A Remark on Propositional Kripke Frames Sound for Intuitionistic Logic

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Abstract

Usually, in the Kripke semantics for intuitionistic propositional logic (or for superintuitionistic logics) partially ordered frames are used. Why? In this paper we propose an intrinsically intuitionistic motivation for that. Namely, we show that every Kripke frame (with an arbitrary accessibility relation), whose set of valid formulas is a superintuitionistic logic, is logically equivalent to a partially ordered Kripke frame.

Keywords: Intuitionistic propositional logic, intermediate logics, Kripke semantics, logical soundness.

While considering the Kripke semantics for intuitionistic propositional logic (or for superintuitionistic logics), only partially ordered (p.o.) Kripke frames are usually used. It is well known that quasi-ordered (i.e., reflexive and transitive) frames can be accepted as well, but this is in essence just the same. In fact, the quotient (modulo the usual equivalence) of a quasi-ordered Kripke frame is a p.o. frame, and intuitionistic formulas 'do not notice' this transformation. On the other hand, it is known (cf. e.g. [5], [4]) that some non-quasi-ordered Kripke frames are sound for intuitionistic logic as well (via the usual definition of intuitionistic validity).

So the conventional restriction of all frames to partial orderings seems to be slightly ad hoc. The most common motivation [1] appeals to modal logic and to Gödel – Tarski

[1] An anonymous referee reminded an informal motivation for using partial orderings in intuitionistic semantics: possible worlds represent knowledge and the accessibility corresponds to acquiring knowledge. The inclusion of sets is definitely a partial ordering, but this informal motivation does not seem convincing to us by different reasons. E.g., the generally accepted 'monotonicity condition' (knowledge grows in future) is controversial, because it ignores delusions and mistakes. Nevertheless, the monotonicity is almost necessary for intuitionistic semantics, because it corresponds to an intuitionistic axiom (cf. [4] or Lemma 2.1(2) in Section 2.1).

translation of intuitionistic logic in logic **S4** (recall that Kripke frames sound for **S4** are exactly the quasi-ordered ones). But this motivation seems rather external for intuitionistic logic, and so it is not quite convincing (all the more, **S4** is not the weakest normal modal logic, in which intuitionistic logic can be embedded via the Gödel – Tarski translation, see [1], [2], and [3]).

One can try to find a more intrinsically intuitionistic argument to explain, why namely quasi-orderings (or equivalently, partial orderings) are in some sense immanent for a sound interpretation of intuitionistic, or of superintuitionistic logics. Here we propose such a motivation.

Namely, we show that every Kripke frame whose set of valid formulas is a superintuitionistic logic, is logically equivalent to a partially ordered frame.

Now let us turn to exact definitions.

1 Intuitionistic sound Kripke frames

We consider *superintuitionistic propositional logics* understood in the usual way, as sets of formulas containing all axioms of *intuitionistic* (or *Heyting*) *propositional logic* **H** and closed under modus ponens and the substitution rule. So **H** is the smallest superintuitionistic logic. It is well known that all consistent superintuitionistic logics are included in *classical logic* $\mathbf{C} = (\mathbf{H} + p\vee\neg p)$; these logics are also called *intermediate*.

For convenience, in addition to connectives $\&, \vee, \rightarrow$, we also use constants $\bot$ (the falsity) and $\top$ (the truth). We use the standard abbreviations $\neg A = (A \rightarrow \bot)$ and $(A\leftrightarrow B) = (A\rightarrow B)\&(B\rightarrow A)$.

Let Var be the set of variables and Im be the set of all implications.

Remark Obviously, one can eliminate the constant $\top$ by replacing it with $(\bot \rightarrow \bot)$ or with $(A \rightarrow A)$ for any formula A. Usually one can also eliminate $\bot$ by using the independent connective $\neg$; then $\bot$ is defined as $A\&\neg A$ or $\neg(A\rightarrow A)$ for an arbitrary A. However, this is not quite semantically adequate, as we will see later (cf. e.g. Example 2 in Section 1.3). On the other hand, our considerations can be easily transferred to the language with the basic connective $\neg$ instead of $\bot$ (with minor modifications [2]).

1.1 We consider propositional *Kripke frames*; such a frame F is a non-empty set W with an arbitrary binary relation R on W. We write $u\in F$ for $u\in W$.

A *valuation* (more precisely, an *intuitionistic valuation*) in a Kripke frame $F = (W, R)$ is a forcing relation $u \vDash A$ between points $u\in F$ and formulas A, satisfying the usual intuitionistic clauses (for all $u\in F$):

$$u \vDash (B\&C) \Leftrightarrow (u \vDash B)\,\&\,(u \vDash C), \quad u \vDash (B \vee C) \Leftrightarrow (u \vDash B) \vee (u \vDash C),$$

$$u \vDash (B\rightarrow C) \Leftrightarrow \forall v\,[\,uRv\ \&\ v \vDash B\ \Rightarrow\ v \vDash C\,], \quad u \nvDash \bot, \quad u \vDash \top,$$

[2] E.g. one can use $\top\rightarrow p$ at some point, where we use $\top\rightarrow\bot$, etc.

and the following condition (for variables $p \in \mathrm{Var}$): [3]

$$\text{(Atomic heredity)} \qquad \forall u, v \in F\ [\, uRv\ \&\ u \vDash p \ \Rightarrow\ v \vDash p\,].$$

Clearly, $u \vDash \neg B \Leftrightarrow \forall v \in R(u)\,[\, v \not\vDash B\,]$ and $u \vDash (A \leftrightarrow B) \Leftrightarrow \forall v \in R(u)\,[\, v \vDash A \Leftrightarrow v \vDash B\,]$.

A formula A is said to be *true* under a valuation in F if $u \vDash A$ for any $u \in F$. A formula A is *valid* in a Kripke frame F if it is true under any valuation in F. Let $\mathbf{L}(F)$ denote the set of all formulas valid in F.

We call a frame F *intuitionistic sound* (or **H**-*sound*, for short) if $\mathbf{L}(F)$ is a superintuitionistic (or equivalently, an intermediate) logic. We say that F is *weakly* **H**-*sound* (or *w***H**-*sound*) if all intuitionistic theorems are valid in F, i.e., if $\mathbf{H} \subseteq \mathbf{L}(F)$.

Later on we will see that a w**H**-sound frame is **H**-sound iff $\mathbf{L}(F)$ is closed under modus ponens. We suppose that Kripke frames violating modus ponens are rather unsatisfactory for intuitionistic logic (even if they validate **H**). [4]

Note that $\mathbf{L}(F)$ is closed under modus ponens iff

$$(\top \to A) \in \mathbf{L}(F) \ \Rightarrow\ A \in \mathbf{L}(F) \quad \text{for any formula } A. \tag{MP}$$

In fact, (MP) implies that $\quad B,\ (B \to A) \in \mathbf{L}(F) \Rightarrow A \in \mathbf{L}(F)$,
because if $B \in \mathbf{L}(F)$, then $\forall u (u \vDash B)$ and so $\forall u [\, u \vDash (B \to A) \ \Leftrightarrow\ u \vDash (\top \to A)\,]$ for every valuation in F.

By the way, the converse implication $\quad A \in \mathbf{L}(F) \ \Rightarrow\ (\top \to A) \in \mathbf{L}(F)$
obviously holds in every Kripke frame F; so the implication in the condition (MP) actually means the equivalence.

By the semantics generated by a class $\mathcal{K}$ of frames we mean the class of logics $\mathcal{S}(\mathcal{K}) = \{\, \mathbf{L}(F) \mid F \in \mathcal{K}\,\}$. So the class of **H**-sound Kripke frames generates the maximal possible Kripke semantics for superintuitionistic logics (with the usual definition of intuitionistic forcing).

