SUPPOSE MORE RABBITS ARE BORN

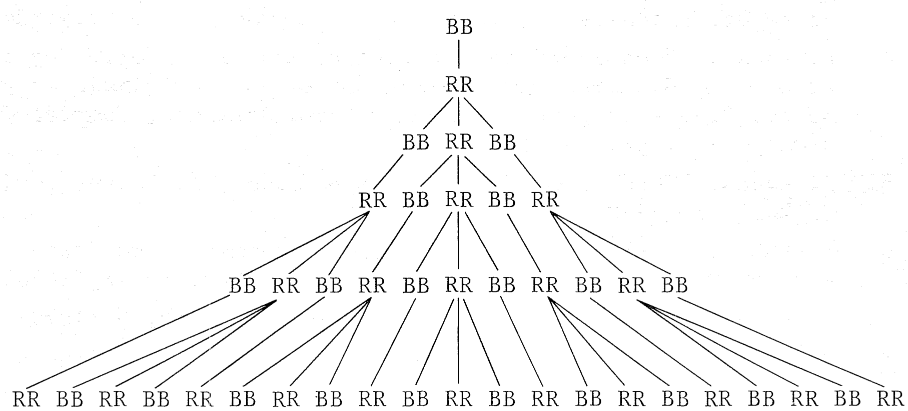
SHARI LYNN LEVINE
Student, Half Hollow Hills H.S. East, Dix Hills, NY 11746

(Submitted November 1986)

How would Fibonacci's age-old sequence be redefined if, instead of bearing one pair of baby rabbits per month, the mature rabbits bear two pairs of baby rabbits per month? The answer is an intriguing sequence that has led to the development of what are herein defined as "multi-nacci sequences of the order q," where q is the number of rabbit pairs per litter. Table 1 illustrates the sequence with $q = 2$.

Table 1

Pair Sequence	Month
1	1
1	2
3	3
5	4
11	5
21	6
43	7



Key: RR = Pair of rabbits ready to reproduce
BB = Pair of bunnies (immature rabbits)

Call this sequence the "Beta-nacci sequence"; note that each term can be generated by adding the preceding term to twice the one before that, i.e.,

$$B_n = B_{n-1} + 2B_{n-2}.$$

Using a similar process, sequences can be developed for situations when 3, 4, 5, and 6 rabbit pairs per litter are born. Call these multi-nacci sequences Gamma-, Delta-, Epsi-, and Zeta-nacci sequences, respectively. Table 2 illustrates the first seven terms in each of these multi-nacci sequences and the general formulas for each sequence.

Table 2

Beta-nacci		Gamma-nacci		Delta-nacci		Epsi-nacci		Zeta-nacci	
n	B_n	n	G_n	n	D_n	n	E_n	n	Z_n
0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1
2	1	2	1	2	1	2	1	2	1
3	3	3	4	3	5	3	6	3	7
4	5	4	7	4	9	4	11	4	13
5	11	5	19	5	29	5	41	5	55
6	21	6	40	6	65	6	96	6	133
7	43	7	97	7	181	7	301	7	463
$B_{n-1} + 2B_{n-2}$		$G_{n-1} + 3G_{n-2}$		$D_{n-1} + 4D_{n-2}$		$E_{n-1} + 5E_{n-2}$		$Z_{n-1} + 6Z_{n-2}$	

SUCCESSIVE TERM RATIOS

When one examines the ratio created from two successive terms of the Fibonacci sequence, as n gets larger, the ratio under investigation approaches the Golden Ratio, $\phi = 1.618033989\ldots$, which is the decimal representation of

$$\phi = (1 + \sqrt{5})/2.$$

For the multi-nacci sequences to be analogous to the Fibonacci sequence, each sequence should also have a unique ratio that is approached when one forms a ratio of one term to its preceding term. Indeed, this is the case. Let S_q = the limit, as $n \to \infty$, of successive term ratios of any multi-nacci sequence of order q. (By this definition, $S_1 = \phi$.) Let ${}_nS_q$ = the successive term ratios of the n^{th} term to its preceding term in any multi-nacci sequence of order q, e.g., ${}_5S_2 = 2.20$. The Beta-nacci sequence ratio is examined in Table 3.

Table 3

n	B_n	${}_nS_2$*	n	B_n	${}_nS_2$*
1	1		7	43	2.048
2	1	1.000	8	85	1.977
3	3	3.000	9	171	2.012
4	5	1.667	10	341	1.994
5	11	2.200	11	683	2.003
6	21	1.909	12	1365	1.999

**To the nearest thousandth.*

Thus, we can see that for the Beta-nacci sequence $S_2 \to 2$.

