a is neither in active assumptions nor in

φ.

$(=I)\ \vdash t = t$
$(=E)\ \varphi[x/t_1], t_1 = t_2, \vdash\ \varphi[x/t_2]$
$(\exists i)\quad a \neq d\ \vdash\ \bot$, where a is neither in active assumptions nor in d.
$(=\Box)\ \varphi\ \vdash\ \Box\varphi$, where φ is $a = b$ or $a \neq b$
$(\imath E)\ Ea, a = \imath x\varphi(x) \vdash \varphi(a) \wedge \forall x(\varphi(x) \to x = a)$
$(\imath I)\ Ea, \varphi(a) \wedge \forall x(\varphi(x) \to x = a) \vdash a = \imath x\varphi(x)$

Note that in the rules for quantifiers and boxed identity only rigid terms are allowed to instantiate variables, whereas in other identity rules all terms may appear. $(\exists i)$ is a special rule which guarantees that all descriptions have some denotation although not necessarily in the actual world. This rule provides a form of "rigidification" of nonrigid terms.

In what follows we will use two equivalent rules for definite descriptions of the form:

$Ea, a = \imath x\varphi(x) \vdash \forall x(\varphi(x) \leftrightarrow x = a)$
$Ea, \forall x(\varphi(x) \leftrightarrow x = a) \vdash a = \imath x\varphi(x)$

Although on the ground of free logic $\forall x(\varphi(x) \leftrightarrow x = a)$ is not equivalent to $\varphi(a) \wedge \forall x(\varphi(x) \to x = a)$, in the presence of Ea they are equivalent since $\varphi(a) \wedge \forall x(\varphi(x) \to x = a)$ implies $\forall x(\varphi(x) \leftrightarrow x = a)$ and the latter with Ea implies $\varphi(a) \wedge \forall x(\varphi(x) \to x = a)$.

Garson's system !S (where S is the name of suitable propositional modal logic) is adequate with respect to relational semantics with varying domains (of objects), actualist quantification and both rigid and nonrigid terms. Semantical characterisation will not be used in the remaining sections. However, for better understanding of the meaning of his system's principles and intuitions behind them, we will recall briefly the notion of a model for this logic. Our characterisation is a slightly modified version of Garson's semantics but giving equivalent results. In particular, for easier comparison with more standard semantics of first order languages we admit variables $x, y, ...$ as occurring free as well, and our free variables $a, b, ...$ treat as individual rigid constants. It does not make any essential differences with Garson's version.

A model is any structure $\mathfrak{M} = \langle \mathcal{W}, \mathcal{R}, D, d, I_w\rangle$, where $\mathcal{W}, \mathcal{R}$ is a standard modal frame, D is a nonempty domain, $d : W \longrightarrow \mathcal{P}(D)$ is a function which assigns a set of (existent) objects to every world, and I_w is a family of world's relative functions of interpretation for predicate symbols, defined as follows:

$I_w(P^n) \subseteq D^n$, for every n-argument predicate and world.

An assignment a is defined in a standard way as $a : VAR \cup CON \longrightarrow D$, similarly for the notion of x-variant. Interpretation $I^a_w(t)$ of a term t in w under an assignment a is just $a(t)$ for elements of VAR and CON. Now, I^a_w for definite descriptions is defined in terms of satisfaction relation, so we recall it first (essential clauses only):

$\mathfrak{M}, a, w \vDash P^n(t_1, ..., t_n)$	iff	$\langle I^a_w(t_1), ..., I^a_w(t_n)\rangle \in I_w(P^n)$
$\mathfrak{M}, a, w \vDash t_1 = t_2$	iff	$I^a_w(t_1) = I^a_w(t_2)$
$\mathfrak{M}, a, w \vDash Et$	iff	$I^a_w(t) \in d(w)$
$\mathfrak{M}, a, w \vDash \neg\varphi$	iff	$\mathfrak{M}, a, w \nvDash \varphi$
$\mathfrak{M}, a, w \vDash \varphi \rightarrow \psi$	iff	$\mathfrak{M}, a, w \nvDash \varphi$ or $\mathfrak{M}, a, w \vDash \psi$
$\mathfrak{M}, a, w \vDash \Box\varphi$	iff	$\mathfrak{M}, a, w' \vDash \varphi$ for any w' such that $\mathcal{R}ww'$
$\mathfrak{M}, a, w \vDash \forall x\varphi$	iff	$\mathfrak{M}, a^x_o, w \vDash \varphi$ for all $o \in d(w)$
$\mathfrak{M}, a, w \vDash \exists x\varphi$	iff	$\mathfrak{M}, a^x_o, w \vDash \varphi$ for some $o \in d(w)$

Now, for any definite description: If there is a unique $o \in d(w)$ such that $\mathfrak{M}, a^x_o, w \vDash \varphi$, then $I^a_w(\imath x\varphi) = o$; otherwise $I^a_w(\imath x\varphi) \notin d(w)$.

Definitions of truth in a model, satisfiability, validity and entailment are standard. Note that we obtain different normal modal logics by restricting $\mathcal{R}$ suitably. Thus for T-modality which we have chosen as a fixed representative, $\mathcal{R}$ must be reflexive. We omit the details of adequacy proof and direct a reader to Garson [11]. One should note that in this semantics improper descriptions are explained as having a nonexistent designatum in respective world. It means that every description has a designatum but not in the sense of Fregean theory of the chosen object where all improper descriptions have a unique designatum. Improper descriptions just have designates somewhere. Such an approach is also in contrast to Fitting and Mendelsohn's solution where one can treat as proper description a term which designates in other world and improper descriptions are terms that do not designate at all. But, as Garson pointed out, this question may be treated as a way of interpretation of worlds in a model rather than an issue requiring a technical regulation in the semantics.

3 Sequent System SC!S

We will use a version of Gentzen's LK calculus but with sequents built not from finite lists but from multisets of formulae and with all rules multiplicative (i.e. context-free in case of many-premiss rules). Since modal details are not essential we just fix rules adequate for logic T, hence the concrete sequent calculus specified below should be named SC!T. Clearly, one can use rules characterising other modal logics, weaker (like K) or stronger (like S4 – as in Thomason's Q3). Note however, that if we want to have our system cut-free, modal rules should be taken only from the, rather modest, list of those systems where cut elimination holds [3].

In what follows, for rules with more than two premisses we will use Γ, Δ in conclusions always to denote multiset unions of $\Gamma_1, ..., \Gamma_n, \Delta_1, ..., \Delta_n$ occurring in premisses. The system consists of the following rules:

$(AX)\quad \varphi \Rightarrow \varphi$ $\qquad\qquad (Cut)\quad \dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$

[3] For a survey see e.g. Fitting [6], Goré [13], Indrzejczak [15], Poggiolesi [27] or Wansing [31], [32].