The following simple technical lemma will be unexpectedly useful further on:

Lemma 1.1 *For any formula A there exists a formula*

$$A' = \mathop{\&}_i \bigvee_j A_{ij}, \quad \text{where } \forall i, j\ [\, A_{i,j} \in \mathrm{Im} \cup \mathrm{Var}\,] \tag{$*$}$$

(*or* $A' = \bot$) [5] *such that:*

(1) $\mathbf{H} \vdash (A \leftrightarrow A')$;

(2) $u \vDash A \ \Leftrightarrow\ u \vDash A'$ *for every valuation in a frame* [6] F *and* $u \in F$.

So $\quad A \in \mathbf{L}(F) \ \Leftrightarrow\ A' \in \mathbf{L}(F) \quad$ for every frame F.

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[3] Here we use the terminology from [5], [4].

[4] If one prefers, another motivation for the validity of modus ponens is related to the notion of strong soundness mentioned in Section 2.5.

[5] Note that $\top$ can be presented by implication $(\bot \to \bot)$, but $\bot$ is not semantically equivalent to $(\top \to \bot)$, as we will see later on (e.g. in a one-element irreflexive frame, cf. Example 2 in Section 1.3).

[6] Here we in general do not suppose that F is (weakly) **H**-sound.

Proof. Note that the formula A is a &,$\vee$-combination of its subformulas from $\mathrm{Im} \cup \mathrm{Var} \cup \{\bot, \top\}$. We transform it into an equivalent conjunctive normal form $(*)$. Now (1) is obvious. And (2) holds, because the 'inner' connectives & and $\vee$ in a formula correspond, by the definition of valuation, to the 'external' conjunction and disjunction satisfying the distributivity laws etc. $\square$

1.2 Let R^+ and R^* be the transitive closure of R (in a frame $F = (W, R)$) and the corresponding quasi-ordering, i.e.,

$$uR^+v \Leftrightarrow \exists n>0\,(u\,R^n\,v), \qquad uR^*v \Leftrightarrow (u=v) \vee (uR^+v).$$

For $u \in F$ define the *cone* $F^u = (W^u, R\,|\,W^u)$, where $W^u = R^*(u) = \{u\} \cup R^+(u)$. A set $W' \subseteq W$ is *open* if it is upward closed, i.e., $\forall u \in W'\, \forall v \in R(u)\,(v \in W')$, [7] or equivalently, iff $\forall u \in W'\,(W^u \subseteq W')$ (in other words, $W' = \bigcup (W^u : u \in W')$). Naturally, an open set W' gives rise to an *open subframe* $F' = (W', R\,|\,W')$ of F. Clearly, the restriction of a valuation in F to a cone F^u or to any open subframe is a valuation again (the inductive clauses are preserved obviously). So we conclude that

$$\mathbf{L}(F) \subseteq \mathbf{L}(F') \quad \text{for any open subframe } F' \text{ of } F,$$

and $\mathbf{L}(F) = \bigcap (\mathbf{L}(F^u) : u \in F)$. Thus

(1) (*F is w$\mathbf{H}$-sound*) $\Leftrightarrow \forall u \in F$ (*F^u is w$\mathbf{H}$-sound*);

(2) $\forall u \in F$ (*F^u is $\mathbf{H}$-sound*) $\Rightarrow$ (*F is $\mathbf{H}$-sound*),

the converse to (2) in general does not hold, as we shall see later on.

It is well known that all p.o. frames are $\mathbf{H}$-sound; moreover, quasi-ordered frames are $\mathbf{H}$-sound as well. Actually, for a quasi-ordered Kripke frame its quotient modulo the equivalence relation $(u \equiv v) \Leftrightarrow (uRv)\,\&\,(vRu)$ is a p.o. frame $S[F] = (F/\equiv)$ (the *skeleton* of F). Clearly, F and $S[F]$ have in essence the same valuations (modulo $\equiv$); thus $\mathbf{L}(F) = \mathbf{L}(S[F])$ in this case.

Now we define the *skeleton* $S[F]$ for an arbitrary frame $F = (W, R)$ as the skeleton of the associated quasi-ordered frame $F^* = (W, R^*)$. In other words, $S[F]$ is the quotient $(W/\equiv)$ modulo the equivalence relation $\equiv$ given by the following three equivalent conditions:

$$(u \equiv v) \text{ iff } [(u=v) \vee (uR^+v \,\&\, vR^+u)] \text{ iff } (v \in F^u \,\&\, u \in F^v) \text{ iff } (F^u = F^v),$$

partially ordered by the relation

$$(u/\equiv)\, R_S\, (v/\equiv) \Leftrightarrow (uR^*v) \Leftrightarrow (v \in F^u).$$

Clearly $\quad (u \equiv v) \Rightarrow (u \vDash p \Leftrightarrow v \vDash p) \quad$ for a valuation in F, so the following valuation in $S[F]$ is well-defined (and satisfies the atomic heredity):

$$(u/\equiv) \vDash_S p \Leftrightarrow u \vDash p \qquad \text{for variables } p\,.$$

Therefore there exists the natural one-to-one correspondence between valuations in F and in $S[F]$. However, in general we cannot guarantee that

$$(u/\equiv) \vDash_S A \Leftrightarrow u \vDash A \tag{S}$$

[7] Obviously, the cone W^u is the least open set containing u.

for non-atomic A, and the equality $\mathbf{L}(F) = \mathbf{L}(S[F])$ in general does not hold. Namely, definitely $\mathbf{L}(F) \neq \mathbf{L}(S[F])$ for every frame F that is not **H**-sound. Also there exist **H**-sound frames such that $\mathbf{L}(F) \neq \mathbf{L}(S[F])$; we will give an example in Section 2.4 (Appendix).

1.3 It is known that there exist **H**-sound not quasi-ordered Kripke frames. Let us begin with some useful and instructive examples.

Example 1 Let F be a two-element frame $W = \{v_1, v_2\}$ with irreflexive and non-transitive relation $R = \{\langle v_1, v_2\rangle, \langle v_2, v_1\rangle\}$, see Figure 1.

Clearly its skeleton $S[F]$ is a one-element reflexive frame, and one can easily check the equivalence (S) by induction on the complexity of A. So we obtain that $\mathbf{L}(F) = \mathbf{L}(S[F]) = \mathbf{C}$, and thus F is **H**-sound.

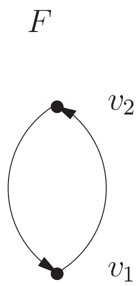

Fig. 1.

Similarly, one can take an arbitrary p.o. frame F' and replace some of its points u by cycles $u_1 \, R \, u_2 \, R \ldots R \, u_n \, R \, u_1$; these cycles can be chosen either reflexive or non-reflexive, transitive or non-transitive; and anyway we obtain a frame F such that $S[F] = F'$, $\mathbf{L}(F) = \mathbf{L}(F')$.

Example 2 Let $\Pi_0 = \{\pi_0\}$ be irreflexive one-element frame (with empty R), see Figure 2.

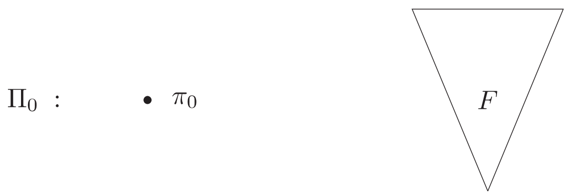

Fig. 2.

Clearly all implications are valid in Π_0, i.e., $\mathrm{Im} \subset \mathbf{L}(\Pi_0)$. [8]

[8] So $\neg(A \to A), (A \to A) \& \neg(A \to A) \in \mathbf{L}(\Pi_0)$, and both these formulas are not semantically equivalent

Lemma 1.2 *(1) Let A' be a formula (*) from Lemma 1.1. Then:*

$$A' \in \mathbf{L}(\Pi_0) \ \Leftrightarrow\ \forall i\, \exists j\, (\, A_{ij} \in \mathrm{Im}\,).$$

(2) $\mathbf{C} \subset \mathbf{L}(\Pi_0)$.

