Colored Ray Configurations

Ruy Fabila-Monroy* Alfredo García† Ferran Hurtado‡ Rafel Jaume§ Pablo Pérez-Lantero¶
Maria Saumell‖ Rodrigo I. Silveira** Javier Tejel† Jorge Urrutia††

Abstract

We study the cyclic sequences induced at infinity by pairwise-disjoint colored rays with apices on a given balanced bichromatic point set, where the color of a ray is inherited from the color of its apex. We derive a lower bound on the number of color sequences that can be realized from any fixed point set. We also examine sequences that can be realized regardless of the point set and exhibit negative examples as well. In addition, we provide algorithms to decide whether a sequence is realizable from a given point set on a line or in convex position.

1 Introduction

Let $P = \{p_1, \ldots, p_n\}$ be a set of n points in the plane and let $Q = \{q_1, \ldots, q_n\}$ be a set of n pairwise-disjoint rays such that q_i has apex p_i, for $i \in \{1, \ldots, n\}$. These rays induce at infinity a cyclic permutation of the numbers $\{1, 2, \ldots, n\}$, defined by the indices of the rays. If we assume that P is in general position, how many different permutations can always be obtained disregarding the geometry of P? Is there an upper bound for their number for all sets of n points? What happens in some particular configuration, for example when P is in convex position? These problems—and several related questions—were introduced by García et al. [7].

A clear motivation for the research in [7] was the extensive investigation that has been going on counting the number of non-crossing geometric graphs of several families, such as spanning cycles, perfect matchings, triangulations and many more, and on estimating how large these numbers can get [1, 4, 9, 14, 15, 16]. On the other hand, arrangements of rays have appeared in graph representation: *Ray Intersection Graphs* are graphs in which there is a node for every ray in a given set and an adjacency when two of the rays intersect [3, 6, 17]. Finally, on the applied side, it is worth mentioning recent work on sensor networks in the plane in which each sensor coverage region is an arbitrary ray [12]. The rays act as barriers for detecting the movement between regions in the arrangement.

In this paper we consider a natural variation on the problem in [7]: the point set consists now of red and blue points, and the ray we shoot from a point inherits its color. We investigate the binary circular sequences that the colored rays induce at infinity: We study how many different color patterns can always be obtained (in order of magnitude) and how many color alternations, depending on the generality of the position of the points (Section 3). We also investigate whether there are color patterns that are forbidden or universal, in the sense of their dependence on the point set (Section 4). In addition, we provide decision algorithms for some particular cases (Section 5) and include remarks on additional issues not developed in this short paper (Section 6).

2 Initial notation and definitions

Henceforth, $\mathbb{N}$ will denote the positive integers. Given $k \in \mathbb{N}$, we denote by $[k]$ the set of integers $\{1, \ldots, k\}$. Let $S = R \cup B$ be a finite bichromatic point set, where R is the set of *red* points and B is the set of *blue* points. We require S to be *balanced* ($|R| = |B|$), which is the variant that has received most attention in the family of problems on *red-blue point sets* [11].

Given any finite point set X of m points in the plane, and a set H of m pairwise-disjoint rays, each one having apex at a point in X, we say that H *is a set of rays from* X. When X is a colored point set, we assign to each ray the color (r or b) of its apex.

Given a set H of rays from S, let $C(S, H)$ denote the circular sequence of length $|S|$ in the alphabet $\{r, b\}$ induced by the rays at infinity, taken in clockwise order. Equivalently, we can take any circle large enough to enclose S and think of $C(S, H)$ as the clockwise sequence

*Dept. de Matemáticas. Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional, México D.F., México, ruyfabila@math.cinvestav.edu.mx

†Dept. de Métodos Estadísticos, IUMA, Universidad de Zaragoza, Spain, {olaverri, jtejel}@unizar.es

‡Dept. de Matemàtica Aplicada II, Universitat Politècnica de Catalunya, Barcelona, Spain, ferran.hurtado@upc.edu

§Institute of Computer Science, Freie Universität Berlin, Germany, jaume@mi.fu-berlin.de

¶Escuela de Ingeniería Civil en Informática, Universidad de Valparaíso, Chile, pablo.perez@uv.cl

‖Dept. of Mathematics, University of West Bohemia, Pilsen, Czech Republic, saumell@kma.zcu.cz

**Dept. de Matemática & CIDMA, Universidade de Aveiro, and Dept. Matemàtica Aplicada II, Universitat Politècnica de Catalunya, rodrigo.silveira@ua.pt

††Instituto de Matemáticas, Universidad Nacional Autónoma de México, México D.F., México, urrutia@matem.unam.mx

of the colored intersection points of the rays with the circle.