$(W\Rightarrow)\ \dfrac{\Gamma\Rightarrow\Delta}{\varphi,\Gamma\Rightarrow\Delta}$ $\quad(\Rightarrow W)\ \dfrac{\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow\Delta,\varphi}$

$(C\Rightarrow)\ \dfrac{\varphi,\varphi,\Gamma\Rightarrow\Delta}{\varphi,\Gamma\Rightarrow\Delta}$ $\quad(\Rightarrow C)\ \dfrac{\Gamma\Rightarrow\Delta,\varphi,\varphi}{\Gamma\Rightarrow\Delta,\varphi}$

$(\neg\Rightarrow)\ \dfrac{\Gamma\Rightarrow\Delta,\varphi}{\neg\varphi,\Gamma\Rightarrow\Delta}$ $\quad(\Rightarrow\neg)\ \dfrac{\varphi,\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow\Delta,\neg\varphi}$

$(\wedge\Rightarrow)\ \dfrac{\varphi,\psi,\Gamma\Rightarrow\Delta}{\varphi\wedge\psi,\Gamma\Rightarrow\Delta}$ $\quad(\Rightarrow\wedge)\ \dfrac{\Gamma\Rightarrow\Delta,\varphi\qquad\Pi\Rightarrow\Sigma,\psi}{\Gamma,\Pi\Rightarrow\Delta,\Sigma,\varphi\wedge\psi}$

$(\vee\Rightarrow)\ \dfrac{\varphi,\Gamma\Rightarrow\Delta\qquad\psi,\Pi\Rightarrow\Sigma}{\varphi\vee\psi,\Gamma,\Pi\Rightarrow\Delta,\Sigma}$ $\quad(\Rightarrow\vee)\ \dfrac{\Gamma\Rightarrow\Delta,\varphi,\psi}{\Gamma\Rightarrow\Delta,\varphi\vee\psi}$

$(\rightarrow\Rightarrow)\ \dfrac{\Gamma\Rightarrow\Delta,\varphi\qquad\psi,\Pi\Rightarrow\Sigma}{\varphi\rightarrow\psi,\Gamma,\Pi\Rightarrow\Delta,\Sigma}$ $\quad(\Rightarrow\rightarrow)\ \dfrac{\varphi,\Gamma\Rightarrow\Delta,\psi}{\Gamma\Rightarrow\Delta,\varphi\rightarrow\psi}$

$(\leftrightarrow\Rightarrow)\ \dfrac{\Gamma\Rightarrow\Delta,\varphi,\psi\qquad\varphi,\psi,\Pi\Rightarrow\Sigma}{\varphi\leftrightarrow\psi,\Gamma,\Pi\Rightarrow\Delta,\Sigma}$ $\quad(\Rightarrow\leftrightarrow)\ \dfrac{\varphi,\Gamma\Rightarrow\Delta,\psi\qquad\psi,\Pi\Rightarrow\Sigma,\varphi}{\Gamma,\Pi\Rightarrow\Delta,\Sigma,\varphi\leftrightarrow\psi}$

$(\Box\Rightarrow)\ \dfrac{\varphi,\Gamma\Rightarrow\Delta}{\Box\varphi,\Gamma\Rightarrow\Delta}$ $\quad(\Rightarrow\Box)\ \dfrac{\Gamma\Rightarrow\varphi}{\Box\Gamma\Rightarrow\Box\varphi}$

$(\forall\Rightarrow)\ \dfrac{\Gamma\Rightarrow\Delta,Ea\qquad\varphi[x/a],\Pi\Rightarrow\Sigma}{\forall x\varphi,\Gamma,\Pi\Rightarrow\Delta,\Sigma}$ $\quad(\Rightarrow\forall)^1\ \dfrac{Ea,\Gamma\Rightarrow\Delta,\varphi[x/a]}{\Gamma\Rightarrow\Delta,\forall x\varphi}$

$(\exists\Rightarrow)^1\ \dfrac{Ea,\varphi[x/a],\Gamma\Rightarrow\Delta}{\exists x\varphi,\Gamma\Rightarrow\Delta}$ $\quad(\Rightarrow\exists)\ \dfrac{\Gamma\Rightarrow\Delta,Ea\qquad\Pi\Rightarrow\Sigma,\varphi[x/a]}{\Gamma,\Pi\Rightarrow\Delta,\Sigma,\exists x\varphi}$

1. where a is not in Γ,Δ and φ.

$(=\Rightarrow)\ \dfrac{t=t,\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow\Delta}$ $\quad(=d\Rightarrow)^2\ \dfrac{a=d,\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow\Delta}$

2. where a is not in Γ,Δ,d.

$$(\Rightarrow=)^3\ \frac{\Gamma_1\Rightarrow\Delta_1,\varphi[x/t_1]\qquad\Gamma_2\Rightarrow\Delta_2,t_1=t_2\qquad\varphi[x/t_2],\Gamma_3\Rightarrow\Delta_3}{\Gamma\Rightarrow\Delta}$$

3. where φ is atomic, t_1,t_2 are any terms.

$(=\Box)\ \dfrac{\Gamma\Rightarrow\Delta,a=b\qquad\Box a=b,\Pi\Rightarrow\Sigma}{\Gamma,\Pi\Rightarrow\Delta,\Sigma}$ $\quad(\neq\Box)\ \dfrac{a=b,\Gamma\Rightarrow\Delta\qquad\Box a\neq b,\Pi\Rightarrow\Sigma}{\Gamma,\Pi\Rightarrow\Delta,\Sigma}$

$$(\Rightarrow\imath)^4\ \frac{\Gamma_1\Rightarrow\Delta_1,Ea\qquad Eb,\varphi[x/b],\Gamma_2\Rightarrow\Delta_2,a=b\qquad Eb,a=b,\Gamma_3\Rightarrow\Delta_3,\varphi[x/b]}{\Gamma\Rightarrow\Delta,a=\imath x\varphi}$$

4. where b is not in Γ,Δ,φ

$$(\imath\Rightarrow)\ \frac{\Gamma_1\Rightarrow\Delta_1,Ea\quad\Gamma_2\Rightarrow\Delta_2,Eb\qquad\Gamma_3\Rightarrow\Delta_3,\varphi[x/b],a=b\qquad\varphi[x/b],a=b,\Gamma_4\Rightarrow\Delta_4}{a=\imath x\varphi,\Gamma\Rightarrow\Delta}$$

A definition of a proof is standard, as well as definitions of principal, side and parametric formulae in rule's applications. It is easy to demonstrate the soundness of this calculus but we'll rather prove it indirectly by showing that it is not stronger than Garson's system (Theorem 2 in the Appendix). As for completeness one may suspect that the presented system is too weak. Thomason [30] introduced a generalised versions of $(\forall I)$ and $(\exists i)$ in order to prove completeness of Q3. Such rules are also used by Goldblatt [12] under the name template rules. Garson [11] avoids rules of this kind because they are derivable in his natural deduction system due to the presence of modal subproofs. In the standard sequent calculus it is not possible to represent modal nesting mechanism involved here, so we cannot derive such rules. Nevertheless, they are admissible and we can add to SC!S suitable counterparts of such rules of the form:

$$(T \Rightarrow \forall)\ \frac{\Rightarrow \varphi_1 \to \Box(\varphi_2 \to \ldots \Box(\varphi_n \to (Ea \to \psi[x/a])\ldots)}{\Rightarrow \varphi_1 \to \Box(\varphi_2 \to \ldots \Box(\varphi_n \to \forall x\psi)\ldots)}$$

$$(T = d \Rightarrow)\ \frac{\Rightarrow \varphi_1 \to \Box(\varphi_2 \to \ldots \Box(\varphi_n \to a \neq d)\ldots)}{\Rightarrow \varphi_1 \to \Box(\varphi_2 \to \ldots \Box(\varphi_n \to \bot)\ldots)}$$

where $n > 1$ (for $n = 1$ both rules are derivable), a is not in any φ_i, ψ, d.