Proof. The 'if' part of (1) is obvious. Also, a formula $\bigvee_j p_j$ with variables p_j clearly does not belong to $\mathbf{L}(\Pi_0)$ and is not classically valid. Hence the 'only if' part of (1) follows as well. And if $A' \in \mathbf{C}$, then $A' \in \mathbf{L}(\Pi_0)$ by (1). □

Thus the frame Π_0 is w**H**-sound. However it is not **H**-sound; in fact, modus ponens fails in $\mathbf{L}(\Pi_0)$, since $(\top \to \bot) \in \mathbf{L}(\Pi_0)$ and $\bot \notin \mathbf{L}(\Pi_0)$.

On the other hand, for any partially ordered, and moreover, for any **H**-sound frame F the disjoint union (F, Π_0) of F and Π_0 (see Figure 2) is **H**-sound, since

$$\mathbf{L}(F, \Pi_0) = \mathbf{L}(F) \cap \mathbf{L}(\Pi_0) = \mathbf{L}(F)$$

(recall that here $\mathbf{L}(F) \subseteq \mathbf{C} \subset \mathbf{L}(\Pi_0)$).

Example 3 Let F be a p.o. frame. Take a frame $(\Pi_0 + F)$ obtained by adding a minimal irreflexive point π_0 to F (see Figure 3). Now, to extend relation R from F to $(\Pi_0{+}F)$, we have to define (in an arbitrary way) a non-empty set $R(\pi_0) \subseteq F$ (what does π_0 'see'?).[9]

$\Pi_0 + F$:

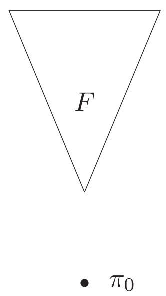

Fig. 3.

One can put, say, $R(\pi_0) = F$; then R is transitive on $(\Pi_0 + F)$. And moreover, obviously, R is transitive iff $R(\pi_0)$ is an open subset of F.

Clearly, for any choice of $R(\pi_0)$, all implications valid in F are valid in $(\Pi_0{+}F)$ as well, i.e., $\mathrm{Im} \cap \mathbf{L}(\Pi_0{+}F) = \mathrm{Im} \cap \mathbf{L}(F)$. So we obtain:

Lemma 1.3 $\mathbf{H} \subseteq \mathbf{L}(\Pi_0{+}F)$*, i.e.,* $(\Pi_0{+}F)$ *is* w**H***-sound for any p.o. frame F (and for any choice of $R(\pi_0)$).*

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to $\bot$ in this frame (cf. Remark at the beginning of Section 1). Moreover, one can easily see that there does not exist a &,∨,→,¬-formula equivalent to $\bot$; in fact, any formula of this kind is true at π_0 if all variables are true, while $\bot$ is false (cf. also Lemma 1.1).

[9] If $R(\pi_0)$ is empty, then we have a disjoint union (F, Π_0).

Proof. One can show that $\mathbf{H} \vdash A \;\Rightarrow\; A \in \mathbf{L}(\Pi_0 + F)$,
by induction on the complexity of A (not by induction on its proof!); for the case $A = A_1 \vee A_2$ use the disjunction property of $\mathbf{H}$. □

Thus, if $\mathbf{L}(F) = \mathbf{H}$, then $\mathbf{L}(\Pi_0 + F) = \mathbf{H}$, and so $(\Pi_0 + F)$ is $\mathbf{H}$-sound.

Similarly, if the logic $\mathbf{L}(F)$ has the disjunction property, then one can show that $\mathbf{L}(\Pi_0 + F) = \mathbf{L}(F)$. On the other hand, let F be a one-element reflexive frame, then $(\Pi_0 + F)$ is not $\mathbf{H}$-sound; in fact, modus ponens fails, since

$$(\top \to p \vee \neg p) \in \mathbf{L}(\Pi_0 + F) \quad \text{and} \quad p \vee \neg p \notin \mathbf{L}(\Pi_0 + F).$$

Similarly, one can show that $(\Pi_0 + F)$ is not $\mathbf{H}$-sound for any rooted finite F, and moreover, for any rooted F of finite height (if $\pi_0 \, R \, u_0,\ u_0$ being the root of F). [10]

The considered examples show us that there exist very simple and small (actually, ≤ 3-element) $\mathbf{H}$-sound frames, which are: (a) non-transitive and non-reflexive, (b) transitive and non-reflexive, (c) non-transitive and reflexive. Also there exist frames of kinds (a) and (b) that are w$\mathbf{H}$-sound, but not $\mathbf{H}$-sound (on the other hand, every reflexive w$\mathbf{H}$-sound frame is $\mathbf{H}$-sound, cf. Proposition 1 in Section 2.3).

In Section 2, basing on the presented examples, we shall describe the classes of w$\mathbf{H}$-sound and $\mathbf{H}$-sound frames, and establish Reducibility Theorem:

Theorem *For every* $\mathbf{H}$*-sound Kripke frame* F *there exists a partially ordered frame* F' *such that* $\mathbf{L}(F) = \mathbf{L}(F')$.

This statement shows that the semantics of all $\mathbf{H}$-sound Kripke frames equals the usual semantics of partially ordered Kripke frames. In other words, non-transitive or non-reflexive frames give nothing new for superintuitionistic logics (if we deal with the usual definition of intuitionistic forcing).

2 Description of intuitionistic sound frames

In this section we describe the classes of $\mathbf{H}$-sound and w$\mathbf{H}$-sound frames. As the main result, we obtain Reducibility Theorem.

2.1 Let $F = (W, R)$ be a Kripke frame. We call $u \in W$ a *parasite* if $\neg\exists w\,(wRu)$; in other words, parasites are minimal irreflexive points 'stuck' to the frame from below. A parasite u is *isolated* if $R(u) = \varnothing$, i.e., if its cone $F^u = \{u\}$ is isomorphic to Π_0 (see Example 2). Let $\Pi[F]$ and $\Pi_0[F]$ be the sets of all parasites and of isolated parasites from F, respectively. The *essential part* of F is $E[F] = F \backslash \Pi[F]$; obviously, $E[F]$ is an open subframe of F.

If $E[F] = \varnothing$, then $F = \Pi_0[F]$ and $\mathbf{L}(F) = \mathbf{L}(\Pi_0)$, see Example 2.

A frame F is called *co-serial* if $\forall u \in F\, \exists w \in F\, (wRu)$, i.e., iff $\Pi[F] = \varnothing$ (or equivalently, $E[F] = F$).

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[10] All mentioned claims will be proved later, in Section 2.4 (Appendix).

Now we recall (and slightly reformulate) some notions introduced in [5]. A Kripke frame $F = (W, R)$ is called *weakly reflexive* if $\forall u \in F\,(uR^+u)$ and *weakly transitive* if $\forall u, v \in F\,[\,(uR^2v) \Rightarrow \exists w\,(uRw \equiv v)\,]$. A frame F is *weakly quasi-ordered* if it is weakly reflexive and weakly transitive. Clearly, reflexivity, transitivity, and quasi-ordering imply the corresponding weak properties. Every weakly reflexive frame F is co-serial. Also note that a transitive frame is weakly reflexive iff it is reflexive (since $R^+ = R$ in a transitive frame).

Remark By the way, we can give a 'uniform' presentation of these notions.
Put $[\Omega_n]$: $\forall u, v \in F\,[\,(uR^nv) \Rightarrow \exists w\,(uRw \equiv v)\,]$ for $n \geq 0$.

Clearly $[\Omega_2]$ is the definition of weak transitivity. Moreover,

(I) F is weakly transitive iff $\forall n > 0\,[\Omega_n]$, i.e.,

$\forall u, v \in F\,[\,(uR^+v) \Rightarrow \exists w\,(uRw \equiv v)\,]$.

