

ABOUT THE LINEAR SEQUENCE OF INTEGERS SUCH THAT EACH TERM IS THE SUM OF THE TWO PRECEDING

ANDRÉ GOUGENHEIM
30 Boulevard Flandrin, Paris, France

1. The sequences of integers such that each term is equal to the sum of both preceding are infinite in number. Two of these have been especially investigated: the Fibonacci sequence, conceived at the beginning of the 13th Century by the mathematician Leonardo of Pisa, better known as Fibonacci, the Lucas sequence pointed out at the end of the last century by the French mathematician Lucas and named for him. Both sequences gave rise to many works which showed manifold properties of these sequences and conduced to strides in the numbers theory.

The present research work doesn't mean to go back on these questions, but it tends to make known how the use of the hyperbolic functions make much easier general feature works on the linear sequences defined at the beginning of the present paper, and from which Fibonacci and Lucas sequences are only special cases.¹ The author has recently had recourse to these functions in a very different field, that of mathematic geography, and he has been the first to show that their utilization simplified notably the determination of the conformal representations of the sphere or ellipsoide on the plane, that it lightened very much the algebraic expression of these representations and that it helped to state precisely the relationships existing between the different systems.

2. The sequences concerned are defined by the general relation:

$$(1) \quad z_n = z_{n-1} + z_{n-2} \quad ,$$

in which z_n indicates the term of rank n .

¹Lucas developed a very different generalization of both sequences. It will be reminded in paragraph 6.

Each sequence can therefore be characterized by two arbitrary integers which we call z_0 and z_1 and which don't seem, a priori, to be part of the sequence because they are not squaring with the definition (1); but, actually, they, too, enter into the sequence since it is possible to extend it without end, in the opposite direction, starting from the arbitrary terms z_0 and z_1 .

3. The shape of the relation (1) between the successive terms of the sequences suggests immediately the use of circular or hyperbolic lines (functions) for expressing each term according to its place in the sequence. As it is a question of indefinitely increasing sequences, it is obviously suitable to have recourse to hyperbolic lines.

Let us write the relation (1) in the form:

$$(2) \quad z_{n+1} - z_{n-1} = z_n,$$

and designate by m , λ , and ϕ , three constants to fix ulteriorly in terms of sequence's data. Let us set besides: either

$$z_{n+1} = m \operatorname{sh} \lambda (n + \phi + 1) \quad \text{and} \quad z_{n-1} = m \operatorname{sh} \lambda (n + \phi - 1)$$

or

$$z_{n+1} = m \operatorname{ch} \lambda (n + \phi + 1) \quad \text{and} \quad z_{n-1} = m \operatorname{ch} \lambda (n + \phi - 1).$$

Then the relation (2) conduces to:

$$z_n = 2m \operatorname{sh} \lambda \operatorname{ch} \lambda (n + \phi)$$

for the first case, or

$$z_n = 2m \operatorname{sh} \lambda \operatorname{sh} \lambda (n + \phi)$$

for the second case.

Let us define now the parameter λ by $\operatorname{sh} \lambda = 1/2$, from which it comes $\operatorname{ch} \lambda = \sqrt{5}/2$ and

$$\left(e = \frac{1 + \sqrt{5}}{2} \right) \text{ (golden number)}$$

Both expressions of z_n become simplified and it is obvious moreover that the terms of the sequence can be represented alternatively by hyperbolic sines and cosines

$$(2 \text{ bis}) z_n = m \operatorname{ch} \lambda (n + \phi)$$

or

$$z_n = m \operatorname{sh} \lambda (n + \phi)$$

or, generally, speaking

$$z_n = m \frac{e^{\lambda(n+\phi)} \pm e^{-\lambda(n+\phi)}}{2} .$$

The parameters m and ϕ are easily obtained with the help of initial data z_0 and z_1 , but it is obviously necessary to consider two cases according as one adopts for z_0 , a hyperbolic sine or cosine, and the inverse for z_1 . In the first case, the terms with an even index agree with hyperbolic sines, those with an odd index are represented by cosines. In the second case, the inverse occurs. To make a distinction between both cases, we shall write:

$$A = z_1 + z_0 e^{-\lambda} \qquad B = z_1 - z_0 e^{\lambda}$$

from what, taking the value of λ into consideration,

$$A - B = 2z_0 \operatorname{ch} \lambda \qquad A + B = 2z_1 - z_0 \qquad AB = z_1^2 - z_0 z_1 - z_0^2 .$$