However, such rules are necessary only if we prove completeness by means of canonical models and using Thomason's strategy of saturation. For sequent calculus without these rules (and without cut) we can adapt completeness proof provided by Garson for his tableau system which works without the need of using template rules. Thus we conclude that the system is equivalent to Garson's !S (!T in particular) and admits cut elimination; proofs of both results are in the Appendix. Note also that even the version with added template rules admits cut elimination.

Below, in order to see how the system works we will provide some examples of proofs. For better readability we underline side-formulae of all rule-applications.

$$(\Rightarrow=)\ \frac{Ea \Rightarrow \underline{Ea} \qquad a = t \Rightarrow \underline{a = t} \qquad \underline{Et} \Rightarrow Et}{(\exists \Rightarrow)\ \dfrac{\underline{Ea}, \underline{a = t} \Rightarrow Et}{\exists x(x = t) \Rightarrow Et}}$$

This proof works for any term, rigid or nonrigid, but for the converse we must provide two different proofs. In case of rigid terms it is trivial:

$$\frac{Ea \Rightarrow \underline{Ea} \qquad \dfrac{a = a \Rightarrow a = a}{\Rightarrow \underline{a = a}}\,(=\Rightarrow)}{Ea \Rightarrow \exists x(x = a)}\,(\Rightarrow \exists)$$

However, for descriptions we must apply $(= d \Rightarrow)$:

$$
(= d \Rightarrow)\,\dfrac{(C \Rightarrow)\,\dfrac{(\Rightarrow \exists)\,\dfrac{(\Rightarrow=)\,\dfrac{Ed \Rightarrow \underline{Ed} \qquad a = d \Rightarrow \underline{a = d} \qquad \underline{Ea} \Rightarrow Ea}{a = d, Ed \Rightarrow \underline{Ea}} \qquad a = d \Rightarrow \underline{a = d}}{\underline{a = d}, a = d, Ed \Rightarrow \exists x(x = d)}}{\underline{a = d}, Ed \Rightarrow \exists x(x = d)}}{Ed \Rightarrow \exists x(x = d)}
$$

The next proof of $E\imath xAx \Rightarrow A\imath xAx$ is much more involved and the converse is not provable. First we construct a proof:

$$
(W \Rightarrow)\,\dfrac{(\Rightarrow=)\,\dfrac{Aa \Rightarrow \underline{Aa} \qquad a = \imath xAx \Rightarrow \underline{a = \imath xAx} \qquad \underline{A\imath xAx} \Rightarrow A\imath xAx}{Aa, a = \imath xAx \Rightarrow A\imath xAx}}{\underline{a = b}, \underline{Aa}, a = \imath xAx \Rightarrow A\imath xAx}
$$

let S_1 denote the last sequent (the root) of this proof-tree; it is then used to obtain:

$$
(\imath \Rightarrow)\,\dfrac{Ea \Rightarrow \underline{Ea} \quad Eb \Rightarrow \underline{Eb} \qquad (\Rightarrow W)\,\dfrac{a = b \Rightarrow a = b}{a = b \Rightarrow \underline{a = b}, \underline{Aa}} \qquad S_1}{b = \imath xAx, Ea, \underline{Eb}, a = b, a = \imath xAx \Rightarrow A\imath xAx}
$$

again the last sequent S_2 of the above is used to obtain:

$$
(C \Rightarrow)\,\dfrac{(\Rightarrow=)\,\dfrac{Ea \Rightarrow \underline{Ea} \qquad a = b \Rightarrow \underline{a = b} \qquad S_2}{\underline{Ea}, \underline{a = b}, b = \imath xAx, Ea, a = b, a = \imath xAx \Rightarrow A\imath xAx}}{b = \imath xAx, Ea, \underline{a = b}, a = \imath xAx \Rightarrow A\imath xAx}
$$

similarly (the root) S_3 is applied in:

$$
(= d \Rightarrow)\,\dfrac{(C \Rightarrow)\,\dfrac{(\Rightarrow=)\,\dfrac{b = \imath xAx \Rightarrow \underline{b = \imath xAx} \qquad a = \imath xAx \Rightarrow \underline{a = \imath xAx} \qquad S_3}{\underline{b = \imath xAx}, \underline{a = \imath xAx}, b = \imath xAx, Ea, a = \imath xAx \Rightarrow A\imath xAx}}{\underline{b = \imath xAx}, Ea, a = \imath xAx \Rightarrow A\imath xAx}}{\underline{Ea}, a = \imath xAx \Rightarrow A\imath xAx}
$$

and S_4 is eventually used in:

$$
(= d \Rightarrow)\,\dfrac{(C \Rightarrow)\,\dfrac{(\Rightarrow=)\,\dfrac{a = \imath xAx \Rightarrow \underline{a = \imath xAx} \qquad E\imath xAx \Rightarrow \underline{E\imath xAx} \qquad S_4}{\underline{a = \imath xAx}, E\imath xAx, a = \imath xAx \Rightarrow A\imath xAx}}{E\imath xAx, \underline{a = \imath xAx} \Rightarrow A\imath xAx}}{E\imath xAx \Rightarrow A\imath xAx}
$$

Notice that a and b were new in both applications of $(= d \Rightarrow)$ and that all applications of $(\Rightarrow=)$ were on atomic formulae, as required.

We finish this section with an interesting and very useful result. It shows that in the present setting E cannot be in general defined in terms of identity.

Theorem 3.1 (i) $\vdash \Gamma \Rightarrow \Delta, Et$ *iff* $\vdash \Gamma \Rightarrow \Delta, \exists xx = t$

(ii) *If* $\vdash Et, \Gamma \Rightarrow \Delta$, *then* $\vdash \exists xx = t, \Gamma \Rightarrow \Delta$

(iii) *If* $\vdash \exists xx = t, \Gamma \Rightarrow \Delta$, *then* $\vdash Et, \Gamma \Rightarrow \Delta$, *provided* t *is nonrigid.*

Proof. By induction on the height of proofs. The basis of the equivalence and both implications hold by the provability of sequents establishing equivalence of Et and $\exists xx = t$ stated above. So we need to prove only the inductive steps.

In case of the left-right direction of the first equivalence, and of the second item, a proof is trivial since Et may occur only as a parametric formula or introduced by weakening and, due to context insensitivity of almost all rules, we can safely replace it with $\exists x x = t$. Note that neither $(\Rightarrow \Box)$, nor any rule calling for fresh constant might make any harm. The right-left direction of the first item, and the third item require additional work since we must take into account also cases when $\exists x x = t$ is the principal formula of the last applied rule. In the first case we have:

$$(\Rightarrow \exists)\ \frac{\Gamma_1 \Rightarrow \Delta_1, Ea \qquad \Gamma_2 \Rightarrow \Delta_2, a = t}{\Gamma \Rightarrow \Delta, \exists x x = t}$$

Now, from both premisses together with $Et \Rightarrow Et$ we obtain by $(\Rightarrow=)$ $\Gamma \Rightarrow \Delta, Et$.