In fact, we can establish $[\Omega_n]$ by induction on n. The case $n = 1$ is obvious; take $w = v$. The induction step is straightforward, as well. Namely, if $uR^{n-1}u'R^2v$, then we find w' such that $u'Rw' \equiv v$,and by induction hypothesis uR^nw' implies $\exists w\,[uRw \equiv w'(\equiv v)]$.

Also we have

(II) F is weakly reflexive iff $[\Omega_0]$, i.e.,

$\forall u \in F\,\exists w\,(uRw \equiv u)$.

In fact, if uR^+u, i.e., $uRwR^nu$ for some w and $n \geq 0$, then $w \equiv u$.

Therefore,

(III) F is weakly quasi-ordered iff $\forall n \geq 0\,[\Omega_n]$, i.e.,

$\forall u, v \in F\,[\,(uR^*v) \Rightarrow \exists w\,(uRw \equiv v)\,]$.

Recall that R is transitive iff $R^+ = R$ and R is a quasi-ordering iff $R^* = R$. So we see that a frame F is weakly reflexive, weakly transitive, or weakly quasi-ordered iff the composition of R and $\equiv$ is reflexive, transitive, or quasi-ordered, respectively. [11] In other words, weak reflexivity means 'reflexivity up to equivalence' (and similarly for weak transitivity).

Let A be a formula. A valuation $\vDash$ in a frame F is called *A-hereditary* if

$$\forall u, v \in F\,[\,uRv \,\&\, u \vDash A \;\Rightarrow\; v \vDash A\,], \qquad (A\text{-heredity})$$

or equivalently, $\forall u \in F\,[\,(u \vDash A) \;\Rightarrow\; \forall v \in F^u\,(v \vDash A)\,]$.
By definition, all our valuations are p-hereditary for variables p. A frame F is called *A-hereditary* if all valuations in F are A-hereditary; F is called **H**-*hereditary* if it is A-hereditary for all formulas A.

Lemma 2.1 *Let $F = (W, R)$ be a Kripke frame.*

(1) $((\top \to p) \to p) \in \mathbf{L}(F)$ *iff* $E[F]$ *is weakly reflexive.*

(2) $(A \to (\top \to A)) \in \mathbf{L}(F)$ *iff* $E[F]$ *is A-hereditary (for a formula A).*

(3) The following conditions are equivalent:

[11] By the way, (I) implies that $(\equiv) \circ R \circ (\equiv)$ equals $R \circ (\equiv)$ etc. (in a weakly transitive frame).

(*i*) F *is* **H***-hereditary;*

(*ii*) F *is* $(p \to q)$*-hereditary;*

(*iii*) F *is weakly transitive.*

Therefore, *if* F *is w***H***-sound, then* $E[F]$ *is weakly quasi-ordered.*

Note that only ($\Rightarrow$)-parts of (1) and (2) are used for the latter statement; actually, the converse statements are only to complete the picture.[12]

Proof. (1) ($\Rightarrow$). Let wRu and $\neg(uR^+u)$. Take a valuation in F such that $v \vDash p \;\Leftrightarrow\; uR^+v$. Then $u \vDash (\top \to p)$ and $u \nvDash p$, thus $w \nvDash (\top \to p) \to p$.

($\Leftarrow$). Let $E[F]$ be weakly reflexive, and wRu, $u \vDash (\top \to p)$ (for a valuation in F). Then $v \vDash p$ for any $v \in R(u)$, and thus by (Atomic heredity), for any $v \in R^+(u)$. Also uR^+u since $u \in E[F]$, and hence $u \vDash p$.

(2) ($\Rightarrow$). Let $u \in E[F]$, uRv, and $u \vDash A$; also let wRu and $w \vDash (A \to (\top \to A))$ (for a valuation in F). Then $u \vDash (\top \to A)$, and so $v \vDash A$.

($\Leftarrow$). Let wRu and $u \vDash A$. Then $u \in E[F]$, and so by A-heredity it follows that $\forall v \in R(u)\,(v \vDash A)$, i.e., $u \vDash (\top \to A)$.

(3) (ii) $\Rightarrow$ (iii). Let $uRv'Rv$. Consider a valuation in F such that

$$w \vDash p \;\Leftrightarrow\; w \in F^v \qquad \text{and} \qquad w \nvDash q \;\Leftrightarrow\; v \in F^w.$$

Then $v \vDash p$, $v \nvDash q$, thus $v' \nvDash (p \to q)$, and so, by $(p \to q)$-heredity, $u \nvDash (p \to q)$. Hence there exists $w \in R(u)$ such that $w \vDash p$ and $w \nvDash q$, i.e., $w \equiv v$.

(iii) $\Rightarrow$ (i). Let F be weakly transitive. We establish A-heredity by induction on the complexity of A. Clearly, it is sufficient to consider the induction step for $A = (A_1 \to A_2)$.

Let uRv', $v' \nvDash A$, i.e., $v \vDash A_1$ and $v \nvDash A_2$ for some $v \in R(v')$. Then by the weak transitivity, we have $w \in R(u)$ such that $w \equiv v$, i.e., $w \in F^v$ and $v \in F^w$. Hence by A_1-heredity and A_2-heredity, $w \vDash A_1$ and $w \nvDash A_2$. Thus $u \nvDash A$. ☐

2.2 We see that weak transitivity expresses heredity of valuations (for all formulas). Similarly, weak reflexivity is related to another property of valuations introduced in [5] and called converse heredity.[13] Namely, a frame F is *conversely A-hereditary* if the following condition holds (for any valuation in F):

$$\forall u \in F\, [\forall v \in R(u)\,(v \vDash A) \;\Rightarrow\; u \vDash A\,]. \qquad \text{(converse } A\text{-heredity)}$$

A frame F is *conversely* **H***-hereditary* if this property holds for all formulas.

Clearly, converse A-heredity means that

$$[\, u \vDash (\top \to A) \;\Rightarrow\; u \vDash A\,, \quad \text{for every valuation} \vDash \text{and point } u\,].$$

Therefore: $((\top \to A) \to A) \in \mathbf{L}(F) \;\Leftrightarrow\; (\,E[F]$ *is conversely A-hereditary*$\,)$.[14]

[12] The statement (3) actually reformulates Proposition 15 from [5]; we repeat its short proof here, to make our exposition self-contained and to reveal that $(p \to q)$-heredity is sufficient for the weak transitivity. Similarly, (2) almost reformulates Proposition 9 from [4] (note that in [4] the heredity in $E[F]$ is called the conditional heredity in F); here we give a simple proof omitted in [4].

[13] We will not use this notion, but we mention it here to explain the sense of the notions we consider.

[14] Cf. Proposition 10 in [4]; note that there converse heredity in $E[F]$ is called conditional converse heredity in F.

Moreover, for any frame F:

F is weakly reflexive iff *F is conversely p-hereditary for variables p*
(adapt the proof of Lemma 2.1(1), [15] or see Proposition 21 in [5]).

However, *weak reflexivity is not sufficient for converse* **H**-*heredity.*

Namely, take e.g. a weakly reflexive frame F shown at Figure 4,
a formula $A = \neg\neg p$, and a valuation F such that $u \vDash p \Leftrightarrow u = v_3$.
Then $u \vDash \neg p \Leftrightarrow u = v_1$, hence $v_2 \nvDash A$ and $\forall u \in R(v_2)(=\{v_1, v_3\})\,[u \vDash A]$.

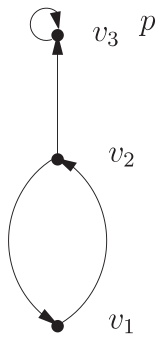

Fig. 4.