Suppose, now, that one intends to adopt hyperbolic sines for the terms with an even index. It comes:

$$m \operatorname{sh} \lambda \phi = z_0 = \frac{A - B}{2 \operatorname{ch} \lambda} \qquad m \operatorname{ch} \lambda (\phi + 1) = z_1,$$

from what

$$m \operatorname{ch} \lambda \phi = \frac{z_1 - z_0 \operatorname{sh} \lambda}{\operatorname{ch} \lambda} = \frac{A - B}{2 \operatorname{ch} \lambda} ,$$

and therefore,

$$m e^{\lambda \phi} = A / \operatorname{ch} \lambda \quad m e^{-\lambda \phi} = B / \operatorname{ch} \lambda \quad m = \sqrt{AB} / \operatorname{ch} \lambda \quad e^{\lambda \phi} = \sqrt{A/B}$$

B must so be positive, and we have consequently:

$$z_1 > z_0 e ,$$

either

$$z_1 > z_0 \frac{1 + \sqrt{5}}{2}$$

or

$$2z_1 - z_0 > z_0 \sqrt{5} .$$

A parallel argument shows that if a hyperbolic cosine is adopted for the terms with an even index, $-B$ takes the place of B in the formulas of m and of $e^{\lambda \phi}$, and that consequently, B must be negative and

$$z_1 < z_0 \frac{1 + \sqrt{5}}{2} .$$

For example, the sequences defined by $z_0 = 3$ and $z_1 = 1$, or by $z_0 = 2$ and $z_1 = 3$ must be represented by

$$z_n = m \operatorname{ch} \lambda (n + \phi)$$

when n is even, whereas a hyperbolic sine is necessary for the sequence defined by $z_0 = 1$ and $z_1 = 2$.

Using the formulas of m and ϕ , we get the general expression

$$(1) \quad z_n = \frac{1}{2 \operatorname{ch} \lambda} [Ae^{\lambda n} - B(-e^{-\lambda})^n]$$

Before going further in the study of the sequences, we deal first with the special case of ϕ integer; then, this parameter can be taken cipher, which is equivalent to shifting the number of the terms, the n^{th} term receiving the index $n - 1$. The condition $\phi = 0$ produces $A = B$ if B is positive, $A = -B$ in the opposite case. Both cases correspond respectively to the Lucas and Fibonacci sequences.

The knowledge of both these sequences makes it much easier to set up formulas of the general sequence z . We add, besides, a special sequence G which also appears in the relations.

4. The Fibonacci Sequence. For this sequence, $A = B$, and consequently, $z_0 = 0$ and $z_1 = A$. Hence, for the general term,

$$z_n = \frac{z_1}{\operatorname{ch} \lambda} [e^{\lambda n} - (-e^{-\lambda})^n]$$

As no motive exists for keeping the same factor z_1 in all terms of the sequence, we can take $z_1 = 1$. Therefore, we have, with the symbol F instead of z :

$$(3) \quad F_n = \frac{\operatorname{sh} \lambda n}{\operatorname{ch} \lambda}$$

if n is even,

$$F_n = \frac{\operatorname{ch} \lambda n}{\operatorname{ch} \lambda}$$

if n is odd.

¹Substituting to the quantities A and B in this formula, their expressions in the terms of z_0 and z_1 , one may obtain a relation which is no other than the relation (5), given further and then more directly obtained.

It would be possible to more quickly obtain these relations by departing from the usual definition $z_0 = 0$, $z_1 = 1$, and writing

$$m \operatorname{sh} \lambda \phi = 0 \quad m \operatorname{ch} \lambda (\phi + 1) = 1 ,$$

relations giving $\phi = 0$ and $m = 1/\operatorname{ch} \lambda$.

As

$$e^\lambda = \frac{1 + \sqrt{5}}{2} \quad -e^{-\lambda} = \frac{1 - \sqrt{5}}{2} ,$$

the expressions of the general term become:

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right]$$

or, more symmetrically,

$$F_n = \frac{\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n}{\frac{1 + \sqrt{5}}{2} - \frac{1 - \sqrt{5}}{2}}$$

and numerically,

$$F_n = \frac{(1,618 \dots)^n - (-0,618 \dots)^n}{2,236 \dots} .$$

As $\operatorname{sh} kx$, with k integer, is always divisible by $\operatorname{sh} x$, and as $\operatorname{ch} kx$ is divisible by $\operatorname{ch} x$ when k is odd, the term F_{kn} is always divisible by F_n , which is also shown by the general formula. Specifically, the even terms have an index divisible by 3; the terms divisible by 3 have an index divisible by 4; the terms divisible by 5 have an index divisible by 5; and so on.