In the second case we have:

$$(\exists \Rightarrow)\ \frac{Ea, a = t, \Gamma \Rightarrow \Delta}{\exists x x = t, \Gamma \Rightarrow \Delta}$$

with a fresh, and we proceed as follows:

$$\cfrac{(\Rightarrow=)\ \cfrac{a = t \Rightarrow a = t \qquad Et \Rightarrow Et \qquad Ea, a = t, \Gamma \Rightarrow \Delta}{(C \Rightarrow)\ \cfrac{a = t, a = t, Et, \Gamma \Rightarrow \Delta}{(= d \Rightarrow)\ \cfrac{a = t, Et, \Gamma \Rightarrow \Delta}{Et, \Gamma \Rightarrow \Delta}}}}{}$$

a is again fresh. Notice that the application of $(= d \Rightarrow)$ was necessary here which explains the proviso in the last item.

$\Box$

4 Comments on Rules

We finish the presentation of the sequent calculus for !S with some remarks concerning the shape of rules and possible extensions of the system. Both the selection of these particular rules for identity and the shape of rules for iota-operator were dictated by the need of proving cut elimination. The problems of possible applications for proof-search was not our concern here; we only pause to mention that standard tableau system may be easily obtained on the basis of SC!S rules. The remarks below help to make clear if they offer some advantages in actual proof construction.

The rules for iota-operator were constructed by decomposition of Garson's natural deduction rules in such a way as to obtain a well-behaved pair of rules. By this we mean some requirements put on the rules of standard sequent calculus analysed in Wansing [31] and Poggiolesi [27] like symmetry, explicitness and separation. Rules which exhibit all these properties jointly, may be called canonical, after Avron and Lev [1]. The rules for boolean connectives and quantifiers in SC!S provide a clear example of canonical rules, whereas $(\Rightarrow \Box)$ fails to be separated and explicit. The rules for iota-operator are not canonical either, although they satisfy conditions of explicitness and symmetry. They are not fully separated since in the principal formulae identity is present in addition to iota-operator, and identity is treated here as a logical constant.

From the point of view of cut elimination proof, more important is the fact that both rules for description satisfy a property of reductivity. It was stated in general form for rules of hypersequent calculi in Metcalfe, Olivetti and Gabbay [25] and we may roughly define it as follows: A pair of introduction rules $(\Rightarrow \star)$, $(\star \Rightarrow)$ for a constant $\star$ is reductive if an application of cut on cut formulae introduced by these rules may be replaced by the series of cuts made on less complex formulae, in particular on their subformulae [4]. Reductivity permits induction on the complexity of cut formula in the course of proving cut elimination (see the proof in the Appendix as an exemplification). Hence it is important that all rules for connectives (including necessity) and for free quantifiers, as well as both rules for descriptions, satisfy this property.

In case of the rules for identity the situation is worse but it is not a fault of this system only. In fact, it is possible to demonstrate that identity cannot be formalised by means of canonical rules (see Indrzejczak [20]). As for the rules applied in SC!S we can observe that they are in a (reversed) sense symmetric, but they are not explicit and, in case of $(= \Box)$ and $(\neq \Box)$, also not separated. The problem of their reductivity simply does not arise since they are not introduction rules and, in fact, it is an advantage here allowing cut elimination. Still, $(\Rightarrow=), (= \Box)$ and $(\neq \Box)$ may seem like some restricted forms of cut in disguise. It is true to some extent but it is a reasonable price for possibility of proving general cut elimination. One may ask however, if some other choices do not provide better solution. Suppose we will use additional axiomatic sequents of the form $\Rightarrow t = t$, $\Rightarrow a = d$ (with a not in d) and $t_1 = t_2, \varphi[x/t_1] \Rightarrow \varphi[x/t_2]$. In such case it is not possible to eliminate cuts with at least one premiss being of such form and having definite description as one of the arguments of identity, while the other premiss having this identity is deduced by one of the rules for $\imath$. Similar problem arises for sequents of the form $a = b \Rightarrow \Box a = b$, $a \neq b \Rightarrow \Box a \neq b$, although here it is generated not by identities with descriptions but by $(\Rightarrow \Box)$. The problem is due to the fact that $(\Rightarrow \Box)$ is context sensitive and, in general, not permutable with other rules, so after reduction of the height we cannot apply the rule with the same result. The same problem is encountered if we use rules of the form:

$$(\Rightarrow= \Box)\ \frac{\Gamma \Rightarrow \Delta, a = b}{\Gamma \Rightarrow \Delta, \Box a = b} \qquad\qquad (\Rightarrow\neq \Box)\ \frac{\Gamma \Rightarrow \Delta, a \neq b}{\Gamma \Rightarrow \Delta, \Box a \neq b}$$

Notice however that if instead of our $(\Rightarrow \Box)$ we will use a rule for transitive logic like S4, of the form:

$$\frac{\Box\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}$$

[4] Again, one can refer here to Avron and Lev [1], where the criterion of coherency of canonical rules yields the same result.

the above rules will work. We decided however to choose rules which are insensitive to the changes in background modal rules and work uniformly with any modal logic for which cut-free sequent calculus exists.

One could also think about using some other rules for expressing Leibniz Law instead of our $(\Rightarrow=)$. For instance, Negri and von Plato's rule seems to be a reasonable option:

$$\frac{t_1 = t_2, \varphi[x/t_1], \varphi[x/t_2], \Gamma \Rightarrow \Delta}{t_1 = t_2, \varphi[x/t_1], \Gamma \Rightarrow \Delta}$$

But such choice also does not work if $t_1 = t_2$ is a cut formula with description. Consider a situation where it is principal in both premisses but in the left premiss introduced by $(\Rightarrow \imath)$; in such cases there is no possibility of replacing this cut with cuts made on premisses of these two rule's applications. Similarly, if we introduce a rule used in Indrzejczak [19] for the formalization of Fregean description theory:

$$(\Rightarrow=')\ \frac{\Gamma \Rightarrow \Delta, \varphi[x/t_1] \qquad \Pi \Rightarrow \Sigma, t_1 = t_2}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \varphi[x/t_2]}$$

This time we have a problem if $\varphi[x/t_2]$ is an identity statement with description and cut formula in the right premiss is deduced by $(\imath \Rightarrow)$. The same remarks apply to other possible rules – in fact there are four more (see p.79 of Indrzejczak [15] or [20]). In general, in cases where cut-formula is an identity with description, principal in both premisses, and in one premiss introduced by the rule for identity whereas in the other by the rule for $\imath$, there is no possibility of reduction of the height or complexity of cut formula.

The most important disadvantage is that we must sacrifice subformula property in the strict form. In proofs there may appear not only subformulae of formulae from the root but also atomic formulae as well as boxed identities and negated identities. But note that this is only a little more (boxed identities and their negations) than in case of sequent calculi for several axiomatic theories provided by Negri and von Plato [26].

5 Extensions and Alternatives

SC!T may be easily extended in at least three ways: by enriching the language, by strenghtening the theory of descriptions, by strengthening the background modal logic. The last option is obvious. One can easily provide modal rules for most known normal modal logic but not many of them admit cut elimination. Standard sequent calculi are rather weak tool in this respect and to provide cut-free characterisation of such logics like B or S5, not to mention bimodal temporal logics, one must use non-standard, generalised framework like hypersequent calculi or nested calculi (see the references listed in footnote 3.). On the other hand, we can develop in such a way a suitable theory of description also in the setting of weaker modal logics, like regular, monotonic or even congruent. Cut-free characterisation of many such logics was provided in Indrzejczak [14] and [16], and may be lifted from propositional level to first-order level with descriptions.