On the other hand, weak reflexivity together with weak transitivity imply converse **H**-heredity; in other words,

every weakly quasi-ordered frame is conversely **H**-*hereditary.*

In fact, [16] let $\forall v \in R(u)\,(v \vDash A)$. Since uR^+u (by weak reflexivity), take $v \in R(u)$ such that $vR^n u$ for some $n \geq 0$ (i.e., $u \in F^v$) . Then $v \vDash A$, and by A-heredity (see Lemma 2.1(3)) $u \vDash A$.

Therefore, we obtain:

Claim *For a frame F the following conditions are equivalent:*

(i) *F is* **H**-*hereditary and conversely* **H**-*hereditary*;

(ii) *F is $(p \rightarrow q)$-hereditary and conversely p-hereditary*;

(iii) *F is weakly quasi-ordered.*

Remark [17] By the way, Lemma 2.1(2) makes a hint, how one can try to modify the notion of intuitionistically acceptable valuations. Namely, one can admit *weak valuations*

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[15] Lemma 2.1(1) readily gives the same equivalence for $E[F]$ (and so for any co-serial frame F). However the argument actually goes through for an arbitrary F as well.

[16] The statement actually follows from [5, Propositions 21 and 19], but in an indirect way, involving some complicated and refined notions, which relate to rudimentary Kripke models; so we give a simple and straightforward direct argument here.

[17] The reader may skip this remark and turn to Section 2.3 that contains the proof of Reducibility Theorem.

satisfying (Atomic heredity) only on $E[F]$, but not necessarily on F. In fact, this form of heredity actually corresponds to the well-known intuitionistic axiom $p \to (q \to p)$. However this modification gives nothing new, because the following statement holds:

(+) *for any*[18] *Kripke frame F, the set of formulas true under all weak valuations equals our $\mathbf{L}(F)$, defined via traditional valuations from Section 1.1.*

Proof. In fact, it is sufficient to show that every formula refuted under a weak valuation, does not belong to $\mathbf{L}(F)$, i.e., it is refuted by a valuation defined in Section 1.1. This statement easily follows from Lemma 1.1. Namely, let a weak valuation $\vDash$ in F satisfying (Atomic heredity) on $E[F]$, be given. Take a valuation $\vDash'$ in F equal to $\vDash$ on $E[F]$ and such that $u \nvDash 'p$ for all $u \in \Pi[F]$ and all variables p. Clearly, the valuation $\vDash'$ satisfies (Atomic heredity) on F. Now

$$u \vDash A \;\Leftrightarrow\; u \vDash' A \quad \text{for all } u \in E[F] \text{ and all formulas } A.$$

Also $\quad u \vDash A \;\Leftrightarrow\; u \vDash' A \;$ for $u \in \Pi[F]$ and $A \in \mathrm{Im}$. Hence we conclude that for every $u \in \Pi[F]$ and for every formula A' of the form (*) (from Lemma 1.1):

$$u \nvDash A' \quad \text{implies} \quad u \nvDash ' A'.$$

Therefore: $\quad u \nvDash A \;\Rightarrow\; u \nvDash ' A \quad$ for every A and $u \in F$. □

Actually, the statement (+) together with Lemma 2.1(2) allows to show (in the standard way) that

the set $\mathbf{L}(F)$ is substitution closed for every w$\mathbf{H}$-sound frame F.[19]

Therefore: $\quad$ *F is $\mathbf{H}$-sound* iff
(*F is w$\mathbf{H}$-sound and $\mathbf{L}(F)$ is closed under modus ponens*).

Later on we will obtain another proof of the latter equivalence (see Proposition 3).

2.3. Clearly, the frames considered in Example 1 are weakly quasi-ordered; so this example makes a hint, how to reduce an arbitrary weakly quasi-ordered frame to a quasi-ordered one. Namely, a weakly reflexive frame $F = (W, R)$ gives rise to a quasi-ordered frame $F^* = (W, R^*) = (W, R^+)$ (recall that R^+ is reflexive here, i.e., $R^+ = R^*$).

Lemma 2.2 *Let $F = (W, R)$ be a weakly quasi-ordered frame and $F^* = (W, R^*)$ be the corresponding quasi-ordered frame. Then a forcing relation $\vDash$ is a valuation in F iff it is a valuation in F^*.*

Hence $\mathbf{L}(F) = \mathbf{L}(F^*)$, and so this set is an intermediate logic. Therefore,

all weakly quasi-ordered frames are $\mathbf{H}$-sound.

Note that $S[F] = S[F^*]$, thus $\mathbf{L}(F) = \mathbf{L}(S[F])$ for a weakly quasi-ordered F.

Proof. [20] Note that a valuation is uniquely determined for <u>all</u> formulas by its restriction

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[18] not necessarily (weakly) $\mathbf{H}$-sound

[19] In fact, if $[B/q]A \notin \mathbf{L}(F)$, i.e., $v \nvDash [B/q]A$ for a valuation $\vDash$ in F, then $v \nvDash ' A$ for a weak valuation $\vDash'$ such that $u \vDash' q \;\Leftrightarrow\; u \vDash B$; by Lemma 2.1(2), this $\vDash'$ satisfies (Atomic heredity) on $E[F]$. Thus $A \notin \mathbf{L}(F)$ by (+).

[20] Cf. the proof of Proposition 6 from [5].

to Var, by applying the usual inductive clauses, see Section 1.1. Thus, it is sufficient to establish the 'only if' part. [21]

Clearly, the atomic heredity in F and in F^* is just the same. So we have to check the inductive clause for implication, i.e., to show that for any valuation in F and for all formulas B, C the following equivalence holds:

$$\forall v \in R(u)\,[\,v \vDash B \;\Rightarrow\; v \vDash C\,] \;\Leftrightarrow\; \forall v \in R^+(u)\,[\,v \vDash B \;\Rightarrow\; v \vDash C\,].$$

In fact, let uR^+v, $v \vDash B$, $v \nvDash C$. Then $uR^n v' R v$ for some v' and $n \geq 0$ (i.e., $v' \in F^u$). Then $v' \nvDash (B \to C)$ in F, and by **H**-heredity of F (see Lemma 2.1(3)), $u \nvDash (B \to C)$ in F, i.e., there exists $v'' \in R(u)$ such that $v'' \vDash B$, $v'' \nvDash C$. $\Box$

Recall that a frame F is called co-serial if $E[F] = F$; all weakly reflexive frames are co-serial. So by applying Lemmas 2.1 and 2.2, we readily obtain the following description of **H**-sound frames without parasites:

Proposition 1 *For a frame F the following conditions are equivalent:* [22]

(1) *F is co-serial and* **H***-sound;*

(2) *F is co-serial and w***H***-sound;*

(3) *F is co-serial and* $(\top \to p) \to p$, $(p \to q) \to (\top \to (p \to q)) \in \mathbf{L}(F)$;

(4) *F is weakly quasi-ordered.*

Thus in particular,

*all transitive co-serial (w)***H***-sound frames are quasi-ordered.*

Now let us consider the general case.

Proposition 2 *For a frame F the following conditions are equivalent:*

(1) *F is w***H***-sound;*

(2) $(\top \to p) \to p$, $(p \to q) \to (\top \to (p \to q)) \in \mathbf{L}(F)$;

(3) *$E[F]$ is weakly quasi-ordered.* [23]

Proof. (2) $\Rightarrow$ (3) follows from Lemma 2.1.