Likewise, when n becomes very great, which makes $\operatorname{th} \lambda n$ very near from the unity, the ratio of consecutive terms draws near to $\operatorname{ch} \lambda + \operatorname{sh} \lambda$, i.e.,

$$e^\lambda = \frac{1 + \sqrt{5}}{2} = 1,618 \dots$$

So the successive terms of the Fibonacci sequence are:

$n =$	0	1	2	3	4	5	6	7	8	9	10	11	12	...
$F_n =$	0	1	1	2	3	5	8	13	21	34	55	89	144	...
$\frac{\text{sh}\lambda n}{\text{ch}\lambda} =$	0		1		3		8		21		55		144	...
$\frac{\text{ch}\lambda n}{\text{ch}\lambda} =$		1		2		5		13		34		89		...

5. The Lucas Sequence. We have seen that, for this sequence, $A = -B$, from which $z_0 = A/\text{ch}\lambda$ and $z_1 = z_0/2$, and, for the general term, using the symbol L for the terms of the sequence, and taking $z_1 = 1$, as in the Fibonacci sequence, and for the same reason,

$$L_n = e^{\lambda n} + (-e^{-\lambda})^n = \left(\frac{1 + \sqrt{5}}{2}\right)^n + \left(\frac{1 - \sqrt{5}}{2}\right)^n$$

and we have

$$(4) \quad L_n = 2 \text{ch}\lambda n$$

for n even, and

$$L_n = 2 \text{sh}\lambda n$$

for n odd. It would also be possible to get these expressions directly from the relations

$$m \text{ch}\lambda\phi = z_0 = 2$$

and

$$m \operatorname{sh} \lambda (\phi + 1) = z_1 = 1$$

which give $\phi = 0$ and $m = 2$.

If one considers the product kn , the term L_{kn} is divisible by L_n when k is odd. Particularly, the terms having an index odd multiple of 3 are divisible by 4, whereas, as $\operatorname{ch} 6\lambda$ is equal to 9, odd integers, the terms having for index a multiple of 6 and consequently for expression $2\operatorname{ch} 6\lambda n$, are divisible by 2, and by no other power of this number, whatever the evenness of n may be.

The Lucas sequence, therefore, is as follows:

$n =$	0	1	2	3	4	5	6	7	8	9	10	11	12	...	
$L_n =$	2	1	3	4	7	11	18	29	47	76	123	199	312	...	322
$2\operatorname{ch} n =$	2		3		7		18		47		123		312	...	322
$2\operatorname{sh} n =$		1		4		11		29		76		199		...	

6. The previous expressions of F_n and L_n in terms of

$$\frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \frac{1 - \sqrt{5}}{2}$$

are connected with more general results set up by Edouard Lucas, who considers the functions U_n and V_n defined by

$$U_n = \frac{a^n - b^n}{a - b}$$

and

$$V_n = a^n + b^n .$$

Lucas shows that $U_{2n} = U_n V_n$ (a similar formula is given further in paragraph 9) and that, on the other hand, he can write $U_n = 2 \sin n$ and $V_n = 2 \cos n$; for n real, the circular trigonometric lines fit, whereas for n imaginary, one must use hyperbolic functions.

It is also interesting to consider the quadratic equation having the roots a and b . In the special case where U_n and V_n agree, respectively, with F_n and L_n , this equation is $x^2 - x - 1 = 0$.

7. Connected Sequences. One can easily set up the relation:

$$(5) \quad z_n = z_0 F_{n-1} + z_1 F_n$$

permitting to deal with all sequences defined by relation (1) as soon as the Fibonacci sequence has been investigated.