Given a positive integer n, we call any circular sequence of $2n$ elements in the alphabet $\{r, b\}$ consisting of n red elements and n blue elements a *configuration*. We assume hereafter that any configuration C starts with a red element and ends with a blue one. Notice that C can be partitioned into $2k$ blocks $c_1, c_2, \dots, c_{2k}$ for some $k \in \mathbb{N}$, where for $i \in [2k]$ all elements of c_i are red if i is odd, and all elements are blue if i is even. We say that k is the *alternation number* of C. Hence, C can be identified with the tuple $(r_1, b_1, r_2, b_2, \dots, r_k, b_k)$, where r_i is the number of (red) elements in c_{2i-1} and b_i the number of (blue) elements in c_{2i}, for $i \in [k]$. Let $\Gamma(n)$ denote the number of configurations or, equivalently, the number of binary balanced necklaces. It holds that

$$\frac{1}{2n}\binom{2n}{n} \le \Gamma(n) = \frac{1}{2n}\sum_{d|n}\varphi(d)\binom{2n/d}{n/d} \le \binom{2n}{n},$$

where φ is Euler's totient function [13]. Consequently, we have $\Gamma(n) = \Theta^*(4^n)$.[1]

Given S and a configuration C, we say that C is *realizable* (or *feasible*) from S if there exists a set H of pairwise-disjoint rays from S such that $C = C(S, H)$. We also say in this situation that C is *realized* by H (from S). See Figure 1 for an example. We say that a configuration is *universal* if it is feasible from any point set of the corresponding size.

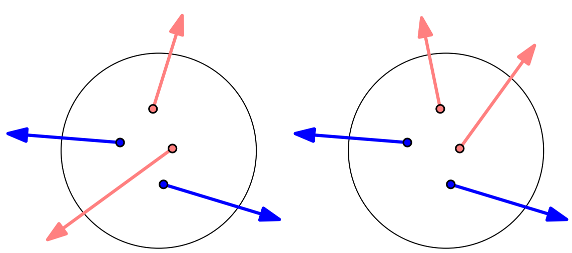

Figure 1: A point set and a realization of the configurations *rbrb* (left) and *rrbb* (right).

Given a directed line ℓ, let ℓ^+ and ℓ^- denote the sets of points to the right and to the left of ℓ, respectively. Given a point p and a vector v in the plane, let $h(p, v)$ denote the ray $\{p + t \cdot v \mid t \in \mathbb{R}, t \ge 0\}$ with apex p. Let H' be a set of rays such that for every pair $h_1, h_2 \in H'$ the intersection $h_1 \cap h_2$ is either empty or contains an infinite number of points. In this case we say that H' is a set of *non-crossing* rays. For any such a set H' there exists a small enough angle $\delta > 0$, such that for all possible angles $\varepsilon \in (0, \delta)$ the set H'_ε of rays—called a *perturbation* of H', and obtained by performing a counterclockwise rotation of angle ε on every ray of H' around its apex—is a set of pairwise-disjoint rays.

[1] The $*$ means that subexponential factors are omitted.

Let $\gamma(S)$ denote the number of different feasible configurations $C(S, H)$ over all the sets H of rays from S. Observe that this number can be just a constant: for example, placing n red points on the x-axis followed by n blue points, we obtain a set T in which the only feasible configuration is (n, n), and hence $\gamma(T) = 1$.

To avoid these trivial situations, we consider only point sets in *general position*, that is, without collinear triples. However, we obtain better bounds if we assume *strong general position*, in which we require that no two different pairs of points in S define parallel lines.

Let $\gamma^{\min}_{\mathrm{sgp}}(n)$ and $\gamma^{\max}_{\mathrm{sgp}}(n)$ be the minimum and the maximum of $\gamma(S)$, respectively, taken over all balanced bichromatic sets S of $2n$ points in the plane in strong general position. The notations $\gamma^{\min}_{\mathrm{gp}}(n)$ and $\gamma^{\max}_{\mathrm{gp}}(n)$ correspond *mutatis mutandis* to the case in which only general position is required.

