ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-169 Proposed by Francis DeKoven, Highland Park, Illinois.

Show $n^2 + 1$ is a prime if and only if $n \neq ab + cd$ with $ad - bc = \pm 1$ for integers a, b, c, d.

H-170 Proposed by H. W. Gould, West Virginia University, Morgantown, West Virginia.

Define the power sequence P to be the sequence of natural numbers which are perfect powers m^r, $r > 1$, arranged in increasing order of magnitude. Define the first term in the sequence as $P_1 = 1$. Then $P = 1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, \cdots$. Find a formula for the n^{th} term, P_n, of the power sequence. Determine the asymptotic behavior of P_n. Define $\psi(n)$ to be the number of terms in the power sequence $\leqslant p_n$ and relatively prime to p_n. Then the consecutive values of $\psi(n)$ are $1, 1, 3, 2, 5, 5, 4, 2, 9, 5, 8, \cdots$. Find a formula for $\psi(n)$ and determine the behavior of this function ψ. Find suitable generating series for p_n and $\psi(n)$. Finally, find a formula for the n^{th} non-power; i.e., for the n^{th} term in the sequence complementary to P. Note: It may, or may not, be a good idea to include $P_1 = 1$ in the sequence defined above.

H-171 Proposed by Douglas Lind, Stanford University, Stanford, California.

Does there exist a continuous real-valued function f defined on a compact interval I of the real line such that

$$\int_I f(x)^n dx = F_n .$$

What if we require f only be measurable?

SOLUTIONS

SUB MATRICES

H-139 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Put

$$A_n = \begin{bmatrix} F_n & F_{n+1} & \cdots & F_{n+k-1} \\ F_{n+k-1} & F_n & \cdots & F_{n+k-1} \\ \cdots & \cdots & \cdots & \cdots \\ F_{n+1} & F_{n+2} & \cdots & F_n \end{bmatrix}$$

$$M = \begin{bmatrix} A_n & A_{n+k} & \cdots & A_{n+(m-1)k} \\ A_{n+(m-1)k} & A_n & \cdots & A_{n+(m-2)k} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n+k} & A_{n+2k} & \cdots & A_n \end{bmatrix}$$

Evaluate det M.

For $m = k = 2$, the problem reduces to H-117 (Fibonacci Quarterly, Vol. 5, No. 2 (1967), p. 162).

Solution by the Proposer.

Put $\epsilon = e^{2\pi i/k}$, $\omega = e^{2\pi i/m}$ and define

$$P = (\epsilon^{ij}) \qquad (i,j = 0, 1, \cdots, k-1) ,$$

$$U = (\omega^{rs}P) \qquad (r,s = 0, 1, \cdots, m-1) .$$

Also put

$$M = (B_{r-s}) \qquad (r,s = 0,1, \cdots, m-1) ,$$

where $B_0, B_1, \cdots, B_{m-1}$ are arbitrary square matrices of order k and $B_{r+m} = B_r$. Then

$$MU = \left(\sum_t B_{r-t}\omega^{ts}P\right) = \left(\sum_t B_t\omega^{-ts}\omega^{rs}P\right),$$

$$UMU = \left(P\sum_{u,t}\omega^{ru}\omega^{us}B_t\omega^{-ts}\,P\right) \;;$$

Since

$$\sum_{u=0}^{m-1}\omega^{u(r+s)} = \begin{cases} m & (m \mid r+s) \\ 0 & (m \nmid r+s) \end{cases},$$

it follows that

$$|UMU| = \prod_{s=0}^{m-1}\left|P\left(\sum_t B_t\omega^{-ts}\right)P\right|$$

$$= |P|^{2m}\prod_{s=0}^{m-1}\left|\sum_{t=0}^{m-1} B_t\omega^{-ts}\right| .$$

On the other hand,

$$U^2 = \left(\sum_t \omega^{(r+s)t}P^2\right) ,$$

so that

$$|U^2| = m^m|P|^{2m} .$$

Therefore, since $|P| \neq 0$,

(1) $$|M| = \prod_{s=0}^{m-1} \left| \sum_{t=0}^{m-1} B_t \omega^{ts} \right| .$$

Now take

$$B_t = A_{n+tk} \qquad (t = 0, 1, \cdots, m - 1) .$$

Then

(2) $$\sum_{t=0}^{m-1} B_t \omega^{ts} = \sum_{t=0}^{m-1} A_{n+tk} \omega^{ts} .$$

We shall limit ourselves to the case $k = 2$, so that

$$A_n = \begin{bmatrix} F_n & F_{n+1} \\ F_{n+1} & F_n \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} .$$

