ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. Hillman

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108.

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should include solutions.

Anyone desiring acknowledgment of contributions should enclose a stamped, self-addressed card (or envelope).

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n, satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

Also, $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-664 *Proposed by Mohammad K. Azarian, U. of Evansville, Evansville, IN*

Let $a_0 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$ for n in $\{0, 1, \ldots\}$. Show that

$$\lim_{n\to\infty} a_n = \sum_{i=0}^{\infty}\left[\sum_{j=0}^{i}\binom{i}{j}\right]^{-1}.$$

B-665 *Proposed by Christopher C. Street, Morris Plains, NJ*

Show that $AB = 9$, where

$$A = (19 + 3\sqrt{33})^{1/3} + (19 - 3\sqrt{33})^{1/3} + 1,$$

$$B = (17 + 3\sqrt{33})^{1/3} + (17 - 3\sqrt{33})^{1/3} - 1.$$

B-666 *Taken from solutions to B-643 by Russell Jay Hendel, Dowling College, Oakdale, NY, and by Lawrence Somer, Washington, D.C.*

For primes p, prove that

$$\binom{n}{p} \equiv [n/p] \pmod{p},$$

where $[x]$ is the greatest integer in x.

B-667 *Proposed by Herta T. Freitag, Roanoke, VA*

Let p be a prime, $p \neq 2$, $p \neq 5$, and m be the least positive integer such that $10^m \equiv 1 \pmod{p}$. Prove that each m-digit (integral) multiple of p remains a multiple of p when its digits are permuted cyclically.

B-668 *Proposed by A. P. Hillman in memory of Gloria C. Padilla*

Let h be the positive integer whose base 9 numeral

100101102...887888

is obtained by placing all the 3-digit base 9 numerals end-to-end as indicated.

(a) What is the remainder when h is divided by the base 9 integer 14?
(b) What is the remainder when h is divided by the base 9 integer 81?

B-669 *Proposed by Gregory Wulczyn, Lewisburg, PA*

Do the equations

$$25F_{a+b+c}F_{a+b-c}F_{b+c-a}F_{c+a-b} = 4 - L_{2a}^2 - L_{2b}^2 - L_{2c}^2 + L_{2a}L_{2b}L_{2c},$$
$$L_{a+b+c}L_{a+b-c}L_{b+c-a}L_{c+a-b} = -4 + L_{2a}^2 + L_{2b}^2 + L_{2c}^2 + L_{2a}L_{2b}L_{2c},$$

hold for all even integers a, b, c?

SOLUTIONS

Circulant Determinant for F_{n+1}

B-640 *Proposed by Russell Euler, Northwest Missouri State U., Marysville, MO*

Find the determinant of the $n \times n$ matrix (x_{ij}) with $x_{ij} = 1$ for $j = i$ and for $j = i - 1$, $x_{ij} = -1$ for $j = i + 1$, and $x_{ij} = 0$, otherwise.

Solution by Paul S. Bruckman, Edmonds, WA

Let A_n denote the given matrix and D_n its determinant. Clearly, $D_1 = 1$, and $D_2 = 2$. We may expand D_n along its first row; doing so, we see that $D_n = D_{n-1} + B_{n-1}$, where B_n is the determinant of the $n \times n$ matrix obtained by replacing $x_{21} = 1$ by 0 in A_n, all other entries unchanged. Expanding B_{n-1} along its first column, we see that $B_{n-1} = D_{n-2}$. Therefore, we obtain the recurrence relation:

(1) $\quad D_n = D_{n-1} + D_{n-2}, \ n = 3, 4, \ldots .$

Together with the initial values of D_n, we see that

(2) $\quad D_n = F_{n+1} \ (n = 1, 2, \ldots).$

Also solved by R. André-Jeannin, C. Ashbacher, Piero Filipponi, Russell Jay Hendel, Hans Kappus, L. Kuipers, Y. H. Harris Kwong, Carl Libis, Alex Necochea, Bob Prielipp, H.-J. Seiffert, Sahib Singh, and the proposer.