We shall consider now that this relation (5) defined a function $G_n(z, F)$ of both sequences, and we shall spread it to any sequences y and z , writing:

$$G_n(y, z) = z_0 y_{n-1} + z_1 y_n \cdot$$

Through the relation (5), one shows without difficulty that $G_n(y, z) = G_n(z, y)$, and consequently,

$$G_n(y, z) = y_0 z_{n-1} + y_1 z_n \cdot$$

More generally, and if q is any integer, we find more:

$$G_n(y, z) = z_q y_{n-q-1} + z_{q+1} y_{n-q} \cdot$$

One can also show that $G_n = G_{n-1} + G_{n-2}$, and therefore that the sequence G is a linear sequence of the family (1) concerned and has the following terms:

$$G_0 = y_1 z_0 + z_1 y_0 - y_0 z_0$$

and

$$G_1 = y_0 z_0 + y_1 z_1 .$$

With the symbols of paragraph 3, we can show that, on the other hand,

$$\begin{aligned} A(y, z) &= A(y)A(z) & m(y, z) &= m(y)m(z)\text{ch}\lambda \\ B(y, z) &= B(y)B(z) & \phi(y, s) &= \phi(y) + \phi(z) \end{aligned}$$

hence,

$$(5 \text{ bis}) \quad G_n(y, z) = m(y)m(z) \text{ch}\lambda \text{sh}\lambda[n + \phi(y) + \phi(z)]$$

or

$$G_n(y, z) = m(y)m(z) \text{ch}\lambda \text{ch}\lambda[n + \phi(y) + \phi(z)] ,$$

accordingly as to whether G_1 is superior to

$$G_0 \frac{1 + \sqrt{5}}{2}$$

We have first, $G_n(z, F) = z_n$. The sequence $G_n(z, L)$ affords a special interest because

$$G_n(z, L) = L_0 z_{n-1} + L_1 z_n = 2z_{n-1} + z_n = z_{n-1} + z_{n+1}$$

which gives, in particular:

$$(6) \quad L_n = G_n(L, F) = F_{n-1} + F_{n+1} .$$

When the sequences y and z are the same, one may obtain, using $G_n(z)$ instead of $G_n(z, z)$,

$$G_n(z) = z_0 z_{n-1} + z_1 z_n = z_q z_{n-q-1} + z_{q+1} z_{n-q} .$$

In this sequence,

$$G_0(z) = z_0(2z_1 - z_0)$$

$$G_1(z) = z_0^2 + z_1^2$$

$$G_2(z) = z_1(2z_0 + z_1) .$$

We find also:

$$(6 \text{ bis}) \quad G_n(z) = m^2 \text{ch} \lambda \text{ch} \lambda (n + 2\phi) \quad \text{or} \quad m^2 \text{ch} \lambda \text{sh} \lambda (n + 2\phi) ,$$

according to the value of the ratio G_1/G_0 .

Consequently, through these relations,

$$(7) \quad \begin{aligned} G_n(F) &= F_n \\ G_n(L) &= 5F_n \\ z_n G_n(z, L) &= G_{2n}(z) . \end{aligned}$$

Likewise, the sequence z_n can be connected to the same sequence z_{n+p} shifted by an integer P . As $m(z) = m(z_{+p})$ and

$$(z, z_{+p}) = (z) + (z_{+p}) ,$$

it comes

$$(7 \text{ bis}) \quad \begin{aligned} G_n(z, z_{+p}) &= m^2 \text{ch} \lambda \text{ch} \lambda (n + p + 2\phi) \quad \text{if } n \text{ and } p \text{ have different} \\ &\hspace{15em} \text{evenness} \\ G_n(z, z_{+p}) &= m^2 \text{ch} \lambda \text{sh} \lambda (n + p + 2\phi) \quad \text{if } n \text{ and } p \text{ have the same} \\ &\hspace{15em} \text{evenness} \end{aligned}$$

Obviously, the terms of the connected sequences G must be, like those of the other linear sequences, alternatively a hyperbolic sine and a hyperbolic cosine.

8. Sundry Relations. Having resort to formulas interconnecting hyperbolic lines, we can set up many relations between the terms of the linear sequences of type (1).