Then

$$A_n P = \begin{bmatrix} F_{n+2} & -F_{n-1} \\ F_{n+2} & F_{n-1} \end{bmatrix} ,$$

$$\left(\sum_{t=0}^{m-1} A_{n+tk} \omega^{-ts} \right) P = \begin{bmatrix} \Sigma F_{n+2t+2} \omega^{ts} & -\Sigma F_{n+2t-1} \omega^{ts} \\ \Sigma F_{n+2t+2} \omega^{ts} & \Sigma F_{n+2t-1} \omega^{ts} \end{bmatrix}$$

so that

(3) $$\left| \sum_{t=0}^{m-1} A_{n+2t} \omega^{-ts} \right| = -\left(\sum_{t=0}^{m-1} F_{n+2t+2} \omega^{ts} \right) \left(\sum_{t=0}^{m-1} F_{n+2t-1} \omega^{ts} \right) .$$

Now

$$\sum_{t=0}^{m-1} F_{n+2t}\omega^{ts} = \frac{1}{\alpha - \beta}\left\{\alpha^n \frac{1 - \alpha^{2m}}{1 - \alpha^2\omega^s} - \beta^n \frac{1 - \beta^{2m}}{1 - \beta^2 \ s}\right\}$$

$$= \frac{F_n - F_{n+2m} - (F_{n-2} - F_{n+2m-2})\omega^s}{(1 - \alpha^2\omega^s)(1 - \beta^2\omega^s)}$$

and

$$\prod_{s=0}^{m-1}\sum_{t=0}^{m-1} F_{n+2t}\omega^{ts} = \frac{(F_n - F_{n+2m})^m - (F_{n-2} - F_{n+2m-2})^m}{2 - L_{2m}} .$$

It therefore follows from (1), (2), and (3), that

$$|M| = \frac{(-1)^m}{(L_{2m} - 2)^2}\left\{(F_{n+2} - F_{n+2m+2})^m - (F_n - F_{n+2m})^m\right\}$$

$$\cdot\left\{(F_{n-1} - F_{n+2m-2})^m - (F_{n-3} - F_{n+2m-3})^m\right\}.$$

It can be verified that when $m = 2$, the right member reduces to $F_{2n+6}F_{2n}$ in agreement with H-117.

The result for arbitrary k is presumably very complicated.

SUM DIFFERENCE

H-141 Proposed by H. T. Leonard, Jr., and V. E. Hoggatt, Jr., San Jose State College, San Jose, California. (Corrected Version)

Show that

(a) $$\frac{F_{2n} + 2^nF_n}{2} = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} L_{2(n-(2k+1))}F_{2k+1}$$

(b) $$\frac{L_{2n} - L_n}{2} = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} L_{2k+1}$$

(c) $$\frac{L_{2n} + L_n}{2} = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k} L_{2k}$$

Solution by David Zeitlin, Minneapolis, Minnesota.

Let $\alpha \neq \beta$ be the roots of $z^2 - z - 1 = 0$ $(\alpha > \beta)$.

(b) and (c). We have

(1) $$(1 + x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i = \sum_{k=0}^{[n/2]} \binom{n}{2k} x^{2k} + \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} x^{2k+1} .$$

Since $L_n = \alpha^n + \beta^n$, $1 + \alpha = \alpha^2$ (also for β), we add (1), for $x = \alpha$, to (1) for $x = \beta$ to obtain

(2) $$L_{2n} = \sum_{k=0}^{[n/2]} \binom{n}{2k} L_{2k} + \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} L_{2k+1} .$$

Since $\alpha + \beta = 1$, we obtain, by addition of (1), for $x = -\alpha$, to (1), for $x = -\beta$,

(3) $$L_n = \sum_{k=0}^{[n/2]} \binom{n}{2k} L_{2k} - \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} L_{2k+1} .$$

Addition of (2) and (4) gives (c); subtraction of (3) from (2) gives (b).

