HEPTAGONAL NUMBERS IN THE PELL SEQUENCE AND DIOPHANTINE EQUATIONS $2x^2 = y^2(5y-3)^2 \pm 2$

B. Srinivasa Rao

1-5-478/1, New Maruthinagar, Dilsukhnagar, Hyderabad-500 060, A.P., India

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1. INTRODUCTION

A positive integer N is called a **heptagonal (generalized heptagonal) number** if $N = \frac{m(5m-3)}{2}$ for some integer $m > 0$ (for any integer m). The first few are 1, 7, 18, 34, 55, 81, $\cdots$, and are listed in [3] as sequence number $A000566$. These numbers have been identified in the Fibonacci and Lucas sequence (see [4] and [5]). Now, in this paper we consider the **Pell sequence** $\{P_n\}$ defined by

$$P_0 = 0,\ P_1 = 1 \text{ and } P_{n+2} = 2P_{n+1} + P_n \text{ for any integer } n \tag{1}$$

and show that 0, 1 and 70 are the only generalized heptagonal numbers in $\{P_n\}$. This can also solve the Diophantine equations of the title. Earlier, McDaniel [1] has proved that 1 is the only triangular number in the Pell sequence and in [2] it is established that 0, 1, 2, 5, 12 and 70 are the only generalized Pentagonal Numbers in $\{P_n\}$.

2. IDENTITIES AND PRELIMINARY LEMMAS

We recall that the **associated Pell sequence** $\{Q_n\}$ is defined by

$$Q_0 = Q_1 = 1 \text{ and } Q_{n+2} = 2Q_{n+1} + Q_n \text{ for any integer } n, \tag{2}$$

and that it is closely related to the Pell sequence $\{P_n\}$. We have the following well-known properties of these sequences: For all integers m, n, k and t,

$$\left.\begin{array}{c} P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}} \text{ and } Q_n = \frac{\alpha^n + \beta^n}{2} \\ \text{where } \alpha = 1 + \sqrt{2} \text{ and } \beta = 1 - \sqrt{2} \end{array}\right\} \tag{3}$$

$$P_{-n} = (-1)^{n+1} P_n \text{ and } Q_{-n} = (-1)^n Q_n \tag{4}$$

$$Q_n^2 = 2P_n^2 + (-1)^n \tag{5}$$

$$Q_{3n} = Q_n(Q_n^2 + 6P_n^2) \tag{6}$$

$$P_{m+n} = 2P_m Q_n - (-1)^n P_{m-n} \tag{7}$$

$$P_{n+2kt} \equiv (-1)^{t(k+1)} P_n \pmod{Q_k} \tag{8}$$

$$2|P_n \text{ iff } 2|n \text{ and } 2 \nmid Q_n \text{ for any } n \tag{9}$$

$$3|P_n \text{ iff } 4|n \text{ and } 3|Q_n \text{ iff } n \equiv 2 \pmod 4 \tag{10}$$

$$5|P_n \text{ iff } 3|n \text{ and } 5 \nmid Q_n \text{ for any } n \tag{11}$$

$$9|P_n \text{ iff } 12|n \text{ and } 9|Q_n \text{ iff } n \equiv 6 \pmod{12}. \tag{12}$$

If m is odd, then

$$\left.\begin{array}{rl} (i) & Q_m \equiv \pm 1 \pmod 4 \text{ according as } m \equiv \pm 1 \pmod 4, \\ (ii) & P_m \equiv 1 \pmod 4, \\ (iii) & Q_m^2 + 6P_m^2 \equiv 7 \pmod 8. \end{array}\right\} \tag{13}$$

Since an integer N is generalized heptagonal if and only if $40N + 9$ is the square of an integer congruent to $7 \pmod{10}$, we have to first identify those n for which $40P_n + 9$ is a perfect square. We begin with

Lemma 1: Suppose $n \equiv \pm 1 \pmod{2^2 \cdot 5}$. Then $40P_n + 9$ is a perfect square if and only if $n = \pm 1$.

Proof: If $n = \pm 1$, then by (4) we have $40P_n + 9 = 40P_{\pm 1} + 9 = 7^2$.