(3) $\Rightarrow$ (1). If $E[F] = \varnothing$, i.e., $F = \Pi_0[F]$, then $\mathbf{L}(F) = \mathbf{L}(\Pi_0) \supset \mathbf{C}$, by Lemma 1.2. If $E[F]$ is a non-empty weakly quasi-ordered frame, then $\mathbf{L}(E[F])$ is an intermediate

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By the way, that proposition actually states that $\mathbf{L}(F^*) \subseteq \mathbf{L}_r(F)$ for an arbitrary frame F, where $\mathbf{L}_r(F)$ is the set of formulas true under all rudimentary Kripke models in F (recall that in [5] a valuation is called *rudimentary* if it satisfies A-heredity and converse A-heredity for all formulas A). One can show that $\mathbf{L}(F) = \mathbf{L}_r(F) = \mathbf{L}(F^*)$ for weakly quasi-ordered frames F (because all valuations in these frames are rudimentary, cf. Claim in Section 2.2). However, this argument is rather detour, and we prefer a simple direct argument to prove Lemma 2.2 without using the notion of rudimentary Kripke models etc.

[21] Then the 'if' part readily follows as well. In fact, for a valuation $\vDash$ in F^* take its restriction to Var and prolong it to a valuation in F. Now, by the 'only if' part, this is a valuation in F^*, and so it equals the original valuation $\vDash$.

[22] In fact, Lemma 2.1 gives (3) $\Rightarrow$ (4) and Lemma 2.2 gives (4) $\Rightarrow$ (1).

[23] For a reader familiar with [4] note that these conditions actually mean that all valuations in F form conditionally rudimentary Kripke models; therefore, this proposition indirectly follows from [4, Proposition 12 etc.].

logic and $\mathbf{H} \subseteq \mathbf{L}(F \backslash \Pi_0[F]) \subseteq \mathbf{L}(E[F]) \subseteq \mathbf{C}$, cf. Lemma 1.3. [24] Now, if $\Pi_0[F] \neq \varnothing$, then $\mathbf{L}(F) = \mathbf{L}(F \backslash \Pi_0[F]) \cap \mathbf{L}(\Pi_0) = \mathbf{L}(F \backslash \Pi_0[F])$, since $\mathbf{C} \subset \mathbf{L}(\Pi_0)$. □

Therefore,
to describe *the semantics of (weakly)* **H**-*sound Kripke frames* it is sufficient to consider only *(weakly)* **H**-*sound frames without isolated parasites.*

Note that Proposition 2 implies in particular that
*a transitive frame F is w***H***-sound* iff *$E[F]$ is quasi-ordered.*

Proposition 3 *For a frame F the following conditions are equivalent:*

(1) *F is* **H**-*sound;*

(2) *F is w***H***-sound and $\mathbf{L}(F)$ is closed under modus ponens;*

(3) *F is w***H***-sound and $\mathbf{L}(F) = \mathbf{L}(E[F])$.*

Proof. (3) $\Rightarrow$ (1) follows from Proposition 2 (the implication (1) $\Rightarrow$ (3)) and Proposition 1 (the implication (4) $\Rightarrow$ (1)).

(2) $\Rightarrow$ (3). If $A \in \mathbf{L}(E[F])$, then $(\top \rightarrow A) \in \mathbf{L}(F)$. Thus $A \in \mathbf{L}(F)$ by (MP). □

Therefore we have established Reducibility Theorem stated at the end of Section 1. In fact, if a frame F is **H**-sound, then $E[F]$ is weakly quasi-ordered and $\mathbf{L}(F) = \mathbf{L}(E[F]) = \mathbf{L}(F')$ for a partially ordered (p.o.) frame $F' = S[E[F]]$.

2.4 APPENDIX

A description of **H**-soundness given in Proposition 3, unlike Propositions 1 and 2, is slightly implicit. Namely, it involves a vague condition

$$\mathbf{L}(F) = \mathbf{L}(E[F]). \qquad (\lambda)$$

By Lemma 1.1, the condition (λ) means that

$$A' \in \mathbf{L}(E[F]) \;\Rightarrow\; A' \in \mathbf{L}(F) \quad \text{ for any } A' = \bigvee_j A_j,\ A_j \in \mathrm{Im} \cup \mathrm{Var}.$$

Open Problem *Try to find a more explicit description of* **H**-*soundness; in other words, represent in a more 'convenient' form the condition (λ) for frames F with weakly quasi-ordered essential part $E[F]$.*

We suppose, Example 3 (from Section 1.3) makes a hint that this problem may not have a satisfactory solution. However to conclude our considerations we mention here some straightforward approaches to the problem (and explain that they do not give a general description). So the whole problem seems to be too hopeless and does not worth serious efforts.

[24] Note that $\mathbf{L}(F \backslash \Pi_0[F]) = \bigcap \left(\mathbf{L}(\{w\} \cup E[F]) : w \in \Pi[F] \backslash \Pi_0[F] \right)$ and every $(\{w\} \cup E[F])$ is a frame of the kind $(\Pi_0 + E[F])$ described in Example 3.

First, it is sufficient to consider only frames described in Example 3, i.e.:

$$\text{frames } F \text{ with empty } \Pi_0(F), \text{ one-element } \Pi(F)=\{\pi_0\}, \text{ and p.o. } E[F]. \qquad (\varphi)$$

In fact, for a frame F with a weakly quasi-ordered $E[F]$ we consider a frame $\tilde{F} = S[E[F]] \cup \Pi[F]$, in which $\Pi[\tilde{F}] = \Pi[F]$, $E[\tilde{F}] = S[E[F]]$, and

$$w\tilde{R}(u/\equiv) \Leftrightarrow \exists u' \equiv u\ (wRu') \qquad \text{for } w \in \Pi[F],\ u \in E[F].$$

There exists a natural one-to-one correspondence between valuations in $\tilde{F}$ and in F (cf. the end of Section 1.2); namely, the correspondent valuations satisfy the condition (S) on $E[F]$ and coincide on $\Pi[F]$ (recall that $E[F]$ is **H**-hereditary, so forcing for all formulas in $E[F]$ does not distinguish $\equiv$-equivalent points). Hence $\mathbf{L}(F) = \mathbf{L}(\tilde{F})$.

Now[25]

$$\mathbf{L}(\tilde{F}) = \bigcap (\mathbf{L}(\tilde{F}_w) : w \in \Pi[F]),$$

where $\tilde{F}_w = E[\tilde{F}] \cup \{w\}$ for $w \in \Pi[F]$ is an open subframe of $\tilde{F}$.

Thus $\quad \mathbf{L}(\tilde{F}) = \mathbf{L}(E[\tilde{F}]) \Leftrightarrow \forall w \in \Pi[F]\, (\mathbf{L}(\tilde{F}_w) = \mathbf{L}(E[\tilde{F}])),$

since $\mathbf{L}(\tilde{F}_w) \subseteq \mathbf{L}(E[\tilde{F}])$ for any $w \in \Pi[F]$.

Therefore we conclude that

$$F \textit{ is } \mathbf{H}\textit{-sound} \quad \text{iff} \quad \textit{all } \tilde{F}_w \textit{ are } \mathbf{H}\textit{-sound.}$$

Clearly, every $\tilde{F}_w$ is a frame of the form (φ).

Hence now we consider only frames F of the form (φ).

Recall that a frame F of this form can be described by a non-empty[26] subset $R(\pi_0) \subseteq E[F]$, cf. Example 3.

We say that $R(\pi_0) (\subseteq E[F])$ *violates* a formula $A' = \bigvee_j (B_j \to C_j)$ if there exists a valuation in $E[F]$ such that $\quad \forall j\, \exists u \in R(\pi_0)\, [\, u \models B_j,\ u \not\models C_j\,]$.

Clearly, $R(\pi_0)$ violates A' iff

there exists a valuation in F such that $\pi_0 \not\models A' \vee \bigvee_k p_k$

for an arbitrary (perhaps, empty) list of variables $(p_k : k)$.[27]

Thus, if $(A' \vee \bigvee_k p_k) \in \mathbf{L}(E[F]) \backslash \mathbf{L}(F)$, then $R(\pi_0)$ violates A'. Hence we obtain:

Lemma 2.3 *Let F be a frame of the form (φ). Then:*

$\mathbf{L}(F) = \mathbf{L}(E[F])$ *(i.e., F is **H**-sound)* *iff*

[*for any formula $A = A' \vee A''$, where $A' = \bigvee_j (B_j \to C_j)$, $A'' = \bigvee_k p_k$:*

$$\textit{if } R(\pi_0) \textit{ violates } A', \quad \textit{then } A \notin \mathbf{L}(E[F])\,]. \qquad (\lambda^*)$$

Actually in (λ^*) we may assume that A'' is the disjunction of all variables occurring in A' (in fact, if (λ^*) holds for this A'', then it readily holds for an arbitrary A'' as well).