As for the first option, it is not problematic to add lambda operator in such a way as to cover Garson's system λS which permits expression of scope differences like de re/de dicto distinction. In Garson [11] it is formalised by means of one axiom:

$\lambda x\varphi(x)(a) \leftrightarrow \varphi(a)$

where $\lambda x\varphi(x)$ is a predicate abstracted from a formula φ. It is rather restricted use of lambda operator but still permitting important extension of expressive power. In the setting of sequent calculus it is enough to add two rules:

$$(\Rightarrow \lambda)\ \frac{\Gamma \Rightarrow \Delta, \varphi(a)}{\Gamma \Rightarrow \Delta, \lambda x\varphi(x)(a)} \qquad (\lambda \Rightarrow)\ \frac{\varphi(a), \Gamma \Rightarrow \Delta}{\lambda x\varphi(x)(a), \Gamma \Rightarrow \Delta}$$

It is obvious that their addition cannot spoil the proof of cut elimination since they are also reductive.

Finally, we will take a look at possible strengthenings of this system and some alternatives. If we do not consider elements connected with the treatment of modalities and rigid/nonrigid term distinctions but focus only on the rules for descriptions it is obvious that Thomason's and Garson's theory of descriptions is essentially the minimal free description theory MFD of Lambert [23], [24]. We have a strengthening of the latter not only in the sense that it is developed on the basis of modal logic since it is a conservative extension. More important is the addition of a rule $(\exists i)$ which in our system is covered by $(= d \Rightarrow)$. Yet this may be still considered to be rather weak theory of definite descriptions in the sense that it is rather concerned with proper definite descriptions. There are some costs of that, e.g. the law of extensionality of the form: $\forall x(\varphi \leftrightarrow \psi) \rightarrow \imath x\varphi = \imath x\psi$ which is technically useful, cannot be proved although we can prove its weaker version: $E\imath x\varphi \rightarrow (\forall x(\varphi \leftrightarrow \psi) \rightarrow \imath x\varphi = \imath x\psi)$. In the setting of classical logic, Fregean approach with the chosen object being denotation of all improper descriptions provides a solution to this problem. It may be criticised as artificial (Garson's theory does not equate all improper descriptions) but technically it seems to be more plausible. Fregean approach was developed formally by Kalish and Montague [21] and recently also received a cut-free sequent calculus in Indrzejczak [19]. A counterpart of this approach in the setting of free logic is even easier to formulate and was provided by Scott [29]. It is enough to add an axiom $\neg Ed \rightarrow d = \imath xx \neq x$. Since a converse of this implication also holds for Scott's logic we can express it in the setting of SC by means of two rules:

$$(\Rightarrow id)\ \frac{Ed, \Gamma \Rightarrow \Delta,}{\Gamma \Rightarrow \Delta, d = \imath x(x \neq x)} \qquad (id \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, Ed}{d = \imath x(x \neq x), \Gamma \Rightarrow \Delta}$$

Again it is clear that our proof of cut elimination still holds for such extension. One can prove in this logic a full version of the law of extensionality for descriptions. Lambert [24] considered also other extensions of MFD, weaker than Scott's logic, and some of them may be also expressed quite easily in our framework by means of reasonably simple rules. We illustrate the issue with two examples. The logic FD1 is an extension of MFD by means of the addition of cancellation law: $t = (\imath x\varphi = t)$, An effect of this axiom is directly obtained by means of the rule similar to $(==\!\!>)$ and $(= d \Rightarrow)$ but with such identity in the antecedent of the premiss. If we are interested just in having extensionality principle for descriptions we can add it as an axiom to MFD. This way we obtain the system FDV due to van Fraasen. The same effect can be obtained in SC by the addition of the rule:

$$(\Rightarrow d_1 = d_2)\ \frac{\varphi[x/a], \Gamma \Rightarrow \Delta, \psi[x/a] \qquad \psi[x/a], \Pi \Rightarrow \Sigma, \varphi[x/a]}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \imath x\varphi = \imath x\psi}$$

where a is not in $\varphi, \psi, \Gamma, \Delta, \Pi, \Sigma$.

The proof of cut elimination presented in the Appendix works for all such extensions of SC!S.

There are at least two approaches to definite descriptions in modal logic framework which are significantly different from the hierarchy of theories developed in free logic setting; one is due to Goldblatt [12], and the other to Fitting and Mendelsohn [7]. The former uses two-sorted language to make a distinction between rigid and nonrigid terms. In many respects the strategy of expressing relations between both kinds of terms is similar to the approach of Thomason/Garson and a rule which is a counterpart of $(\exists i)$ is applied. However, there are also strong differences. Goldblatt's theory is based on a semantics where individual variables range not over objects from D but over substances which are defined as partial functions from W to D. One of the consequences of this choice is invalidation of reflexivity of identity; a weaker axiom $t = t' \rightarrow t = t$ is postulated instead. Note also that $t = t$ is equivalent in his system to Et, hence this axiom may be expressed as $t = t' \rightarrow Et$. Although semantical machinery of Goldblatt's approach is a bit more complicated, on the syntactical side it has no serious consequences. The only axiom for definite descriptions[5] has the form: $\imath x\varphi = t \leftrightarrow Et \wedge \forall x(\varphi \leftrightarrow x = t)$, where t is rigid and does not have free x. Note that t may be also a rigid description. Goldblatt shows that in his approach descriptions cannot be eliminated in the Russelian way; it would be possible only for rigid descriptions and on the condition that some chosen nonexistent object would be added in the spirit of Fregean approach.

We can easily obtain an equivalent formalization of Goldblatt's axiom in standard sequent calculus by means of the following rules:

$$(\Rightarrow \imath)^1\ \frac{\Gamma_1 \Rightarrow \Delta_1, Et \qquad Ea, \varphi[x/a], \Gamma_2 \Rightarrow \Delta_2, t = a \qquad Ea, t = a, \Gamma_3 \Rightarrow \Delta_3, \varphi[x/a]}{\Gamma \Rightarrow \Delta, t = \imath x\varphi}$$

[5] Strictly speaking it works for modal logics characterised by Kripkean models. Golblatt considers also logics characterised by non-Kripkean models, where truth conditions for quantifiers are weaker, and such logics require also a template rule for description.

1. where a is not in Γ, Δ, φ

$$(\imath \Rightarrow)\ \frac{Et, \Gamma_1 \Rightarrow \Delta_1, Ea \qquad Et, \Gamma_2 \Rightarrow \Delta_2, \varphi[x/a], t = a \qquad Et, \varphi[x/a], t = a, \Gamma_3 \Rightarrow \Delta_3}{t = \imath x\varphi, \Gamma \Rightarrow \Delta}$$

However, one should remember that also rules for identity and quantifiers need to be changed in order to comply with Goldblatt's system. For completeness we need template rules like $(T \Rightarrow \forall)$ and $(T = d \Rightarrow)$. We do not develop this system here for the lack of space; only suitable rules for definite descriptions were displayed above just for comparison with Garson's approach.