It can be shown that for the Gamma-nacci, Delta-nacci, Epsi-nacci, and Zeta-nacci sequences, the following ratios are approached:

Gamma-nacci: $S_3 \to 2.30277$
Delta-nacci: $S_4 \to 2.56155$
Epsi-nacci: $S_5 \to 2.79129$
Zeta-nacci: $S_6 \to 3.00000$

The technique of the proofs of these ratios is illustrated below using the Gamma-nacci sequence.

Let $A = G_{n-2}$ when n is very large. Then, the next term in the sequence, G_{n-1}, will be approximately $S_3(A)$, and the next term, G_n, will be $(S_3)^2A$.

Remember that, by definition, $G_n = G_{n-1} + 3G_{n-2}$.

But this is $(S_3)^2A = S_3A + 3A$, whose solution is $S_3 = (1 \pm \sqrt{13})/2$.

Disregarding the $-\sqrt{13}$, because there are no negative rabbits,

$S_3 = 2.30277\ldots$.

Note that the equation $(S_3)^2 - S_3 - 3 = 0$ bears a striking resemblance to the equation $(S_1)^2 - S_1 - 1 = 0$ that generates ϕ. In fact, an entire family of equations can be created which when solved yield the ratios indicated earlier. Specifically, the general equation is $(S_q)^2 - S_q - q = 0$, and the ratio

$S_q = (1 + \sqrt{1 + 4q})/2$.

SPECIAL RECIPROCAL PROPERTIES

One special property of the Golden Ratio is that it is its own reciprocal after one has been subtracted from it. With the multi-nacci sequences, some more general questions can be investigated, such as: "What number, when one is subtracted from it, is twice its own reciprocal, or three times its own reciprocal, or four times its own reciprocal?" The answers, in this order, are the Beta-nacci, Gamma-nacci, and Delta-nacci successive term ratios: S_2, S_3, and S_4, respectively.

The proof of a generalized version of this question is very straightforward.

$$
\begin{aligned}
(S_q)^2 - S_q - q &= 0 \\
(S_q)^2 &= S_q + q \\
S_q &= 1 + \frac{q}{S_q} \\
(S_q - 1) &= q\left(\frac{1}{S_q}\right)
\end{aligned}
$$

Thus, the special reciprocal property of the Fibonacci sequence is but one of a more general set of reciprocal properties of the ratio limits of the multi-nacci sequences.

BETA-NACCI SEQUENCE PROPERTIES

In particular, the Beta-nacci sequence has been given additional examination because it appears to have many interesting properties.

Table 4

n	B_n	$2B_n$	2^n	$\sum_{n=0}^{n} B_n$	$(B_n)^2$	$(B_{n-1})(B_{n+1})$
0	0	0	1	0	0	
1	1	2	2	1	1	0
2	1	2	4	2	1	3
3	3	6	8	5	9	5
4	5	10	16	10	25	33
5	11	22	32	21	121	105
6	21	42	64	42	441	473
7	43	86	128	85	1849	1785
8	85	170	256	170	7225	7353
9	171	342	512	341	29241	28985
10	341	682	1024	682	116281	116793
11	683	1366	2048	1365	466489	465465

Notice that in Table 4 the sum of any two successive terms in the B_n column is a power of 2, or

$$B_n + B_{n-1} = 2^{n-1}. \tag{1}$$

The $\sum_{n=0}^{n} B_n$ column is remarkably like the B_n. In fact,

$$\sum_{n=0}^{n} B_{n-1} = B_n + \frac{(-1)^n - 1}{2}.$$

Examining the $2B_n$ column, it appears that there is a difference of ±1 between the entries in the B_n and $2B_{n-1}$ locations. That is,

$$B_n - 2B_{n-1} = (-1)^{n-1}. \tag{2}$$

Because in the Fibonacci sequence there is a relationship between $(F_n)^2$ and $(F_{n-1})(F_{n+1})$, the Beta-nacci numbers have been examined for a similar relationship. From Table 4 entries, the results of $(B_n)^2 - (B_{n-1})(B_{n+1})$ are +1, -2, +4, -8, +16, -32, +64, -128, +256, -512, and +1024, so that $(B_n)^2 - (B_{n-1})(B_{n+1}) = (-2)^{n-1}$. Fibonacci numbers have the same relationship using the base of (-1) instead of (-2). It can be shown that $(T_n)^2 - (T_{n-1})(T_{n+1}) = (-q)^{n-1}$, where T is any term of a multi-nacci series of order q.