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[25] Here we suppose that $\Pi[F] \neq \varnothing$, since otherwise $F = E[F]$ is definitely **H**-sound.

[26] If $R(\pi_0) = \varnothing$, then $\mathbf{L}(F) = \mathbf{L}(E[F])$, cf. Example 2, and F is **H**-sound again.

[27] For the 'only if' part one can prolong a given valuation in $E[F]$ (actually refuting A' at π_0) to a valuation in F; namely, put $\pi_0 \not\models p$ for all variables p.

Clearly, if $R(\pi_0)$ violates A', then $(B_j \to C_j) \notin \mathbf{L}(E[F])$ for any j. Therefore if $\mathbf{L}(E[F])$ has the disjunction property, then (λ^*) holds, and so F is $\mathbf{H}$-sound.

Also note that F is $\mathbf{H}$-sound e.g. if $E[F]$ is an infinite chain (recall that its logic $[\,\mathbf{H} + (p \to q) \vee (q \to p)\,]$ lacks the disjunction property). In fact, if $R(\pi_0)$ violates A', then $A' \notin \mathbf{L}(E[F])$. Hence A' is falsified in any sufficiently large ($\geq n$-element for some n) cone in $E[F]$. Now, take $u, v \in F$ such that uRv, $u \neq v$, and $A' \notin \mathbf{L}(F^v)$, then $A = (A' \vee A'') \notin \mathbf{L}(F^u)$, since all variables can be refuted at u.

Another sufficient condition for $\mathbf{H}$-soundness is in terms of $R(\pi_0)$ rather than $E[F]$. We say that a set $R(\pi_0) \subseteq E[F]$ is *co-directed* if for any finite $X \subseteq R(\pi_0)$ there exists $v_X \in E[F] \backslash X$ such that $\forall u \in X\,(v_X R u)$. Clearly, if $R(\pi_0)$ is co-directed and $R(\pi_0)$ violates A', then $v_X \not\models A = A' \vee A''$ for some X and a valuation $\models$ in $E[F]$. Thus any F with a co-directed $R(\pi_0)$ satisfies (λ^*), and so it is $\mathbf{H}$-sound.

Now we are ready to describe the $\mathbf{H}$-soundness for a natural particular case.

Proposition 4 *Let F be a frame of the form (φ) with a rooted $E[F]$. Then:*

$$F \text{ is } \mathbf{H}\text{-sound} \quad \textit{iff} \quad [\,u_0 \notin R(\pi_0)\,] \textit{ or } [\,\mathbf{L}(E[F]) = \mathbf{L}(E[F] \backslash \{u_0\})\,],$$

where u_0 is the root of $E[F]$.

Proof. If $u_0 \notin R(\pi_0)$, then $R(\pi_0)$ is co-directed (take $v_X = u_0$ for any X).

Now let $\mathbf{L}(E[F]) = \mathbf{L}(E[F] \backslash \{u_0\})$. If $R(\pi_0)$ violates A', then $u_0 \not\models A'$ for a valuation in $E[F]$, i.e., $A' \notin \mathbf{L}(E[F])$. Thus there exists a valuation $\models'$ in $E[F] \backslash \{u_0\}$ such that $u \not\models\,' A'$ for some $u \neq u_0$. Then $u_0 \not\models\,' A = A' \vee A''$ in $E[F]$, since we may assume that all variables are false at u_0. Hence $A \notin \mathbf{L}(E[F])$.

Finally let $u_0 \in R(\pi_0)$ and $B \in \mathbf{L}(E[F] \backslash \{u_0\}) \setminus \mathbf{L}(E[F])$. Consider a variable p non-occurring in B, put $A' = (p \to B)$, $A = p \vee A'$. Then $A \in \mathbf{L}(E[F])$ and $R(\pi_0)$ violates A' with a valuation such that $u_0 \not\models B$, $u_0 \models p$. $\square$

Remark By the way, one can rewrite the latter condition (in Proposition 4) in a more 'syntactic' form, using δ-operation of Hosoi – Ono. Namely,

$$[\,\mathbf{L}(E[F]) = \mathbf{L}(E[F] \backslash \{u_0\})\,] \text{ iff } [\,A \in \mathbf{L}(E[F]) \Leftrightarrow \delta A \in \mathbf{L}(E[F])\,\textit{for every formula } A\,],$$

where $\delta A = p \vee (p \to A)$ for a variable p non-occurring in A.[28]

In fact, recall that clearly: $\delta A \in \mathbf{L}(E[F])$ iff $A \in \mathbf{L}(E[F] \backslash \{u_0\})$.

The proposition implies that F is not $\mathbf{H}$-sound if $E[F]$ is a rooted frame of finite height (and its root belongs to $R(\pi_0)$). In fact, here frames $E[F]$ and $E[F] \backslash \{u_0\}$ have different heights, so their logics are distinguishable by a well-known axiom. A similar argument is applicable if, say, $E[F] \backslash X$ is linearly ordered, where X is a non-empty (and non-linear!) downward closed set of finite height (and in many other cases).

But note that we cannot reduce a general case to the case of rooted $E[F]$. For example, let $E[F]$ be a disjoint union of finite cones, say, Jaśkowski's trees, or finite chains, or finite binary trees (and $R(\pi_0) = E[F]$). Then F is $\mathbf{H}$-sound, since $\mathbf{L}(E[F]) =$

[28] Note also that the implication $(\Rightarrow)$ is obvious, so one can read $(\Leftarrow)$ for $(\Leftrightarrow)$ here.

H (or Dummett's logic, or Gabbay – de Jongh's logic of binary trees, [29] respectively), but all its (non-open!) subframes $F'\cup\{\pi_0\}$ for cones F' in $E[F]$ are not **H**-sound.

So we do not hope to find a general description of **H**-sound frames essentially less cumbersome than the technical condition (λ^*). And such a description seems useless, because we have proved that all these frames are actually reducible to partially ordered ones.

Example Proposition 4 gives us an **H**-sound frame such that $\mathbf{L}(F) \neq \mathbf{L}(S[F])$, mentioned in Section 1.2. Namely, let F be a frame of the form (φ) such that $E[F]$ is the set of all proper subsets of a 3-element set I (partially ordered by inclusion), see Figure 5, and $R(\pi_0)$ is the set of all 2-element subsets of I. Then F is **H**-sound, since the root $\varnothing$ of $E[F]$ does not belong to $R(\pi_0)$. Now, π_0 becomes a reflexive minimal point in $S[F]$, and its cone is the tree $T_{2,3}$ of height 2 and branching 3. Thus $\mathbf{L}(S[F]) = \mathbf{L}(E[F])\cap\mathbf{L}(T_{2,3})$, and one can easily show that the Kreisel – Putnam's formula

$$K = (\neg p\to q\vee r) \to (\neg p\to q)\vee(\neg p\to r)$$

belongs to $\mathbf{L}(E[F])\backslash\mathbf{L}(S[F]) = \mathbf{L}(F)\backslash\mathbf{L}(S[F])$.