Conversely, suppose $n \equiv \pm 1 \pmod{2^2 \cdot 5}$ and $n \notin \{-1, 1\}$. Then n can be written as $n = 2 \cdot 3^r \cdot 5m \pm 1$, where $r \geq 0, 3 \nmid m$ and $2|m$. Then $m \equiv \pm 2 \pmod 6$. Taking

$$k = \begin{cases} 5m & \text{if } m \equiv \pm 8 \text{ or } \pm 14 \pmod{30} \\ m & \text{otherwise} \end{cases}$$

we get that

$$k \equiv \pm 2, \pm 4 \text{ or } \pm 10 \pmod{30} \text{ and that } n = 2kg \pm 1, \text{ where } g \text{ is odd.} \tag{14}$$

In fact, $g = 3^r \cdot 5$ or 3^r. Now, by (8), (14) and (4) we get

$$40P_n + 9 = 40P_{2kg \pm 1} + 9 \equiv 40(-1)^{g(k+1)} P_{\pm 1} + 9 \pmod{Q_k} \equiv -31 \pmod{Q_k}.$$

Therefore, the Jacobi symbol

$$\left(\frac{40P_n + 9}{Q_k}\right) = \left(\frac{-31}{Q_k}\right) = \left(\frac{Q_k}{31}\right). \tag{15}$$

But modulo 31, $\{Q_n\}$ has periodic with period 30. That is, $Q_{n+30t} \equiv Q_n \pmod{31}$ for all integers $t \geq 0$. Thus, by (14) and (4), we get $Q_k \equiv 3$, 17 or 15 $\pmod{31}$ and in any case

$$\left(\frac{Q_k}{31}\right) = -1. \tag{16}$$

From (15) and (16), it follows that $\left(\frac{40P_n+9}{Q_k}\right) = -1$ for $n \notin \{-1, 1\}$ showing $40P_n + 9$ is not a perfect square. Hence the lemma.

Lemma 2: Suppose $n \equiv 6 \pmod{2^2 \cdot 5^3 \cdot 7^2}$. Then $40P_n + 9$ is a perfect square if and only if $n = 6$.

Proof: If $n = 6$, then $40P_n + 9 = 40P_6 + 9 = 53^2$.

Conversely, suppose $n \equiv 6 \pmod{2^2 \cdot 5^3 \cdot 7^2}$ and $n \neq 6$. Then n can be written as $n = 2 \cdot 5^3 \cdot 7^2 \cdot 2^r \cdot m + 6$, where $r \geq 1, 2 \nmid m$. And since for $r \geq 1$, $2^{r+60} \equiv 2^r \pmod{2790}$, taking

$$k = \begin{cases} 5^3 \cdot 2^r & \text{if } r \equiv 13 \pmod{60} \\ 5 \cdot 2^r & \text{if } r \equiv 4, 6, 16, 23, 24, 25, 27, 28, 29, 30, 51, 53, 55, 57 \text{ or } 58 \pmod{60} \\ 7^2 \cdot 2^r & \text{if } r \equiv 9, 18, 34, 38, 39, 43 \text{ or } 56 \pmod{60} \\ 7 \cdot 2^r & \text{if } r \equiv 11, \pm 19, 42 \text{ or } 48 \pmod{60} \\ 2^r & \text{otherwise} \end{cases}$$

we get that

$$\begin{aligned} k \equiv\, & 2, 4, 8, 32, 70, 94, 112, 128, 226, 256, 350, 376, 386, 448, 466, 698, \\ & 700, 826, 862, 934, 940, 944, 962, 970, 994, 1024, 1058, 1090, 1118, \\ & 1148, 1166, 1250, 1306, 1322, 1396, 1400, 1442, 1504, 1570, 1652, \\ & 1682, 1802, 1834, 1862, 1876, 1888, 1924, 1940, 2078, 2236, 2296, \\ & 2326, 2434, 2686, 2732 \text{ or } 2768 \pmod{2790} \end{aligned} \tag{17}$$

and

$$n = 2kg + 6, \text{ where } g \text{ is odd and } k \text{ is even.} \tag{18}$$

Now, by (8) and (18), we get

$$40P_n + 9 = 40P_{2kg+6} + 9 \equiv 40(-1)^{g(k+1)} P_6 + 9 \pmod{Q_k} \equiv -2791 \pmod{Q_k}.$$

Hence, the Jacobi symbol

$$\left(\frac{40P_n + 9}{Q_k}\right) = \left(\frac{-2791}{Q_k}\right) = \left(\frac{Q_k}{2791}\right) \tag{19}$$

But modulo 2791, the sequence $\{Q_n\}$ has period 2790. Therefore, by (17), we get

$$\begin{aligned} Q_k \equiv\, & 3, 17, 577, 489, 2583, 1422, 2410, 591, 1025, 811, 662, 127, 2248, \\ & 915, 1961, 2486, 113, 1934, 817, 1248, 544, 1680, 1969, 2679, \\ & 1288, 2585, 21, 2047, 1642, 158, 823, 1381, 2549, 2262, 1843, 418, \\ & 525, 2677, 2557, 831, 1330, 862, 1088, 952, 786, 1397, 523, 2759, \\ & 1761, 115, 2480, 1778, 1303, 2397, 1669 \text{ or } 647 \pmod{2791} \end{aligned}$$

respectively and for all these values of k, the Jacobi symbol

$$\left(\frac{Q_k}{2791}\right) = -1. \tag{20}$$

From (19) and (20), it follows that $\left(\frac{40P_n+9}{Q_k}\right) = -1$ for $n \neq 6$ showing $40P_n + 9$ is not a perfect square. Hence the lemma.