3 Bounds on $\gamma(S)$ and on the alternation number

Theorem 1 *For every bichromatic point set $S = R \cup B$ in strong general position with $|R| = |B| = n$, it holds $\gamma(S) = \Omega(2^{\sqrt{n}}/n)$. Hence, $\gamma^{\min}_{\mathrm{sgp}}(n) = \Omega(2^{\sqrt{n}}/n)$.*

Proof. By the Ham-Sandwich Cut Theorem [8], there exists a (directed) line ℓ such that $|R^+| = |B^-| = \lfloor n/2 \rfloor$, where $R^+ = R \cap \ell^+$ and $B^- = B \cap \ell^-$. Let $m = \lfloor n/2 \rfloor$. We can assume, via a virtual rotation of the coordinate system, that ℓ is the positively oriented x-axis. Since $|R^+| = |B^-| = m$, there exists a non-crossing geometric perfect matching on $R^+ \cup B^-$, that is, m pairwise-disjoint straight-line segments $e_1, e_2, \dots, e_m$ such that e_i connects an element of R^+ with an element of B^- and also intersects ℓ, for $i \in [m]$. Indeed, it is well-known that given t red points and t blue points in the plane in general position, there is always a crossing-free bichromatic perfect matching (since the bichromatic perfect matching minimizing the total edge length is necessarily crossing-free). Assume without loss of generality that the points $e_1 \cap \ell, e_2 \cap \ell, \dots, e_m \cap \ell$ are sorted from left to right. Using the Erdős-Szekeres Theorem on sequences [5], there exist $k = \Omega(\sqrt{m}) = \Omega(\sqrt{n})$ indices $1 \le i_1 < i_2 < \dots < i_k \le m$ such that the clockwise angles from the segments $e_{i_1}, e_{i_2}, \dots, e_{i_k}$ to ℓ are either monotonically increasing or monotonically decreasing. Assume without loss of generality that the angles are monotonically decreasing and observe that, because of the assumption of strong general position, they decrease *strictly*. Let $p_j \in B^-$ and $q_j \in R^+$ denote the endpoints of e_{i_j}, for $j \in [k]$. Let $H_p = \{h(p_j, p_j - q_j) \mid j \in [k]\}$ and $H_q = \{h(q_j, p_j - q_j) \mid j \in [k]\}$, and observe that the elements of H_p (resp. H_q) are pairwise disjoint. Let H_0 be a set of rays from $S \setminus (\{p_j \mid j \in [k]\} \cup \{q_j \mid j \in [k]\})$ such that every element of H_0 does not intersect, and is not parallel to, any element of $H_p \cup H_q$; it is clear that such a set of rays H_0 always exists, and that $H_p \cup H_q \cup H_0$ is a set of non-crossing rays. Furthermore, we can perturb the elements of $H_p \cup H_q$ in 2^k different ways to obtain a set H of pairwise-disjoint rays from S. The perturbation is as follows: For a small enough angle $\varepsilon > 0$ and $j \in [k]$, rotate both $h(p_j, p_j - q_j)$ and $h(q_j, p_j - q_j)$ with angle ε around their apices, either clockwise or counterclockwise. Then, among all sets H, the configuration $C(S, H)$ is different for at least $2^k/2n = \Omega(2^{\sqrt{n}}/n)$ of them. The claim follows. □

Theorem 2 *For every bichromatic point set $S = R \cup B$ in strong general position with $|R| = |B| = n$, there exists a set H of pairwise-disjoint rays from S such that the alternation number of $C(S, H)$ is $\Omega(\sqrt{n})$. This bound is tight.*

Proof. Observe that the sets of rays from S generated in the proof of Theorem 1 yield $\Omega(\sqrt{n})$ color switches. To prove that this bound is tight, let $n = k^2$ for some $k \in \mathbb{N}$ and R and B be defined as follows. For $i \in [k]$, let $B_i = \{(2(i-1) + j/n^2, 0) \mid j \in [k]\}$, $B = \bigcup_{i \in [k]} B_i$ and $R = \{(j/n, 1) \mid j \in [n]\}$. Let s_i be the smallest segment covering B_i, for $i \in [k]$, and s' the smallest segment covering R. Observe that no two rays from elements of R can intersect the same segment s_i. Furthermore, no two rays from $b_1 \in B_i$ and $b_2 \in B_j$ with $i, j \in [k]$, $i \neq j$, can intersect s'. Therefore, any set H of pairwise-disjoint rays from $S = R \cup B$ is such that $C(S, H)$ has $O(k) = O(\sqrt{n})$ alternations. Finally, observe that some infinitesimal perturbation of the points moves them to strong general position, and still yields the same upper bound construction. □