F_{mn} and L_{mn} as Polynomials in F_m and L_m

B-641 *Proposed by Dario Castellanos, U. de Carabobo, Valencia, Venezuela*

Prove that

$$F_{mn} = \frac{1}{\sqrt{5}}\left[\left(\frac{L_m + \sqrt{5}F_m}{2}\right)^n - \left(\frac{L_m - \sqrt{5}F_m}{2}\right)^n\right],$$

$$L_{mn} = \left(\frac{L_m + \sqrt{5}F_m}{2}\right)^n + \left(\frac{L_m - \sqrt{5}F_m}{2}\right)^n.$$

Solution by Y. H. Harris Kwong, SUNY College at Fredonia, NY

Let $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$. It is known that

$$L_m = \alpha^m + \beta^m \quad \text{and} \quad \sqrt{5}F_m = \alpha^m - \beta^m.$$

Solving for α^m and β^m, we have

$$\alpha^m = \frac{L_m + \sqrt{5}F_m}{2} \quad \text{and} \quad \beta^m = \frac{L_m - \sqrt{5}F_m}{2}.$$

Therefore,

$$F_{mn} = \frac{1}{\sqrt{5}}[\alpha^{mn} - \beta^{mn}] = \frac{1}{\sqrt{5}}\left[\left(\frac{L_m + \sqrt{5}F_m}{2}\right)^n - \left(\frac{L_m - \sqrt{5}F_m}{2}\right)^n\right],$$

$$L_{mn} = \alpha^{mn} + \beta^{mn} = \left(\frac{L_m + \sqrt{5}F_m}{2}\right)^n + \left(\frac{L_m - \sqrt{5}F_m}{2}\right)^n.$$

Editor's note: The proposer asked for a proof that

$$F_{nm} = \frac{1}{\sqrt{5}}\left[\left(\frac{L_m + \sqrt{5}F_m}{2}\right)^n - \left(\frac{L_n - \sqrt{5}F_n}{2}\right)^m\right]$$

and

$$L_{nm} = \left(\frac{L_m + \sqrt{5}F_m}{2}\right)^n + \left(\frac{L_n - \sqrt{5}F_n}{2}\right)^m$$

and the Elementary Problems editor inadvertently interchanged some (but not all) m's and n's.

Also solved by R. André-Jeannin, Paul S. Bruckman, James E. Desmond, Russell Euler, Piero Filipponi, Herta T. Freitag, Guo-Gang Gao, Russell Jay Hendel, Hans Kappus, L. Kuipers, Alex Necochea, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, and the proposer.

$L_{k(2n+1)}$ as a Polynomial in L_{2n+1}

B-642 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

It is known that

$$L_{2(2n+1)} = L_{2n+1}^2 + 2,$$

and it can readily be proven that

$$L_{3(2n+1)} = L_{2n+1}^3 + 3L_{2n+1}.$$

Generalize these identities by expressing $L_{k(2n+1)}$, for integers $k \geq 2$, as a polynomial in L_{2n+1}.

Solution by H.-J. Seiffert, Berlin, Germany

Define the Pell-Lucas polynomials $Q_k(x)$ as in [1], p. 7, (1.2), by

$$Q_0(x) = 2, \; Q_1(x) = 2x, \; Q_{k+2}(x) = 2xQ_{k+1}(x) + Q_k(x).$$

First, we show that

$$Q_k(L_{2n+1}/2) = L_{k(2n+1)} \tag{1}$$

is true for $k = 0, 1$. Assuming (1) holds for all $j = 0, \ldots, k$, we get

$$\begin{aligned} Q_{k+1}(L_{2n+1}/2) &= L_{2n+1}Q_k(L_{2n+1}/2) + Q_{k-1}(L_{2n+1}/2) \\ &= L_{2n+1}L_{k(2n+1)} + L_{(k-1)(2n+1)} = L_{(k+1)(2n+1)}, \end{aligned}$$

where the last equality can easily be proven by using the known Binet form of the Lucas numbers. Thus (1) is established by induction on k. In [1], p. 9, (2.16), it is shown that, for $k > 0$,

$$(2) \qquad Q_k(x) = \sum_{j=0}^{[k/2]} \frac{k}{k-j}\binom{k-j}{j}(2x)^{k-2j},$$

where [ ] denotes the greatest integer function. From (1) and (2), we obtain

$$L_{k(2n+1)} = \sum_{j=0}^{[k/2]} \frac{k}{k-j}\binom{k-j}{j}L_{2n+1}^{k-2j}.$$

1. A. F. Horadam & Bro. J. M. Mahon. "Pell and Pell-Lucas Polynomials." *Fibonacci Quarterly 23.1* (1985).

Also solved by R. André-Jeannin, Paul S. Bruckman, Herta T. Freitag, Russell Jay Hendel, L. Kuipers, Y. H. Harris Kwong, Sahib Singh, Paul Smith, and the proposer.