Lemma 3: Suppose $n \equiv 0 \pmod{2 \cdot 7 \cdot 5^3}$. Then $40P_n + 9$ is a perfect square if and only if $n = 0$.

Proof: If $n = 0$, then we have $40P_n + 9 = 40P_0 + 9 = 3^2$.

Conversely, suppose $n \equiv 0 \pmod{2 \cdot 7 \cdot 5^3}$ and for $n \neq 0$ put $n = 2 \cdot 7 \cdot 5^3 \cdot 3^r \cdot z$, where $r \geq 0$ and $3 \nmid z$. Then $n = 2m(3k \pm 1)$ for some integer k and odd m. We choose m as follows

$$m = \begin{cases} 5^3 \cdot 3^r & \text{if } r \equiv 3 \text{ or } 12 \pmod{18} \\ 5^2 \cdot 3^r & \text{if } r \equiv 1, 7, 10, 14 \text{ or } 16 \pmod{18} \\ 5 \cdot 3^r & \text{if } r \equiv 2, 5 \text{ or } 11 \pmod{18} \\ 7 \cdot 3^r & \text{if } r \equiv 8 \text{ or } 17 \pmod{18} \\ 3^r & \text{otherwise.} \end{cases}$$

Since for $r \geq 0$, $3^{r+18} \equiv 3^r \pmod{152}$, we have

$$m \equiv 1, 23, 31, 45, 53, 75, 81, 107, 121, 147 \text{ or } 151 \pmod{152}. \tag{21}$$

Therefore, by (8), (4), (6) and the fact that m is odd, we have

$$40P_n + 9 = 40P_{2(3m)k \pm 2m} + 9 \equiv 40(-1)^{k(3m+1)}P_{\pm 2m} + 9 \pmod{Q_{3m}}$$

$$\equiv \pm 40P_{2m} + 9 \pmod{Q_m^2 + 6P_m^2}$$

according as $z \equiv \pm 1 \pmod 3$. Letting $w_m = Q_m^2 + 6P_m^2$ and using (5), (7) and (13) we get that the Jacobi symbol

$$\left(\frac{40P_n + 9}{w_m}\right) = \left(\frac{\pm 40P_{2m} + 9}{w_m}\right) = \left(\frac{\pm 80Q_mP_m - 9Q_m^2 + 18P_m^2}{w_m}\right) = \left(\frac{\pm 80Q_mP_m + 72P_m^2}{w_m}\right)$$

$$= \left(\frac{2}{w_m}\right)\left(\frac{P_m}{w_m}\right)\left(\frac{\pm 10Q_m + 9P_m}{w_m}\right) = -\left(\frac{w_m}{\pm 10Q_m + 9P_m}\right). \tag{22}$$

Now, if $3|m$ then by (11), $5|P_m$ and from (22) we get

$$\left(\frac{40P_n + 9}{w_m}\right) = -\left(\frac{w_m}{5}\right)\left(\frac{w_m}{\pm 2Q_m + 9\frac{P_m}{5}}\right)$$

$$= -\left(\frac{\left(\pm 2Q_m + 9\frac{P_m}{5}\right)\left(\pm 2Q_m - 9\frac{P_m}{5}\right) + 681\frac{P_m^2}{25}}{\pm 2Q_m + 9\frac{P_m}{5}}\right)$$

$$= -\left(\frac{681}{\pm 2Q_m + 9\frac{P_m}{5}}\right) = -\left(\frac{681}{\pm 10Q_m + 9P_m}\right).$$

And, if $3 \nmid m$ then by (11), $3 \nmid P_m$ and from (22) we get that

$$\left(\frac{40P_n+9}{w_m}\right) = -\left(\frac{(\pm 10Q_m + 9P_m)(\pm 10Q_m - 9P_m) + 681P_m^2}{\pm 10Q_m + 9P_m}\right) = -\left(\frac{681}{\pm 10Q_m + 9P_m}\right).$$