Without the assumption of strong general position many of the segments in the matching used in the proof of Theorem 1, or even all of them, might be parallel, which disables the construction in that proof. It is easy to see that given a set of n red points above the x-axis and a set of n blue points below the x-axis, whose union is in general position, one can always obtain a bichromatic matching of size at least $\sqrt{n}$, such that the angles defined by the matched segments and the x-axis are different. This combines with the technique of Theorem 1 to yield an $\Omega(2^{n^{1/4}}/n)$ lower bound for the number of different configurations realizable from point sets in general position. We can do better with a related yet different approach.

Theorem 3 *For every bichromatic point set $S = R \cup B$ in general position with $|R| = |B| = n$, it holds $\gamma(S) = \Omega(2^{n^{1/3}}/n)$. Hence, $\gamma_{\text{gp}}^{\min}(n) = \Omega(2^{n^{1/3}}/n)$.*

Proof. We start as in the proof of Theorem 1 and obtain (after a virtual rotation) a bichromatic non-crossing geometric perfect matching of a set R^+ of m red points below the x-axis, and a set B^- of m blue points above the x-axis, with $m = \Theta(n)$. Now, using a generalized version of the Erdős-Szekeres Theorem on sequences[2] [10], there exist $k = \Omega(m^{1/3}) = \Omega(n^{1/3})$ indices $1 \leq i_1 < i_2 < \ldots < i_k \leq m$ such that the clockwise angles from the segments $e_{i_1}, e_{i_2}, \ldots, e_{i_k}$ to the x-axis are either monotonically *strictly* increasing, or monotonically *strictly* decreasing, or all equal. Let S_e denote the set of endpoints of e_{i_j} for $j \in [k]$.

In the first two cases we apply entirely the technique in the proof of Theorem 1. When $e_{i_1}, e_{i_2}, \ldots, e_{i_k}$ are all parallel, let us start with a line ℓ_p through each $p \in S$, in the direction of the segments, and rotate then ℓ_p around p an infinitesimal angle ε for all $p \in S$, in such a way that none of them contains two points. Rotating the whole construction if necessary, assume that the new lines ℓ'_p are vertical. Observe that the lines corresponding to the endpoints of a segment e_{i_j} are now different and consecutive in the horizontal order, for $j \in [k]$. Now shoot vertically a ray downwards from every point in $S \setminus S_e$. For each e_{i_j} with $j \in [k]$ we can independently decide for its endpoints whether we shoot a red ray upwards and a blue ray downwards, or reversely. This yields $\Omega(2^{n^{1/3}}/n)$ different configurations, and the claim is proved. □

4 Realizing configurations

We study in this section point sets realizing many configurations, universal and non-universal configurations. Observe that, as a consequence of Theorem 2, configurations with $\omega(\sqrt{n})$ alternations are not realizable from every point set. Further note that given any point set $S = R \cup B$ with $|R| = |B| = n$, the configuration (n, n) is always realizable: Draw from each red point a ray oriented to the right, and from each blue point a ray oriented to the left. The resulting rays are non-crossing and a perturbation H of them satisfies $C(S, H) = (n, n)$.

Theorem 4 *There exist point sets $S = R \cup B$ in strong general position with $|R| = |B| = n$, such that every configuration is feasible. Hence, $\gamma_{\text{sgp}}^{\max}(n) = \Gamma(n)$.*

Proof. Let point sets $R = \{(1, 1), (2, 1), \ldots, (n, 1)\}$ and $B = \{(1/n, 0), (2/n, 0), \ldots, (n/n, 0)\}$. Let C be any configuration $(r_1, b_1, r_2, b_2, \ldots, r_k, b_k)$ with $k \in \mathbb{N}$. Observe that we can draw a set H_B of rays from B such that the elements of H_B are grouped into k groups, such that the ith group for $i \in [k]$ consists of b_i parallel rays, and the groups split R into k blocks such that the jth block from left to right consists of r_j points, for $j \in [k]$. Namely, let $H_B = \{h((i/n, 0), (t_i, 1)) \mid i \in [n]\}$ where

[2]Let $n > s \cdot r \cdot p$. Any sequence of n numbers contains a strictly increasing subsequence with at least $s + 1$ elements, a strictly decreasing subsequence with at least $r + 1$ elements or a constant subsequence of length greater than p.