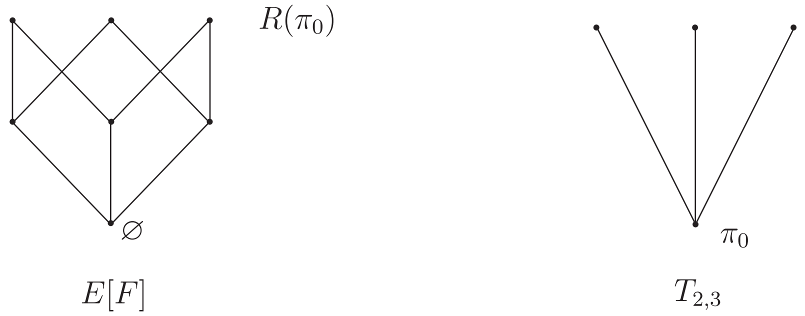

Fig. 5.

2.5 Additional Remark

Anonymous referees proposed to consider a stronger version of intuitionistic soundness, cf. e.g. [5, Proposition 8]. [30]

Recall the well-known notions of *intuitionistic logical consequence*:

$$\Gamma \vdash_{\mathbf{H}} A \ \text{ iff } \quad [\,\mathbf{H} \vdash (\,\&\Gamma_0 \to A\,) \text{ for a finite } \Gamma_0 \subseteq \Gamma\,]$$

and the *semantic consequence* (in a frame F):

$$\Gamma \vDash_F A \ \text{ iff } \quad [\,\forall u(u\vDash\Gamma) \Rightarrow \forall u(u\vDash A), \ \text{ for any valuation } \vDash \text{ in } F\,]$$

(here $u\vDash\Gamma$ for a set Γ of formulas means that $\forall B\in\Gamma(u\vDash B)$).

[29] Recall that it has the disjunction property.

[30] We suppose that this version of soundness is too strong. However, its description is quite simple and readily follows from our previous considerations, so we present it here.

Say that a frame F is *strongly* **H**-*sound* (or *s***H**-*sound*) if

$$\Gamma \vdash_{\mathbf{H}} A \;\Rightarrow\; \Gamma \vDash_F A \qquad \text{for all } \Gamma \text{ and } A. \tag{σ}$$

Obviously, $\top \vdash_{\mathbf{H}} A \Leftrightarrow \mathbf{H} \vdash A$ and $\top \vDash_F A \Leftrightarrow A \in \mathbf{L}(F)$. So

(I) *Every s***H**-*sound frame F is w***H**-*sound.*

Also, clearly: if $\Gamma \vDash_F A$, then $(\Gamma \subseteq \mathbf{L}(F) \Rightarrow A \in \mathbf{L}(F))$.

Hence every s**H**-sound frame satisfies modus ponens, i.e., the condition (MP) from Section 1.1, because $(\top \to A) \vdash_{\mathbf{H}} A$. Therefore by Proposition 3 we obtain

(II) *Every s***H**-*sound frame F is* **H**-*sound.*

It is well known that all p.o. (and thus, all quasi-ordered) frames are s**H**-sound.

Finally, we obtain the following description of s**H**-soundness:

Claim *For a frame F the following conditions are equivalent:*

(1) *F is s***H**-*sound, i.e., it satisfies* (σ) (*with all* Γ *and* A);

(2) *the condition* (σ) *holds with all <u>one-element</u>* Γ (*and all* A);

(3) *F is co-serial and w***H**-*sound;*

(4) *F is weakly quasi-ordered.*

By the way, this claim gives another, slightly indirect, proof of (II), cf. Proposition 1.[31]

Proof. (2) $\Rightarrow$ (3). First, F is w**H**-sound by (I). Now suppose F is not co-serial, i.e, $\Pi[F] \neq \varnothing$. Take a valuation such that $u \vDash p \Leftrightarrow u \in E[F]$. Then we have: $\forall u \in F\,(u \vDash (\top \to p))$ and $u \nvDash p$ for $u \in \Pi[F]$, so $(\top \to p) \nvDash_F p$, while $(\top \to p) \vdash_{\mathbf{H}} p$.

(4) $\Rightarrow$ (1). By Lemma 2.2, the frames F and F^* have the same valuations, so the s**H**-soundness of F readily follows from the s**H**-soundness of F^*.

Now Proposition 1 gives (3) $\Rightarrow$ (4), and concludes the proof. □

So we see that the natural counterpart of <u>Reducibility Theorem</u> for s**H**-soundness holds as well, namely:

*For every s***H**-*sound Kripke frame F there exists a partially ordered frame F' such that*

$$\Gamma \vDash_F A \quad \text{iff} \quad \Gamma \vDash_{F'} A.$$

Put $F' = S[F] (= S[F^*])$.

By the way, we can also mention another, modified version of semantical consequence, called *local* (cf. e.g. $(**)$ after Proposition 8 in [5]):

$$\Gamma \vDash'_F A \;\text{ iff }\; [\, u \vDash \Gamma \Rightarrow u \vDash A, \text{ for any } u \in F \text{ and for any valuation } \vDash \text{ in } F\,].$$

Naturally, a frame F is *s*′**H**-*sound* if

$$\Gamma \vdash_{\mathbf{H}} A \;\Rightarrow\; \Gamma \vDash'_F A \qquad \text{for all } \Gamma \text{ and } A. \tag{σ'}$$

[31] Note that our proof of the claim uses (I) and does not use (II).

Clearly, $\Gamma \vDash'_F A$ implies $\Gamma \vDash_F A$. The converse implication in general does not hold. In fact, obviously, $A \vDash_F (\top \to A)$ for every frame F (and formula A). On the other hand, $A \vDash'_F (\top \to A)$ iff F is A-hereditary (see Section 2.1). Therefore, by Lemma 2.1(3), $\vDash_F$ does not imply $\vDash'_F$ for every F that is not weakly transitive (e.g. for the frame shown at Figure 4, or for 3-element non-transitive chains, reflexive or non-reflexive). By the way, one can show that $\vDash_F$ does not imply $\vDash'_F$ for p.o. frames as well, but we do not know so small and simple counterexamples.[32]

Nevertheless,

a frame is $s'\mathbf{H}$*-sound* iff *it is* $s\mathbf{H}$*-sound.*

In fact, an s$'$**H**-sound frame F is obviously s**H**-sound. On the other hand, if F is s**H**-sound, i.e., weakly quasi-ordered, then (σ') holds, again by Lemma 2.2, since it definitely holds for a quasi-ordered frame F^*.

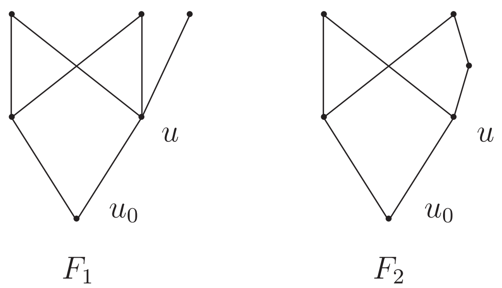

Fig. 6.

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[32] Two simplest p.o. frames that we know are shown at Figure 6; they contain 6 points. Also let us present formulas A_k, B_k such that $B_k \vDash_{F_k} A_k$ and $B_k \nvDash'_{F_k} A_k$ for $k = 1, 2$. Namely,

$$A_1 = \bigvee_{i=0}^{2} \neg C_i, \quad B_1 = \mathop{\&}_{i} (\neg C_i \to \bigvee_{j \neq i} C_j), \quad \text{where} \quad C_0 = p\&q,\ C_1 = p\&\neg q,\ C_2 = \neg p\&q.$$

and $A_2 = \neg p \vee \neg\neg p$, $B_2 = ((\neg\neg p \to p) \to p \vee \neg p)$.
In fact, one can easily construct valuations $\vDash_k$ in F_k such that $u \vDash_k B_k$, $u \nvDash_k A_k$.
On the other hand, $u_0 \vDash B_k \Rightarrow u_0 \vDash A_k$ for any valuation in F_k.
Note that F_2 is the 6-element cone in Nishimura's ladder and $(B_2 \to A_2)$ is the well-known Scott's formula.

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