Both rules for Goldblatt's theory of descriptions are also reductive, so we can expect that cut elimination holds in the way demonstrated in the Appendix. There is one serious difficulty however. In Goldblatt's system rules for quantifiers work not only for rigid but also for nonrigid terms. Moreover, descriptions may be also rigid. It may seem an advantage, in comparison to Garson's system, but in fact it is not, since rules for quantifiers are not reductive. We mean here the fact that in the course of the application of $(\Rightarrow \forall)$ a variable of suitable sort may be instantiated with definite description of arbitrary complexity, so the instance of substituted formula is not less complex and we cannot obtain reduction of the complexity of cut formula. To avoid the problem we should have the instantiation of quantifiers restricted to rigid terms and only nonrigid definite descriptions as in Garson's system. Another possibility would be to apply the solution from Indrzejczak [19] where all terms have the same complexity measure but this solution does not work either. After changing the definition of complexity in such a way the above rules for descriptions do not allow for reduction of cut-degree (see Appendix) since in their premisses a term is unpacked and occurs as a formula which is at least as complex as the identity with description in the conclusion. It was not a problem for Fregean system from [19] since all rules for definite descriptions introduce them only to succedents and the situation with cut on such formulae as principal simply does not arise. Summing up, in contrast to Garson's approach, a version of description theory present in Goldblatt is harder for proving cut elimination theorem, and for the time being we are unable to provide a solution to this problem.

Finally, there is an alternative approach to first-order modal logics with descriptions provided by Fitting and Mendelsohn [7]. As we mentioned in the Introduction, their theory of definite descriptions is perhaps a subtler solution since it does not equate existence and designation and makes distinctions between different kinds of improper descriptions. It is again significantly different than any theory belonging to Lambert hierarchy since, for example $t = t$ is also not a thesis in general but holds only for designating terms. In this respect, their theory is similar to Goldblatt's approach but, in contrast, it is too rich on the syntactical level to be dealt with by means of standard resources which are used in this research. In particular, tableau systems for these logics are based on the complex machinery of labels attached not only to formulae but also to nonrigid terms to fix their denotations in possible worlds.

In fact, insufficiency of the standard apparatus for formalization of Fitting and Mendelsohn's theory is more connected with the way in which nonrigid terms in general are dealt with, than with the specific features of definite descriptions in their theory. In short, nonrigid terms are "rigidified" by means of labels, and lambda operator is used just to permit predication on nonrigid terms, in contrast to Garson's restricted solution.

It seems that the framework of hybrid logics is promising here. One could possibly extend the approach to hybrid first-order modal logic provided by Blackburn and Marx [3] to cover this theory of descriptions, and apply cut-free sequent calculi for hybrid logics developed by Braüner [4] or Indrzejczak [18]. But this is a future project; for the time being we are concerned with the abilities of standard sequent calculi with no labels or nominals.

A Appendix

We will show that for every $\Gamma \vdash \varphi$ derivable in !S we can provide a proof of $\Gamma \Rightarrow \varphi$ in SC!S and conversely. Since for the propositional part as well as for (free) quantifiers and identity rules the equivalence is clear we restrict the consideration to rules for descriptions.

Theorem A.1 *If* $\Gamma \vdash_{!S} \varphi$*, then* $\vdash_{SC!S} \Gamma \Rightarrow \varphi$.

Proof. It is sufficient to show that $(\exists i)$ is a derivable rule in SC!S and that sequents corresponding to both rules for $\imath$ are provable. As for the first:

$$\dfrac{\dfrac{a = d \Rightarrow \neg\neg a = d \qquad \dfrac{\Gamma \Rightarrow \neg a = d}{\neg\neg a = d, \Gamma \Rightarrow}\,(\neg \Rightarrow)}{a = d, \Gamma \Rightarrow}\,(Cut)}{\Gamma \Rightarrow}\,(= d \Rightarrow)$$

The sequents corresponding to Garson's rules are:

$Ea, a = \imath x\varphi(x) \Rightarrow \forall x(\varphi(x) \leftrightarrow x = a)$
$Ea, \forall x(\varphi(x) \leftrightarrow x = a) \Rightarrow a = \imath x\varphi(x)$

We can prove the first one in the following way. First, using $(\imath \Rightarrow)$ twice, we construct:

$$\dfrac{Ea \Rightarrow Ea \qquad Eb \Rightarrow Eb \qquad \varphi(b) \Rightarrow \varphi(b), a = b \qquad \varphi(b), a = b \Rightarrow a = b}{Ea, Eb, a = \imath x\varphi(x), \varphi(b) \Rightarrow a = b}$$

and

$$\dfrac{Ea \Rightarrow Ea \qquad Eb \Rightarrow Eb \qquad a = b \Rightarrow \varphi(b), a = b \qquad \varphi(b), a = b \Rightarrow \varphi(b)}{Ea, Eb, a = \imath x\varphi(x), a = b \Rightarrow \varphi(b)}$$

Both, by the application of $(\Rightarrow\leftrightarrow)$, contractions and $(\Rightarrow \forall)$ yield:

$$\dfrac{Ea, Eb, a = \imath x\varphi(x) \Rightarrow \varphi(b) \leftrightarrow a = b}{Ea, a = \imath x\varphi(x) \Rightarrow \forall x(\varphi(x) \leftrightarrow a = x)}$$

where b is new in the next to last sequent.

For the second sequent we prove first:

$$\cfrac{Eb \Rightarrow Eb \qquad \cfrac{\varphi(b), a = b \Rightarrow a = b \qquad \varphi(b) \Rightarrow \varphi(b), a = b}{\varphi(b) \leftrightarrow a = b, \varphi(b) \Rightarrow a = b}\ (\leftrightarrow\Rightarrow)}{\forall x(\varphi(x) \leftrightarrow a = x), Eb, \varphi(b) \Rightarrow a = b}\ (\forall \Rightarrow)$$

and

$$\cfrac{Eb \Rightarrow Eb \qquad \cfrac{\varphi(b), a = b \Rightarrow \varphi(b) \qquad a = b \Rightarrow \varphi(b), a = b}{\varphi(b) \leftrightarrow a = b, a = b \Rightarrow \varphi(b)}\ (\leftrightarrow\Rightarrow)}{\forall x(\varphi(x) \leftrightarrow a = x), Eb, a = b \Rightarrow \varphi(b)}\ (\forall \Rightarrow)$$

Let S_1 and S_2 denote the roots ot the above proof-trees. Then we obtain:

$$\cfrac{\cfrac{Ea \Rightarrow Ea \qquad S_1 \qquad S_2}{Ea, \forall x(\varphi(x) \leftrightarrow a = x), \forall x(\varphi(x) \leftrightarrow a = x) \Rightarrow a = \imath x\varphi(x)}\ (\Rightarrow \imath)}{Ea, \forall x(\varphi(x) \leftrightarrow a = x) \Rightarrow a = \imath x\varphi(x)}\ (C \Rightarrow)$$

where b is new. □

Theorem A.2 *If* $\vdash_{SC!S} \Gamma \Rightarrow \Delta$, *then* $\Gamma \vdash_{!S} \vee\Delta$, *where* $\vee\Delta$ *is a disjunction of elements of* Δ.