$t_i = r_1 + \ldots + r_s$ for

$$b_1 + \ldots + b_{s-1} < i \leq b_1 + \ldots + b_s.$$

Given H_B, we can draw a set H_R of rays "upwards" from R such that $H = H_R \cup H_B$ is a set of pairwise-disjoint rays from S. It holds $C(S, H) = C$. The constructed point set can be perturbed to lie in strong general position in a way that the proof carries out. □

Proposition 5 *For every bichromatic point set $S = R \cup B$ in general position with $|R| = |B| = n$, the configuration $(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor, \lceil n/2 \rceil, \lceil n/2 \rceil)$ is feasible.*

Proof. Assume n to be odd. The case in which is even is similar. By the Ham-Sandwich Cut Theorem [8], there exists a (directed) line ℓ passing through a red point p and a blue point q, and such that

$$|R \cap \ell^+| = |B \cap \ell^+| = |R \cap \ell^-| = |B \cap \ell^-| = \lfloor n/2 \rfloor.$$

Suppose without loss of generality that ℓ is horizontal and that p is to the right of q, and let

$$\begin{aligned} H' = &\{h(p, p-q), h(q, q-p)\} \\ &\cup \{h(u, p-q) \mid u \in (R \cap \ell^+) \cup (B \cap \ell^-)\} \\ &\cup \{h(u, q-p) \mid u \in (R \cap \ell^-) \cup (B \cap \ell^+)\}. \end{aligned}$$

Observe that H' is a set of non-crossing rays, and a perturbation H of H' is a set of pairwise-disjoint rays from H such that $C(S, H)$ is $(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor, \lceil n/2 \rceil, \lceil n/2 \rceil)$. □

Proposition 6 *For every bichromatic point set $S = R \cup B$ in general position with $|R| = |B| = n$, and any $t \in [n-1]$, either the configuration $(n-1, n-t, 1, t)$ or the configuration $(n-t, 1, t, n-1)$ is feasible.*

Proof. Let $p \in S$ be a point of the convex hull of S. We now show that if $p \in R$, then $(n-1, n-t, 1, t)$ is feasible. If $p \in B$, it can be shown analogously that $(n-t, 1, t, n-1)$ is feasible. Assume then that $p \in R$, and let $q \in B$ be a point such that $|B \cap \ell^-| = t - 1$, where ℓ is the line passing through p and q. Let

$$\begin{aligned} H' = &\{h(p, p-q)\} \\ &\cup \{h(u, q-p) \mid u \in R \setminus \{p\}\} \\ &\cup \{h(u, p-q) \mid u \in B\}, \end{aligned}$$

which is a set of non-crossing rays. A perturbation H of H' is such that $C(S, H) = (n-1, n-t, 1, t)$. □

In contrast with some of the previous configurations having alternation number two, we next show an obstacle for several configurations with alternation number three to be universal.

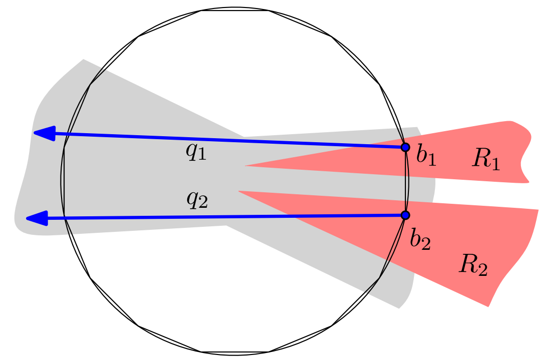

Figure 2: Illustration for the proof of Proposition 7.

Proposition 7 *Let $\mathcal{C}$ be an infinite family of configurations such that any configuration $C \in \mathcal{C}$ of length n has alternation number at least three, every red block of C has size at least n/k and every blue block of C has size at least n/l, where $k, l \in \mathbb{R}$. Then, there exists $n_0 \in \mathbb{N}$ such that any $C \in \mathcal{C}$ of size $n > n_0$ is not universal. In particular, the uniform configuration $(n/k, n/k, ..., n/k, n/k)$ with $k \geq 3$ and $n/k \in \mathbb{N}$ is not universal for big enough n.*

In order to prove the previous proposition, we use two technical lemmas, whose proof is omitted.