(a) We have

$$(4)\qquad (y^2 + x)^n = \sum_{i=0}^{n} \binom{n}{i} y^{2(n-i)} x^i$$

$$= \sum_{k=0}^{[n/2]} \binom{n}{2k} y^{2(n-2k)} x^{2k} + \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} y^{2(n-2k-1)} x^{2k+1} .$$

Since $F_n = (\alpha^n - \beta^n)/(\alpha - \beta)$, we subtract (4), for $x = \beta$, from (4), for $x = \alpha$, and divide the result by $(\alpha - \beta)$ to obtain

$$(5)\qquad \frac{(y^2 - \alpha)^n - (y^2 + \beta)^n}{\alpha - \beta} = \sum_{k=0}^{[n/2]} \binom{n}{2k} y^{2(n-2k)} F_{2k} + \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} y^{2(n-2k-1)} F_{2k+1} .$$

Addition of (5), for $y = \alpha$, to (5), for $y = \beta$, simplifies to

$$(6)\qquad F_{3n} = \sum_{k=0}^{[n/2]} \binom{n}{2k} L_{2(n-2k)} F_{2k} + \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} L_{2(n-2k-1)} F_{2k+1} .$$

Subtraction of (4), for $x = -\beta$, from (4), for $x = -\alpha$, gives

$$(7)\qquad \frac{(y^2 - \alpha)^n - (y^2 - \beta)^n}{\alpha - \beta} = \sum_{k=0}^{[n/2]} \binom{n}{2k} y^{2(n-2k)} F_{2k} - \sum_{k=0}^{[(n-1)/2]}$$

$$- \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} y^{2(n-2k-1)} F_{2k+1} .$$

Addition of (7), for $y = \alpha$, to (7), for $y = \beta$, gives

$$(8)\qquad -2^n F_n = \sum_{k=0}^{[n/2]} \binom{n}{2k} L_{2(n-2k)} F_{2k} - \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} L_{2(n-2k-1)} F_{2k+1} .$$

Subtraction of (8) from (6) gives the desired result.

Also solved by D. Jaiswal (India) and A. C. Shannon (Australia).

ANOTHER SERIES

H-142 Proposed by H. W. Gould, West Virginia University, Morgantown, West Virginia.

With the usual notation for Fibonacci numbers, $F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$, show that

$$\left(\frac{1-\sqrt{5}}{2}\right)^n \sum_{k=0}^{n} \binom{\frac{1+\sqrt{5}}{1-\sqrt{5}}k}{k}\binom{n - \frac{1+\sqrt{5}}{1-\sqrt{5}}k}{n-k} = F_{n+1} ,$$

where

$$\binom{x}{j} = x(x-1)(x-2)\cdots(x-j+1)/j!$$

is the usual binomial coefficient symbol.

Solution by L. Carlitz, Duke University, Durham, North Carolina.

Put

$$\beta = \frac{1+\sqrt{5}}{1-\sqrt{5}} = -\frac{3+\sqrt{5}}{2} = -\left(\frac{1+\sqrt{5}}{2}\right)^2$$

$$u_n = \sum_{k=0}^{n} \binom{\beta k}{k}\binom{n-\beta k}{n-k} .$$

Then

$$\sum_{n=0}^{\infty} u_n t^n = \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} \binom{\beta k}{k}\binom{n-\beta k}{n-k}$$

$$= \sum_{k=0}^{\infty} \binom{\beta k}{k} t^k \sum_{n=0}^{\infty} \binom{n+(1-\beta)k}{n} t^n = \sum_{k=0}^{\infty} \binom{\beta k}{k} t^k (1-t)^{-(1-\beta)k-1} .$$

Now in the formula (see Pólya-Szegö, Aufgaben und Lehrsätze aus der Analysis, Vol. 1, p. 126, No. 216)

$$\sum_{n=0}^{\infty} \binom{\beta n}{n} w^n = \frac{x}{(1 - \beta)x + \beta} ,$$

where $1 - x + wx^{\beta} = 0$, take $x = (1 - t)^{-1}$. Then

$$\frac{x - 1}{x^{\beta}} = t(1 - t)^{\beta - 1} = w.$$

It follows that

$$\sum_{n=0}^{\infty} u_n t^n = (1 - t)^{-1} \sum_{k=0}^{\infty} \binom{\beta k}{k} w^k$$

$$= \frac{1}{1 - t} \frac{x}{(1 - \beta)x + \beta} = \frac{1}{(1 - t)(1 - \beta t)} = \frac{1}{1 - (\beta + 1)t + \beta t^2} ,$$

so that

$$\sum_{n=0}^{\infty} \left(\frac{1 - \sqrt{5}}{2}\right)^n u_n t^n = \frac{1}{1 - t - t^2}.$$

Therefore

$$\left(\frac{1 - \ 5}{2}\right)^n u_n = F_{n+1} \cdot$$

Also solved by D. Jaiswal (India).