In any case,

$$\left(\frac{40P_n+9}{w_m}\right) = -\left(\frac{681}{\pm 10Q_m + 9P_m}\right) = -\left(\frac{\pm 10Q_m + 9P_m}{681}\right). \tag{23}$$

But since modulo 681, the sequence $\{\pm 10Q_m + 9P_m\}$ is periodic with period 152, by (21) it follows that

$$10Q_m + 9P_m \equiv 19, 125, 251, 509, 395, 1, 10, 430, 172, 532, \text{ or } 680 \pmod{681}$$

and

$$-10Q_m + 9P_m \equiv 680, 286, 172, 430, 556, 662, 149, 509, 251, 671 \text{ or } 19 \pmod{681}.$$

In any case

$$\left(\frac{\pm 10Q_m + 9P_m}{681}\right) = 1. \tag{24}$$

Therefore, from (23) and (24) we get $\left(\frac{40P_n+9}{w_m}\right) = -1$. Hence the lemma.

As a consequence of Lemmas 1 to 3 we have the following.

Corollary 1: Suppose $n \equiv 0, \pm 1$ or $6 \pmod{24500}$. Then $40P_n + 9$ is a perfect square if and only if $n = 0, \pm 1$ or 6.

Lemma 4: $40P_n + 9$ is not a perfect square if $n \not\equiv 0, \pm 1$ or $6 \pmod{24500}$.

Proof: We prove the lemma in different steps eliminating at each stage certain integers n congruent modulo 24500 for which $40P_n + 9$ is not a square. In each step we choose an integer m such that the period p (of the sequence $\{P_n\}$ mod m) is a divisor of 24500 and thereby eliminate certain residue classes modulo p. For example

Mod 41: The sequence $\{P_n\}$ mod 41 has period 10. We can eliminate $n \equiv 2, 4$ and $8 \pmod{10}$, since $40P_n + 9 \equiv 7, 38$ and $11 \pmod{41}$ and they are quadratic nonresidue modulo 41. There remain $n \equiv 0, 1, 3, 5, 6, 7$ or $9 \pmod{10}$, equivalently, $n \equiv 0, 1, 3, 5, 6, 7, 9, 10, 11, 13, 15, 16, 17,$ or $19 \pmod{20}$.

Similarly we can eliminate the remaining values of n. After reaching modulo 24500, if there remain any values of n we eliminate them in the higher modulo (That is, in the multiples of 24500). We tabulate them in the following way (Tables A and B).

Period p	**Modulus m**	**Required values of n where** $\left(\frac{40P_n+9}{m}\right)=-1$	**Left out values of n** (mod **t**) **where t is a positive integer**
10	41	± 2 **and** 4	0, ± 1, ± 3, 5, **or** 6 (mod 10)
20	29	± 7 **and** ± 9	0, ± 1, ± 3, ± 5, 6, 10, **or** 16 (mod 20)
100	1549	± 3, ± 5, 10, ± 17, ± 20, ± 21, ± 23, ± 30, ± 35,± 37, 40, ± 43, 46, 56, 86, **and** 96	0,± 1, 6, 16, ± 19, ± 25, 26, ± 39, 50, **or** 76 (mod 100)
	29201	± 15, 36, ± 41, ± 45, 60, 66, **and** 90	
700	349	26, ± 50, ± 61, ± 75, ± 81, ± 99, 126, ± 161, ± 181, 216, ± 219, ± 225, ± 239, ± 261, ± 300, ± 301, 326, ± 339, 376, 426, 576, 616, and 676	0, ± 1, 6, ± 201, **or** 350 (mod 700)
	15401	± 19, ± 39, 76, 100, ± 101, 106, 116, ± 125, ± 139, ± 150, ± 200, 206, 226, 250, 276, ± 319, 406, 416, 506, 606, and 626	
	53549	± 119, ± 275, ± 281, 316, 450, 476, 516, **and** 600	
70	71	± 11, 16, ± 25, 26, 35, **and** 36	
28	13	± 9	
98	1471	± 5, ± 29, ± 33, 34, ± 41, ± 42, **and** ± 43	0, ± 1, 6, 700, 1750, ± 1899, 2450,**or** 2806 (mod 4900)
196	293	14, ± 23, ± 28, ± 51, ± 70, ± 79, ± 83, 84, ± 85, ± 89, 90, **and** 174	
2450	85751	706, ± 1399, **and** 2106	
3500	7001	350, ± 499, ± 699, ± 701, 706, ± 1199, 1400, ± 1401, ± 1601, 2106, **and** 2806	0, ± 1, 6, 1750, 4906, 5600, 6650, ± 6799, 10500, 12250, 17150, 17506, 19600,**or** 22406 (mod 24500)
500	129749	± 99, ± 101, 300, **and** 450	
	286001	50 **and** 200	

Table A.