Binomial Coefficient Congruence

B-643 *Proposed by T. V. Padnakumar, Trivandrum, South India*

For positive integers a, n, and p, with p prime, prove that

$$\binom{n+ap}{p} - \binom{n}{p} \equiv a \pmod{p}.$$

Solution by Y. H. Harris Kwong, SUNY College at Fredonia, NY

A well known result of E. Lucas [2] states that if the p-ary expansions of n and k are $\sum_{i\ge 0} n_i p^i$ and $\sum_{i \ge 0} k_i p^i$, respectively, then

$$\binom{n}{k} \equiv \prod_{i \ge 0}\binom{n_i}{k_i} \pmod{p}.$$

(For a short and simple proof, consult [1].) Suppose the p-ary expansions of a and $m = n + ap$ are $\sum_{i\ge 0} a_i p^i$ and $\sum_{i\ge 0} m_i p^i$, respectively. We have to show that

$$\binom{m}{p} - \binom{n}{p} \equiv \binom{m_1}{1} - \binom{n_1}{1} = m_1 - n_1 \equiv a \equiv a_0 \pmod{p}.$$

But it is clear from $m = n + ap$ that $m_1 \equiv n_1 + a_0 \pmod{p}$, so the proof is completed.

1. N. J. Fine. "Binomial Coefficients Modulo a Prime." *Amer. Math. Monthly 54* (1947):589-92.
2. E. Lucas. *Théorie des nombres*. Vol. I. Paris: Librairie Scientifique et Technique Albert Blanchard, 1961. (Original printing, 1891.)

Also solved by R. André-Jeannin, Paul S. Bruckman, Piero Filipponi, Russell Jay Hendel, Joseph J. Kostal & Subramanyam Durbha, L. Kuipers, Bob Prielipp, Lawrence Somer, and the proposer.

Markov Chain

B-644 *Proposed by H. W. Corley, U. of Texas at Arlington, TX*

Consider three children playing catch as follows. They stand at the vertices of an equilateral triangle, each facing its center. When any child has the ball, it is thrown to the child on her or his left with probability 1/3 and to the child on the right with probability 2/3. Show that the probability that the initial holder has the ball after n tosses is

$$\frac{2}{3}\left(\frac{\sqrt{3}}{3}\right)^n \cos\left(\frac{5n\pi}{6}\right) + \frac{1}{3} \text{ for } n = 0, 1, 2, \ldots .$$

Solution by Hans Kappus, Rodersdorf, Switzerland

More generally, let us assign probabilities p, q $(p + q = 1)$ for throws to the left and right, respectively. Denote by $p_i(n)$ the probability that child i has the ball after n tosses $(i = 1, 2, 3)$ and suppose that child 1 is the initial holder, i.e., impose the initial conditions

(1) $\quad p_1(0) = 1, \; p_1(1) = 0.$

Applying the rule of conditional probability and noting that

$$p_1(n) + p_2(n) + p_3(n) = 1,$$

we have the recursion

$$(2) \quad \begin{cases} p_1(n+1) = q \cdot p_2(n) + p \cdot p_3(n) = -p \cdot p_1(n) + (q - p) \cdot p_2(n) + p \\ p_2(n+1) = p \cdot p_1(n) + q \cdot p_3(n) = (p - q) \cdot p_1(n) - q \cdot p_2(n) + q \end{cases}$$

Eliminating $p_2(n)$ we arrive at the inhomogeneous second-order difference equation

(3) $\quad p_1(n+2) + p_1(n+1) + (1 - 3pq) \cdot p_1(n) = 1 - pq,$

which may be solved by standard methods. The solution turns out to be

(4) $\quad p_1(n) = \frac{2}{3} \cdot (1 - 3pq)^{n/2} \cos n\phi + \frac{1}{3},$

where ϕ is given by

(5) $\quad \cos \phi = -\frac{1}{2} \cdot (1 - 3pq)^{-1/2}, \; \sin \phi = \frac{1}{2}\left(\frac{3 - 12pq}{1 - 3pq}\right)^{1/2}.$

For the special case $p = 1/3$, $q = 2/3$; this is the result of the proposer.

Remark: The process described in the problem is a Markov chain with transition matrix

$$P = \begin{bmatrix} 0 & p & q \\ q & 0 & p \\ p & q & 0 \end{bmatrix}.$$

Also solved by Paul S. Bruckman, Piero Filipponi, and the proposer.

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