NEGATIVE ATTITUDE

H-143 Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tennessee. (Corrected version)

Let $\{H_n\}$ be a generalized Fibonacci sequence and, by the recurrence relation, extend the definition to include negative subscripts. Show that

(i) $$L_{2j+1}\sum_{k=0}^{n} H^2_{(2j+1)k} = H_{(2j+1)(n+1)}H_{(2j+1)n} - H_0H_{-(2j+1)},$$

(ii) $$L_{2j+1}\sum_{k=0}^{n} H_{(2j+1)k} = H_{(2j+1)(n+1)} - H_{-(2j+1)} - H_0 + H_{(2j+1)n},$$

(iii) $$L_{2j}\sum_{k=0}^{n} (-1)^k H^2_{2jk} = (-1)^n H_{2j(n+1)}H_{2jn} + H_0H_{-2j},$$

and derive an expression for

(iv) $$\sum_{k=0}^{n} (-1)^k H_{2jk}.$$

Solution by David Zeitlin, Minneapolis, Minnesota.

Our proof uses the fact that if $P(0) = R(0)$ and $\Delta P(n) = \Delta R(n)$, then $P(n) \equiv R(n)$ (where $\Delta P(n) = P(n+1) - P(n)$). We note that $H_{-n} = (-1)^n (H_0L_n - H_n)$, so that

(A) $$H_0L_{2j+1} = H_{2j+1} - H_{-(2j+1)}$$

and

(B) $$H_0L_{2j} = H_{2j} + H_{-2j}.$$

Proof of (i). For $n = 0$, both sides of (i) are equal by (A). Using the Δ operator, it remains to show that

(1) $$L_a H_{an+a} = H_{an+2a} - H_{an} \qquad (a = 2j+1)\ .$$

We recall now that

(2) $$H_{m+p} = F_{p-1}H_m + F_p H_{m+1}\ ,$$

(3) $$F_{m+1}F_{m-1} - F_m^2 = (-1)^m\ .$$

Thus,

$$H_{an+a} = F_{a-1}H_{an} + F_a H_{an+1}\ ,$$

and

(4) $$L_a H_{an+a} = L_a F_{a-1} H_{an} + F_{2a} H_{an+1}\ ,$$

(5) $$H_{an+2a} - H_{an} = (-1 + F_{2a-1})H_{an} + F_{2a}H_{an+1}\ .$$

By (3),

$$F_{a+1}F_{a-1} - F_a^2 = -1\ ,$$

and so

$$L_a F_{a-1} = F_{a+1}F_{a-1} + F_{a-1}^2 = -1 + F_a^2 + F_{a-1}^2 = -1 + F_{2a-1}\ .$$

Thus, (4) and (5) gives (1) and (i).

Proof of (iii). Both sides of (ii) are equal for $n = 0$ by (B). Using the Δ operator, it remains to show that

(6) $$L_c H_{cn+c} = H_{cn+2c} + H_{cn} \qquad (c = 2j)\ .$$

Proceeding as in the proof of (i), we obtain (6) by noting that $L_cF_{c-1} = 1 + F_{2c-1}$.

Proof of (ii). Identical to the proof of (i).

Derivation of (iv). Using (5) in my paper, "On Summation Formulas for Fibonacci and Lucas Numbers," this Quarterly, Vol. 2, No. 2, 1964, pp. 105-107, we obtain (for $x = p = -1$, $u_n = H_n$, $a = 2j$, and $d = 0$)

$$\text{(iv)} \quad (2 + L_{2j}) \sum_{k=0}^{n} (-1)^k H_{2jk} = (-1)^n (H_{(2j)(n+1)} + H_{2jn}) + H_0 + H_{-2j} .$$

Also solved by A. Shannon (Australia), C. Wall, and M. Yoder.

[Continued from page 267.]

Here $H(4) = 3H(2)$. But $H(2^{e+2}) = 2^e H(4)$.

This leaves us with the following problems: When do Theorems 3.6 and 3.7 hold? When does (2) hold? For the special case $u_{n+1} = u_n + u_{n-1}$, the theorems hold. A rather incomplete proof is given in [2, Theorem 5]. A complete proof is contained in [3] and will be published soon. It would be nice if these results could be established by the simple approach of [1]. Until then, one must be cautious of any results in [1].

REFERENCES

1. Birger Jansson, "Random Number Generators," Victor Pettersons Bokindustri Aktiebolag, Stockholm, 1966.
2. D. D. Wall, "Fibonacci Series Modulo m," American Math. Monthly, 67 (1960), pp. 525-532.
3. A. Andreassian, "Fibonacci Sequences Modulo m," Masters Thesis, American University of Bierut, 1968.