Given a real number $\lambda > 0$, let $K_\lambda(n)$ be the set of nth (complex) roots of the unity, taken as points in the real plane, and scaled by a factor of λ. The *width* of a point set T is the width of the thinnest slab (closed space between two parallel lines) enclosing T.

Lemma 8 *The width of any set $T \subset K_1(n)$ with $3 \leq t = |T| \leq \lfloor n/2 \rfloor$ is at least*

$$\cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{(t-1)\pi}{n}\right).$$

Given a point p outside the unit disk, we define V_p to be the open wedge defined by the rays starting at p and tangent to the unit circle, and containing the origin.

Lemma 9 *For any $p \in K_\lambda(n)$ with $\lambda > 1$, it holds*

$$|(K_\lambda(n) \cap V_p)| \leq \frac{2n}{\pi} \arcsin\left(\frac{1}{\lambda}\right) + 1.$$

We prove now Proposition 7; see Figure 2.

Proof of Proposition 7. Note first that it must be $k, l \geq 3$ since there are at least three blocks of each color. In addition, we also have $n \geq 3k$ and $n \geq 3l$. Let $R = K_1(n)$ and $B = K_\lambda(n)$ with $\lambda > 1$, and X_R and X_B be the circles containing R and B, respectively. Using Lemma 8, it is easy to see that if λ is smaller than

$$g(n) = \left[\cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{(\lceil \frac{n}{k} \rceil - 1)\pi}{n}\right)\right] \left[2 \sin\left(\frac{\pi}{n}\right)\right]^{-1},$$

the rays emanating from a subset $R_1 \subset R$ realizing a block of the configuration will have to cross at least two arcs of X_B of between points of B, since the first factor is a lower bound for the width of R_1 and the distance between two consecutive points of B is $2\lambda \sin(\pi/n)$. Therefore, the ray from at least one point $b_1 \in B$ will have to intersect X_R because, otherwise, it would split the block of R_1. Let $b_1, b_2, b_3 \in B$ be points trapped in three different sets $R_1, R_2, R_3 \subset R$, each of the latter realizing a red block. Note now that it has to be $b_2, b_3 \in V_{b_1} \cup V_{-b_1}$, where $-b_1$ indicates the point in X_B symmetric to b_1 with respect to the origin, since the rays q_2 placed at b_2 and q_3 placed at b_3 should split R and they should not intersect the ray q_1 placed at b_1. Observe that either V_{b_1} or V_{-b_1} must contain at least two of the points b_1, b_2 and b_3. Assume these to be b_1 and b_2. Note then that only the points from the two arcs of X_B between q_1 and q_2 can realize a blue block between R_1 and R_2. With the help of Lemma 9 to bound the number of points of B in these arcs, we have that if λ is larger than

$$f(n) = \left[\sin\left(\frac{\pi}{4n}\left(\left\lceil\frac{n}{l}\right\rceil - 2\right)\right)\right]^{-1},$$

no block of B can be realized between R_1 and R_2. Thus, for n and λ such that $1 < f(n) < \lambda < g(n)$, the configuration is not feasible. Since $g(n) \to \infty$ and $f(n) \to [\sin(\pi/4l)]^{-1} > 1$, the counterexample can be certainly constructed. □

5 Deciding feasibility of configurations

Proposition 10 *Given a point set $S = R \cup B$ on a line ℓ, with $|R| = |B| = n$, and a configuration C, it can be decided in $O(n^2)$ time whether C is feasible for S.*

Proof. Without loss of generality, assume that ℓ is horizontal, and put $S = \{p_1, \ldots, p_{2n}\}$, where the indices are taken from left to right. Note that any realization from S can be perturbed such that all the rays are vertical. The point p_1 must realize a position of its same color in C. Then, p_2 will realize either the previous position of the configuration or the next one, depending on whether the corresponding ray is pointing downwards or upwards. One, two or none of the previous options will be valid depending on whether the color of the previous and next positions of C match the color of p_2. In this way, when we traverse S from left to right choosing the upwards ray or the downwards one for each point, we may be realizing a subsequence of consecutive elements in C. Consider the directed graph having a node for each one of the $O(n^2)$ subsequences of C. Note that we consider as different two equal red-blue patterns if they start at different positions of C. We add an arc from a node corresponding to a subsequence of length $k \geq 0$ to a node corresponding to a subsequence of length $k+1$ if the second subsequence can be obtained from the first one by attaching the color of p_{k+1} before or after it. It is clear that a configuration is feasible for S if and only if there exists a path from the empty sequence to some of the $2n$ linear subsequences of C of length $2n$ in the aforementioned directed graph. Since the out-degree of every node is at most 2, the size of the graph is quadratic and the decision can be made in $O(n^2)$ time. □