Proof. Since one can simulate in Garson's ND system !S all the rules of SC!S we can use the reduct of the latter without the rules for description and $(= d \Rightarrow)$ but with an analog of $(\exists i)$ and two additional axiomatic sequents corresponding to rules:

$Ea, a = \imath x\varphi(x) \Rightarrow \forall x(\varphi(x) \leftrightarrow x = a)$
$Ea, \forall x(\varphi(x) \leftrightarrow x = a) \Rightarrow a = \imath x\varphi(x)$

Call this calculus SC!S*. It is obviously equivalent to !S so it will be enough to demonstrate that the three rules in question are derivable in SC!S*. Derivability of $(= d \Rightarrow)$ is trivial by $(\Rightarrow \neg)$ and $(\exists i)$ but for the remaining rules proofs are more involved. Let us start with $(\Rightarrow \imath)$. On the basis of the first sequent and the first premiss we obtain:

$$\cfrac{\Gamma_1 \Rightarrow \Delta_1, Ea \qquad Ea, a = \imath x\varphi(x) \Rightarrow \forall x(\varphi(x) \leftrightarrow x = a)}{a = \imath x\varphi(x), \Gamma_1 \Rightarrow \Delta_1, \forall x(\varphi(x) \leftrightarrow x = a)}\ (Cut)$$

Three remaining premisses yield:

$$\cfrac{\Gamma_2 \Rightarrow \Delta_2, Eb \qquad \cfrac{\Gamma_3 \Rightarrow \Delta_3, \varphi(b), a = b \qquad \varphi(b), a = b, \Gamma_4 \Rightarrow \Delta_4}{\varphi(b) \leftrightarrow a = b, \Gamma_3, \Gamma_4 \Rightarrow \Delta_3, \Delta_4}\ (\leftrightarrow\Rightarrow)}{\forall x(\varphi(x) \leftrightarrow a = x), \Gamma_2, \Gamma_3, \Gamma_4 \Rightarrow \Delta_2, \Delta_3, \Delta_4}\ (\forall \Rightarrow)$$

By cut on these two root sequents we obtain $a = \imath x\varphi(x), \Gamma \Rightarrow \Delta$.

For the second rule, from the first premiss and the second sequent we obtain:

$$\cfrac{\Gamma_1 \Rightarrow \Delta_1, Ea \qquad Ea, \forall x(\varphi(x) \leftrightarrow a = x) \Rightarrow a = \imath x\varphi(x)}{\forall x(\varphi(x) \leftrightarrow a = x), \Gamma_1 \Rightarrow \Delta_1, a = \imath x\varphi(x)}\ (Cut)$$

its root S in combination with the remaining premisses yields:

$$\dfrac{(Cut)\ \dfrac{(\Rightarrow\forall)\ \dfrac{(C\Rightarrow)\ \dfrac{(\Rightarrow\leftrightarrow)\ \dfrac{Eb, \varphi(b), \Gamma_2 \Rightarrow \Delta_2, a=b \qquad Eb, a=b, \Gamma_3 \Rightarrow \Delta_3, \varphi(b)}{Eb, Eb, \Gamma_2, \Gamma_3 \Rightarrow \Delta_2, \Delta_3, \varphi(b) \leftrightarrow a=b}}{Eb, \Gamma_2, \Gamma_3 \Rightarrow \Delta_2, \Delta_3, \varphi(b) \leftrightarrow a=b}}{\Gamma_2, \Gamma_3 \Rightarrow \Delta_2, \Delta_3, \forall x(\varphi(x) \leftrightarrow a=x)} \qquad S}{\Gamma \Rightarrow \Delta, a = \imath x\varphi(x)}$$

where b is new. □

As a preliminary step for proving cut elimination we need:

Lemma A.3 (Substitution) *If* $\vdash_k \Gamma \Rightarrow \Delta$*, then* $\vdash_k (\Gamma \Rightarrow \Delta)[a/t]$.

Proof. By induction on the height of a proof. It is straightforward but tedious exercise. Note that we provided not sheer admissibility but height-preserving admissibility. □

Moreover, we assume that all proofs satisfy the condition of regularity – every constant which is fresh by side condition on the respective rule must be fresh in the entire proof, not only on the branch where the application of this rule takes place. Clearly, every proof may be systematically transformed into regular proof by Substitution lemma.

Let us define the notions of cut-degree and proof-degree:

(i) Cut-degree is the complexity of cut-formula φ, i.e. the number of connectives and operators occurring in φ and is denoted as $d\varphi$;

(ii) Proof-degree ($d\mathcal{D}$) is the maximal cut-degree in $\mathcal{D}$.

The proof of cut elimination theorem is based on two lemmata which make a reduction first on the right and then on the left premiss of cut. The general strategy of proof was originally developed for hypersequent calculi by Metcalfe, Olivetti and Gabbay [25] and later extensively used in this framework (see e.g. Ciabattoni, Metcalfe, Montagna [5], Indrzejczak [17], Kurokawa [22]). However it is also applicable to standard sequent calculi (see Indrzejczak [18], [20]) and allows for elegant proof which helps to avoid many complexities inherent in other methods of proving cut elimination. Here are the key lemmata:

Lemma A.4 (Right reduction) *Let* $\mathcal{D}_1 \vdash \Gamma \Rightarrow \Delta, \varphi$ *and* $\mathcal{D}_2 \vdash \varphi^k, \Pi \Rightarrow \Sigma$ *with* $d\mathcal{D}_1, d\mathcal{D}_2 < d\varphi$*, and* φ *principal in* $\Gamma \Rightarrow \Delta, \varphi$*, then we can construct a proof* $\mathcal{D}$ *such that* $\mathcal{D} \vdash \Gamma^k, \Pi \Rightarrow \Delta^k, \Sigma$ *and* $d\mathcal{D} < d\varphi$*.*

Lemma A.5 (Left reduction) *Let* $\mathcal{D}_1 \vdash \Gamma \Rightarrow \Delta, \varphi^k$ *and* $\mathcal{D}_2 \vdash \varphi, \Pi \Rightarrow \Sigma$ *with* $d\mathcal{D}_1, d\mathcal{D}_2 < d\varphi$*, then we can construct a proof* $\mathcal{D}$ *such that* $\mathcal{D} \vdash \Gamma, \Pi^k \Rightarrow \Delta, \Sigma^k$ *and* $d\mathcal{D} < d\varphi$*.*

Proof. For lemma 4 by induction on the height of $\mathcal{D}_2$. The basis is trivial. Induction step requires consideration of all cases of possible derivation of $\varphi^k, \Pi \Rightarrow \Sigma$ and the role of cut-formula in the transition. In cases where all occurrences of φ are parametric we simply apply the induction hypotheses to premisses of $\varphi^k, \Pi \Rightarrow \Sigma$ and then apply to them respective rule – it is essentially due to the context independence of almost all rules and regularity of proofs which prevents violation of side conditions. If one of the occurrences of φ in the premiss(es) is a side formula of the last rule we must additionaly apply weakening to restore the lacking formula before the application of a rule. Note also that the situation with $(\Rightarrow \Box)$ as the last applied rule is not of this case since φ, being some $\Box\psi$, is not present in the premiss (ψ is).