Equation (1) shows that the summation of two successive terms in the Beta-nacci sequence is 1, 2, 4, 8, 16, ... If this sequence is studied, it, too, is observed to be a Beta-nacci-type sequence. For example, 16 = 8 + 2(4). In other words, the powers of 2 are a Beta-nacci sequence. (Interestingly, also, is the fact that the powers of 3 are a Zeta-nacci sequence.) Furthermore, if two terms of the 2, 4, 8, 16, ... sequence are summed, a sequence with the terms 3, 6, 12, 24, 48, ... develops. This is also a Beta-nacci-type sequence, i.e., 24 = 12 + 2(6). In fact, summing two successive terms in any multi-nacci sequence creates a new sequence of the same multi-nacci type.

Moreover, summing three successive terms of the Beta-nacci sequence creates the sequence 2, 5, 9, 19, 37, 75, 149, ..., which is yet another Beta-nacci-type sequence, i.e., 37 = 19 + 2(9). Summing any number of successive terms in any multi-nacci sequence results in a new multi-nacci sequence of the same type:

If T_n is the n^{th} term of any type of multi-nacci sequence, then

$$\begin{aligned}
T_n &= T_{n-1} &&+ qT_{n-2}\\
T_{n+1} &= T_n &&+ qT_{n-1}\\
T_{n+2} &= T_{n+1} &&+ qT_n\\
T_{n+3} &= T_{n+2} &&+ qT_{n+1}\\
&\ \vdots && \ \vdots\\
T_{n+m} &= T_{n+m-1} &&+ qT_{n+m-2}\\
\sum_{N=n}^{m} T_N &= \sum_{N=n}^{m} T_{N-1} &&+ q\sum_{N=n}^{m} T_{N-2}
\end{aligned}$$

Similarly, it can be shown that summing the terms in any two or more non-sequential multi-nacci sequences of the same order results in sums which are also a multi-nacci sequence of the same order.

BETA-NACCI n^{th} TERM

In the past, mathematicians have developed formulas for the n^{th} term of the Fibonacci sequence. This is important because, without such a formula, one must enumerate every single term up to the one in question. Thus, the Beta-nacci sequence has been examined for a formula for the n^{th} term.

Using (1) and (2), as defined,

$$\begin{aligned}
B_n + B_{n-1} &= 2^{n-1}\\
B_n - 2B_{n-1} &= (-1)^{n-1}
\end{aligned}$$

we have:

$$B_n = 2B_{n-1} + (-1)^{n-1}$$
$$2B_{n-1} + (-1)^{n-1} + B_{n-1} = 2^{n-1}$$
$$3B_{n-1} = 2^{n-1} - (-1)^{n-1}$$

For ease in examination, let $n - 1 = n$, so $3B_n = 2^n - (-1)^n$. Then

$$B_n = \frac{2^n - (-1)^n}{3}.$$

This formula is much less complicated than one for Fibonacci's n^{th} term.

REPEATING UNITS DIGITS

One can observe that the units digits in the Beta-nacci sequence are 1, 1, 3, 5, 1, 1, 3, 5, 1, 1, 3, 5, ... They repeat every four terms. The units digits of the ΣB_n terms also repeat every four terms as 0, 1, 2, 5, 0, 1, 2, 5, etc. In 1963, Dov Jarden showed in [1] that the units digit of the Fibonacci sequence repeats every 60 terms. Thus, in this regard, Beta-nacci is a vast improvement over Fibonacci. All multi-nacci sequences have units digit repeat periods.

RABBIT PAIRS PER LITTER, q	SEQUENCE	UNITS DIGIT REPEAT PERIOD
1	Fibonacci	60
2	Beta-nacci	4
3	Gamma-nacci	24
4	Delta-nacci	6
5	Epsi-nacci	3
6	Zeta-nacci	20
7	Eta-nacci	12
8	Theta-nacci	24
9	Iota-nacci	6
10	Kappa-nacci	1
11	Lambda-nacci	60

Moreover, the sequence of units digit repeat periods 60, 4, 24, 6, 3, 20, 12, 24, 6, 1 now repeats as we get into the higher-order multi-nacci sequences. The determination of the tens digit repeat periods is left to the reader.

CONCLUSIONS

Fibonacci-type sequences develop from multiple rabbit births. This paper demonstrates that these sequences also have interesting properties of their own which are ripe for future study.

REFERENCE

1. Dov Jarden. "On the Periodicity of the Last Digits of the Fibonacci Numbers." *The Fibonacci Quarterly* 1, no. 4 (1963):21-22.