The preceding algorithm is an adaptation of the algorithm of Akiyama and Urrutia for deciding, given $2n$ points on a circle, n of them being red, and n blue, whether they admit a simple Hamiltonian polygonal path in which the colors of the vertices alternate [2].

Proposition 11 *Given a set of points $S = R \cup B$ in convex position, with $|R| = |B| = n$, and a configuration C, it can be decided in polynomial time whether C is feasible for S.*

We present now some of the ideas of the omitted proof of the previous proposition. We derive a decision algorithm based on dynamic programming. First, the problem is discretized taking advantage of a perturbation argument appearing in [7], which ensures that to decide if a configuration is feasible it is enough to examine only the rays taking directions from a set of $O(n^2)$ vectors. Next, we observe that if a ray q supported by a line ℓ "crosses" the set S, and neither $S^+ = S \cap \ell^+$ nor $S^- = S \cap \ell^-$ is empty, then the position of the configuration realized by q must be "between" the positions realized from S^+ and the positions realized from S^-. Therefore, the points in S^+ and the points in S^- must be able to realize the corresponding parts of the configuration. These subproblems are not completely independent. Indeed, even if all the rays are disjoint from q, the rays from S^+ are pairwise disjoint and so are the rays from S^-, it could be that a ray from S^+ intersects a ray from S^-. In order to decide whether "compatible" realizations exist, we compute the (partial) realizations from S^+ and from S^- that minimize the "angle" with q. If these realizations are not compatible, then no other pair of realizations are. Computing a minimal realization for each subproblem is easy if we assume that there is no other ray "crossing" the point set. Nonetheless, when we allow more than one ray to cross, the computation becomes more involved. More precisely, we need to calculate for any pair of rays and any pair of positions of the configuration, all the partial realizations of the points "between" them that are angularly minimal. Note though that there are two angles to be minimized (in the domination order) in this case and, thus, the number of such realizations is $O(n^2)$. The corresponding angles (and a realization attaining each of them) can be constructed recursively using dynamic programming. The base case is when no ray crosses, in which case it is straightforward to compute all the realizations that are angularly minimal.

6 Remarks

In addition to the results presented above, we also studied a variant of the problem, where the rays are allowed to (properly) intersect. The nature of this problem is significantly different from the disjoint case. For instance, it is easy to see that all configurations of n red and n blue elements can be realized from any point set with that number of red and blue points. However, we can show that for certain configurations any set of realizing rays must have a quadratic number of intersections.

Yet another version of the question is which configurations can be realized by a set of pairwise-crossing rays. This condition is specially interesting, since any configuration realizable by pairwise-crossing rays can be realized by pairwise-disjoint rays as well (the converse is not true). Although there exist universal point sets for this problem, it can be shown that there is no universal configuration.

Acknowledgments. This work started at the *2nd International Workshop on Discrete and Computational Geometry*, held in Barcelona in July 1-5, 2013. R. Fabila-Monroy is supported by CONACyT of Mexico grant 153984. F. Hurtado, R. Silveira, J. Tejel and A. García are partially supported by projects MINECO MTM2012-30951, ESF EUROCORES programme EuroGIGA, CRP ComPoSe: MICINN Project EUI-EURC-2011-4306, F. Hurtado and R. Silveira are partially supported by Gen. Cat. DGR2009SGR1040 and DGR2014SGR46. A. García and J. Tejel are partially supported by Gob. Arag. E58-DGA. R. Jaume is supported by "obra social La Caixa" and the DAAD. P. Pérez-Lantero is partially supported by project CONICYT, FONDECYT/Iniciación 11110069 (Chile), and Millennium Nucleus Information and Coordination in Networks ICM/FIC P10-024F (Chile). M. Saumell is supported by projects NEXLIZ - CZ.1.07/2.3.00/30.0038, which is cofinanced by the European Social Fund and the state budget of the Czech Republic, and ESF EuroGIGA project ComPoSe as F.R.S.-FNRS - EUROGIGA NR 13604. R. Silveira was funded by Portuguese funds through CIDMA and FCT, within project PEst-OE/MAT/UI4106/2014, and by FCT grant SFRH/BPD/88455/2012. J. Urrutia is partially supported by Consejo Nacional de Ciencia y Tecnología, CONACYT (Mexico), grant number CB-2012-01-0178379. We thank Javier Cano, Clemens Huemer, Alberto Márquez, and Vera Sacristán for helpful discussions.