(a) Formulas of addition and subtraction. One finds

$$z_{n+p} + z_{n-p} = \begin{cases} z_n L_p & \text{if } p \text{ is even} \\ (z_{n-1} + z_{n+1})F_p & \text{if } p \text{ is odd} \end{cases}$$

$$z_{n+p} - z_{n-p} = \begin{cases} (z_{n-1} + z_{n+1})F_p & \text{if } p \text{ is even} \\ z_n L_p & \text{if } p \text{ is odd} \end{cases} .$$

These relations can be condensed into the following form:

$$\begin{cases} z_{n+p} + (-1)^p z_{n-p} = z_n L_p \\ z_{n+p} - (-1)^p z_{n-p} = (z_{n+1} + z_{n-1})F_p \end{cases} .$$

One can write them more symmetrically:

$$(8) \quad \begin{cases} z_{n+p} + (-1)^p z_{n-p} = L_p G_n(z, F) \\ z_{n+p} - (-1)^p z_{n-p} = F_p G_n(z, L) \end{cases} \text{ or another way } \begin{cases} = G_p(L, F)G_n(z, F) \\ = G_p(F, F)G_n(z, L) \end{cases} .$$

Each of the above mentioned sums and differences concerns both terms z_{n+p} and z_{n-p} of which the indices are separated from $2p$ which is an even integer.

When the difference, which we call a , between the indices $q+a$ and q of the considered terms is odd, i. e., when we try to compute the sum $z_{q+a} + z_a$ or the difference $z_{q+a} - z_a$, the problem is much more difficult because the terms are expressed, one by a hyperbolic cosine, the other by a sine, and there is no general formula for the addition or subtraction of both lines. Then, it is possible, to make the investigation easier, to pass through the Fibonacci sequence by introducing the following auxiliary linear sequences, of which the number is unlimited and which are only interesting when a is odd. We use the letters x and y to denominate these sequences:

$$x_q(a) = F_{q+a} + F_q$$

$$y_q(a) = F_{q+a} - F_q .$$

Particularly:

$$\begin{aligned}x_q(1) &= F_{q+2} & y_q(1) &= F_{q-1} \\x_q(3) &= 2F_{q+2} & y_q(3) &= 2F_{q+1} \\x_q(5) &= 5F_{q+1} + 4F_q & y_q(5) &= 5F_{q+1} + 2F_q .\end{aligned}$$

Generally speaking, we have

$$F_{q+a} = F_{a-1}F_q + F_aF_{q+1}$$

Hence, with the help of (5),

$$z_{q+a} - z_q = z_0x_{q-1}(a) + z_1x_q(a) = G_q[z, x_q(a)] .$$

In the same way,

$$z_{q+a} - z_q = G_q[z, y_q(a)]$$

(b) Sums or differences of Squares. Using the sums and differences just set up, one finds:

$$(9) \quad \begin{aligned}z_p^2 - z_q^2 &= G_{p+q}(z)F_{p-q} & \text{if } p \text{ and } q \text{ have the same evenness} \\z_p^2 + z_q^2 &= G_{p+q}(z)F_{p-q} & \text{if } p \text{ and } q \text{ have different evenness.}\end{aligned}$$

and, by condensing these relations:

$$z_p^2 - (-1)^{p+q}z_q^2 = G_{p+q}(z)F_{p-q} .$$

The difference of the squares, when $p - q$ is odd, can be written:

$$z_p^2 - z_q^2 = G_q[z, x_q(p - q)]G_q[z, y_q(p - q)]$$

but this way does not lend itself to practical applications. Likewise, for the sum of the squares when $p - q$ is even.

(c) Sums of the terms of Linear Sequences. One easily finds by recurrence the following relation which is suitable to all linear sequences defined by formula (1):

$$Z_{p,q} = \sum_{i=p}^{i=q} z_i = z_{q+2} - z_{p+1} .$$

We have, therefore, in the case of the first $n + 1$ terms, from $p = 0$ to $q = n$:

$$Z_n = z_{n+2} - z_1$$

In addition to this general method, there are, for two special cases, other methods making possible, for instance, to get checking of the computation:

In one of the cases, the number $n + 1$ of the implicated terms is a multiple of 4 and one gets

$$Z_n = \frac{F_{n+1}}{2} \left(\frac{z_{n+5}}{2} + \frac{z_{n+1}}{2} \right) = \frac{F_{n+1}}{2} \frac{G_{n+3}}{2} (z, L)$$

The second special case, which looks more interesting, concerns a number of terms which are multiples of 2 and of no other power of 2. In this case, $n - 1$ is a multiple of 4 and we have, consequently,

$$Z_n = \frac{L_{n+1}}{2} \frac{z_{n+3}}{2} ,$$

i. e., the sum of the $n + 1$ implicated terms is equal to the product of the $(n+5)/2^{\text{th}}$ term of the sequence (index $(n+3)/2$), by the $(n+3)/2^{\text{th}}$ term of the Lucas sequence (index $(n + 1)/2$). There is, therefore, equality between the sum of the first six term ($n = 5$) and the product of the fifth term by 4:

the sum of the first 10 terms ($n = 9$) and the product of the 7th term by 11, the sum of the first 14 terms ($n = 13$) and the product of the 9th term by 29, the sum of the first 18 terms ($n = 17$) and the product of the 11th term by 76.

and so on.