We now eliminate: $n \equiv 1750$, 4906, 5600, 6650, 6799, 10500, 12250, 17150, 17506, 17701, 19600, and 22406 (mod 24500).

Equivalently: $n \equiv 1750$, 4906, 5600, 6650, 6799, 10500, 12250, 17150, 17506, 17701, 19600, 22406, 26250, 29406, 30100, 31150, 31299, 35000, 36750, 41650, 42006, 42201, 44100 and 46906 (mod 49000).

Period p	**Modulus m**	**Required values of n where** $\left(\frac{40P_n+9}{m}\right)=-1$	**Left out values of n** (mod **t**) **where t is a positive integer**
7000	3499	±1750, 3150, ±3299, 3506, 5600 **and** 6650	10500, 35000, **or** 42006 (mod 49000) ⇔10500, 35000, 42006, 59500, 84000, **or** 91006 (mod 98000)
	217001	1406 **and** 2100	
1000	499	±201 **and** 906	
3920	7841	846, 2660, 2806, **and** 3640	84000 (mod 98000)⇔84000, 182000 **or** 280000 (mod 294000)
80	5521	60	
1176	13523	112 **and** 504	**Completely eliminated in modulo** 294000
2100	15749	1400	

Table B.

3. MAIN THEOREM

Theorem 1: **(a)** P_n is a generalized heptagonal number only for $n = 0, \pm 1$ or 6; and **(b)** P_n is a heptagonal number only for $n = \pm 1$.

Proof: Part **(a)** of the theorem follows from Corollary 1 and Lemma 4. For part (b), since, an integer N is heptagonal if and only if $40N + 9 = (10 \cdot m - 3)^2$ where m is a positive integer, we have the following table.

$\boldsymbol{n}$	0	±1	6
$\boldsymbol{P_n}$	0	1	70
$\boldsymbol{40P_n+9}$	3^2	7^2	53^2
$\boldsymbol{m}$	0	1	−5
$\boldsymbol{Q_n}$	1	±1	99

Table C.

4. SOLUTIONS OF CERTAIN DIOPHANTINE EQUATIONS

If D is a positive integer which is not a perfect square it is well known that $x^2 - Dy^2 = \pm 1$ is called the Pell's equation and that if $x_1 + y_1\sqrt{D}$ is the fundamental solution of it (that is, x_1 and y_1 are least positive integers), then $x_n + y_n\sqrt{D} = \left(x_1 + y_1\sqrt{D}\right)^n$ is also a solution of the same equation; and conversely every solution of it is of this form.

Now by (5), we have $Q_n^2 = 2P_n^2 + (-1)^n$ for every n. Therefore, it follow that

$$Q_{2n} + \sqrt{2}P_{2n} \text{ is a solution of } x^2 - 2y^2 = 1, \tag{25}$$

while

$$Q_{2n+1} + \sqrt{2}P_{2n+1} \text{ is a solution of } x^2 - 2y^2 = -1. \tag{26}$$

We have, by (25), (26), Theorem 1, and Table C, the following two corollaries.

Corollary 2: The solution set of the Diophantine equation $2x^2 = y^2(5y-3)^2 - 2$ is $\{(\pm 1, 1)\}$.

Corollary 3: The solution set of the Diophantine equation $2x^2 = y^2(5y-3)^2 + 2$ is $\{(\pm 1, 0),\ (\pm 99, -5)\}$.

REFERENCES

[1] W. L. McDaniel. "Triangular Numbers in the Pell Sequence." *The Fibonacci Quarterly* **34.2** (1996): 105-107.

[2] V. Siva Rama Prasad and B. Srinivasa Rao. "Pentagonal Numbers in the Pell Sequence and Diophantine Equations $2x^2 = y^2(3y-1)^2 \pm 2$." *The Fibonacci Quarterly* **40.3** (2002): 233-241.

[3] N. J. A. Sloane. The On-Line Encyclopedia of Integer Sequences. *http://www.research.att.com/~njas/sequences*, A000566.

[4] B. Srinivasa Rao. "Heptagonal Numbers in the Fibonacci Sequence and Diophantine Equations $4x^2 = 5y^2(5y-3)^2 \pm 16$." *The Fibonacci Quarterly* **41.5** (2003): 414-420.

[5] B. Srinivasa Rao. "Heptagonal Numbers in the Lucas Sequence and Diophantine Equations $x^2(5x-3)^2 = 20y^2 \pm 16$." *The Fibonacci Quarterly* **40.4** (2002): 319-322.

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