References

[1] O. Aichholzer, T. Hackl, C. Huemer, F. Hurtado, H. Krasser, and B. Vogtenhuber. On the number of plane geometric graphs. *Graphs and Combinatorics*, 23(1) (2007), 67-84.

[2] J. Akiyama, and J. Urrutia. Simple alternating path problem. Discrete Mathematics 84 (1) (1990), 101-103.

[3] S. Cabello, J. Cardinal, and S. Langerman, The Clique Problem in Ray Intersection Graphs. *Discrete & Computational Geometry* 50(3) (2013), pp. 771-783.

[4] A. Dumitrescu, A. Schulz, A. Sheffer, and C. D. Tóth, Bounds on the maximum multiplicity of some common geometric graphs, *SIAM Discrete Math.* 27 (2) (2013), 802826.

[5] P. Erdös, G. Szekeres. A combinatorial problem in geometry. *Compositio Mathematica* 2 (1935), pp. 463-470.

[6] S. Felsner, G. B. Mertzios, and I. Musta. On the Recognition of Four-Directional Orthogonal Ray Graphs. *Proc. 38th International Symposium on Mathematical Foundations of Computer Science*, volume 8087 of Lecture Notes in Computer Science, pp. 373-384. Springer, (2013).

[7] A. García, F. Hurtado, J. Tejel, and J. Urrutia. On the number of non-crossing ray configurations. Manuscript, submitted. Abstract with preliminary results in *XII Encuentros de Geometría Computacional*, Valladolid, Spain, 2007, pp. 129-134.

[8] J. E. Goodman and J. O'Rourke, editors. Handbook of discrete and computational geometry. CRC Press, Inc., Boca Raton, FL, USA, second edition, 2004.

[9] M. Hoffmann, M. Sharir, A. Sheffer, C. D. Tóth, and E. Welzl. Counting Plane Graphs: Flippability and its Applications. *Proc. 12th Symp. on Algs. and Data structs.*, (2011), 524–535.

[10] S. Jukna. *Extremal Combinatorics With Applications in Computer Science*, Springer-Verlag (Second Edition), Series *Texts in Theretical Computer Science* XXIV, page 71 exercise 4.12.

[11] A. Kaneko, and M. Kano. Discrete geometry on red and blue points in the plane – a survey. Discrete and Computational Geometry, The Goodman-Pollack Festschrift. Springer, Algorithms and Combinatorics series, Volume 25, 2003, pp. 551-570.

[12] D. Kirkpatrick, B. Yang and S. Zilles. On the barrier-resilience of arrangements of ray-sensors. *Proc. of the XV Sanish Meeting on Computational Geometry*, Seville, Spain, June 2013, pp. 35-38.

[13] C. Moreau. Sur les permutations circulaires distinctes. *Nouvelles annales de mathématiques, journal des candidats aux écoles polytechnique et normale*, Sér. 2, tom. 11 (1872), pp. 309–314

[14] M. Sharir and A. Sheffer. Counting triangulations of planar point sets. *Electr. J. Comb.*, 18(1) (2011).

[15] M. Sharir, A. Sheffer, and E. Welzl. Counting Plane Graphs: Perfect Matchings, Spanning Cycles, and Kasteleyn's Technique. *J. Combinat. Theory A*, 120 (2013), 777–794.

[16] M. Sharir, E. Welzl. On the number of crossing-free matchings, (cycles, and partitions). *SIAM J. Comput.* 36(3) (2006), pp. 695–720.

[17] A. Shrestha, Sa. Tayu, S. Ueno. On orthogonal ray graphs. *Discrete Applied Mathematics* 158(15) (2010), pp. 1650-1659.