In cases where one occurrence of φ in $\varphi^k, \Pi \Rightarrow \Sigma$ is principal we make use of the fact that φ in the left premiss is principal too (note that for C and W it is trivial). We analyse two cases.

Case of $\forall x\varphi(x)$:

$$\dfrac{\dfrac{Ea, \Gamma \Rightarrow \Delta, \varphi(a)}{\Gamma \Rightarrow \Delta, \forall x\varphi(x)} \qquad \dfrac{\forall x\varphi(x)^i, \Pi_1 \Rightarrow \Sigma_1, Eb \qquad \varphi(b), \forall x\varphi(x)^j, \Pi_2 \Rightarrow \Sigma_2}{\forall x\varphi(x)^k, \Pi \Rightarrow \Sigma}}{\Gamma^k, \Pi \Rightarrow \Delta^k, \Sigma}$$

where $k = i + j + 1$ and a is fresh, hence by Substitution Lemma we have:

$Eb, \Gamma \Rightarrow \Delta, \varphi(b)$

By the induction hypothesis we have:

$\Gamma^i, \Pi_1 \Rightarrow \Delta^i, \Sigma_1, Eb$
$\varphi(b), \Gamma^j, \Pi_2 \Rightarrow \Delta^j, \Sigma_2$

Now we can build a proof:

$$\dfrac{\Gamma^i, \Pi_1 \Rightarrow \Delta^i, \Sigma_1, Eb \qquad \dfrac{Eb, \Gamma \Rightarrow \Delta, \varphi(b) \qquad \varphi(b), \Gamma^j, \Pi_2 \Rightarrow \Delta^j, \Sigma_2}{Eb, \Gamma^{j+1}, \Pi_2 \Rightarrow \Delta^{j+1}, \Sigma_2}}{\Gamma^k, \Pi \Rightarrow \Delta^k, \Sigma}$$

Case of $a = \imath x\varphi(x)$:

In the right premiss we have $a = \imath x\varphi(x)^k, \Pi \Rightarrow \Sigma$ deduced from:

$a = \imath x\varphi(x)^i, \Pi_1 \Rightarrow \Sigma_1, Ea$
$a = \imath x\varphi(x)^j, \Pi_2 \Rightarrow \Sigma_2, Eb$
$a = \imath x\varphi(x)^n, \Pi_3 \Rightarrow \Sigma_3, \varphi(b), a = b$
$a = \imath x\varphi(x)^m, \varphi(b), a = b, \Pi_4 \Rightarrow \Sigma_4$

where $k = i + j + n + m + 1$ and by the induction hypothesis we obtain:

(a) $\Gamma^i, \Pi_1 \Rightarrow \Delta^i, \Sigma_1, Ea$
(b) $\Gamma^j, \Pi_2 \Rightarrow \Delta^j, \Sigma_2, Eb$
(c) $\Gamma^n, \Pi_3 \Rightarrow \Delta^n, \Sigma_3, \varphi(b), a = b$
(d) $a = b, \varphi(b), \Gamma^m, \Pi_4 \Rightarrow \Delta^m, \Sigma_4$

In the left premiss we have:

$$\dfrac{\Gamma_1 \Rightarrow \Delta_1, Ea \qquad Ec, \varphi(c), \Gamma_2 \Rightarrow \Delta_2, a = c \qquad Ec, a = c, \Gamma_3 \Rightarrow \Delta_3, \varphi(c)}{\Gamma \Rightarrow \Delta, a = \imath x\varphi(x)}$$

where c is not in Γ, Δ, φ hence by Substitution Lemma we obtain:

(e) $Eb, \varphi(b), \Gamma_2 \Rightarrow \Delta_2, a = b$
(f) $Eb, a = b, \Gamma_3 \Rightarrow \Delta_3, \varphi(b)$

These sequents may be combined, by cuts and contractions, in the following way:

$$\dfrac{\dfrac{\Gamma^n, \Pi_3 \Rightarrow \Delta^n, \Sigma_3, \varphi(b), a = b \qquad \dfrac{\Gamma^j, \Pi_2 \Rightarrow \Delta^j, \Sigma_2, Eb \qquad Eb, a = b, \Gamma_3 \Rightarrow \Delta_3, \varphi(b)}{a = b, \Gamma^j, \Pi_2, \Gamma_3 \Rightarrow \Delta^j, \Sigma_2, \Delta_3, \varphi(b)}}{\Gamma^n, \Pi_3, \Gamma^j, \Pi_2, \Gamma_3 \Rightarrow \Delta^n, \Sigma_3, \Delta^j, \Sigma_2, \Delta_3, \varphi(b), \varphi(b)}}{\Gamma^n, \Pi_3, \Gamma^j, \Pi_2, \Gamma_3 \Rightarrow \Delta^n, \Sigma_3, \Delta^j, \Sigma_2, \Delta_3, \varphi(b)}$$

and

$$\dfrac{\dfrac{\dfrac{\Gamma^j, \Pi_2 \Rightarrow \Delta^j, \Sigma_2, Eb \qquad Eb, \varphi(b), \Gamma_2 \Rightarrow \Delta_2, a = b}{\varphi(b), \Gamma^j, \Pi_2, \Gamma_2 \Rightarrow \Delta^j, \Sigma_2, \Delta_2, a = b} \qquad a = b, \varphi(b), \Gamma^m, \Pi_4 \Rightarrow \Delta^m, \Sigma_4}{\varphi(b), \varphi(b), \Gamma^m, \Pi_4, \Gamma^j, \Pi_2, \Gamma_2 \Rightarrow \Delta^m, \Sigma_4, \Delta^j, \Sigma_2, \Delta_2}}{\varphi(b), \Gamma^m, \Pi_4, \Gamma^j, \Pi_2, \Gamma_2 \Rightarrow \Delta^m, \Sigma_4, \Delta^j, \Sigma_2, \Delta_2}$$

By cut on the last two sequents and several contractions we obtain $\Gamma^k, \Pi \Rightarrow \Delta^k, \Sigma$. Note that all cuts are of lower degree hence we are done. □

The proof of the Left Reduction Lemma is similar but on the height of $\mathcal{D}_1$. The only difference is that now we do not assume that cut-formula in the right premiss is principal. Therefore, when cut-formula is principal in the left premiss we apply first the induction hypothesis and next the rule in question to side-formulae. The new proof of the left premiss satisfies the assumption of the Right Reduction Lemma, so we can safely apply it and, possibly after some applications of structural rules, obtain the result. If the last rule was $(\Rightarrow=)$ or $(= \Box)$ and one of the active formulae from succedent was involved, then we obtain the result by the induction hypothesis from this single premiss, like in case of contraction. Hence no essentially new cases appear and we can skip detailed analysis. Eventually, on the basis of the Left Reduction Lemma we obtain cut elimination by successive decreasing of cut degree in the input proof. Therefore:

Theorem A.6 (Cut Elimination) *If $\Gamma \Rightarrow \Delta$ is provable, then it is provable without applications of (Cut).*

Note that this result applies also to SC!S in extended form, i.e. with template rules $(T \Rightarrow \forall)$ and $(T = d \Rightarrow)$. Since the only possible rule applied to their premiss is $(\Rightarrow\rightarrow)$ both cases are treated as cases of implication and no special operations are needed.

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