(d) Sums of the two-by-two terms. Let us add first the $(n/2) + 1$ terms with an even index, from 0 to n . We find

$$S_n = (z_0 - z_1) + z_{n+1} = z_{n+1} - z_{-1} .$$

For the $(n' + 1)/2$ terms with an odd index n' , from z_1 to z_n , we get likewise

$$S_{n'} = z_{n'+1} - z_0 .$$

One can easily check the accuracy of both expressions of S_n and $S_{n'}$ by taking $n' = n - 1$. Adding both sums, one must find again the sum Z_n of the $n + 1$ terms of the sequence z , from z_0 to z_n .

We have indeed, on the one hand for the number of terms

$$\left(\frac{n}{2} + 1\right) + \frac{n' + 1}{2} = n + 1 ,$$

and, on the other hand, for the formulas of the sums

$$S_{n'} = z_n - z_0$$

and

$$S_n + S_{n'} = z_{n+1} - z_{-1} + z_n - z_0 = z_{n+2} - z_{-1} = Z_n .$$

Moreover, man can try to get the sums of the two-by-two terms between the indices p and q , both even or both odd. The number of these terms is $(p - q)/2$, and their sum is the difference between the sums S_q and S_p made from the beginning of the sequence to q and to p . As p and q have

the same evenness, one obtains

$$S_q - S_p = z_{q+1} - z_{p+1}$$

from which it comes, by using the formulas (8):

$$(10) \quad \left\{ \begin{array}{ll} S_q - S_p = \frac{F_{q-p}}{2} \frac{G_{q+p}}{2} + 1^{(z, L)} & \text{if } \frac{q-p}{2} \text{ is even} \\ S_q - S_p = \frac{L_{q-p}}{2} \frac{G_{q+p}}{2} + 1^{(z, F)} & \text{if } \frac{q-p}{2} \text{ is odd} \end{array} \right.$$

9. Application to the Fibonacci and Lucas Sequences. The relations, which we shall get by application of the formula of the previous paragraphs, could be obtained by using the formulas (3) and (4), which give the terms of both the Fibonacci and Lucas sequences in the shape of hyperbolic lines of the index n . We think, nevertheless, more into the spirit of the present paper to consider both sequences as special cases of very great simplicity.

We have previously seen that in (6), $L_n = F_{n-1} + F_{n+1}$. Substituting F to z in the formulas (7), one finds $F_n L_n = F_{2n}$.

(a) Formulas of addition and subtraction. The Formulas (8) give:

$$\begin{array}{ll} F_{n+p} + (-1)^p F_{n-p} = F_n L_p & L_{n+p} + (-1)^p L_{n-p} = L_n L_p \\ F_{n-p} - (-1)^p F_{n-p} = F_p L_n & L_{n+p} - (-1)^p L_{n-p} = 5F_n F_p \end{array} .$$

In particular,

$$\left\{ \begin{array}{l} F_{n+1} + F_{n-1} = L_n \\ F_{n+1} - F_{n-1} = F_n \end{array} \right. \quad \left\{ \begin{array}{l} L_{n+1} + L_{n-1} = 5F_n \\ L_{n+1} - L_{n-1} = L_n \end{array} \right. ,$$

and consequently,

$$F_{n+1}^2 - F_{n-1}^2 = F_n L_n = F_{2n} \quad L_{n+1}^2 - L_{n-1}^2 = 5F_n L_n = 5F_{2n} .$$

(b) Sums or differences of squares. One finds with the help of formulas (9):

$$F_p^2 - (-1)^{p+q} F_q^2 = F_{p+q} F_{p-q} \quad L_p^2 - (-1)^{p+q} L_q^2 = 5F_{p+q} F_{p-q} ,$$

from which we deduce, among others

$$\begin{aligned} F_n^2 + F_{n+1}^2 &= F_{2n+1} & L_n^2 + L_{n+1}^2 &= 5F_{2n+1} \\ F_n^2 + (-1)^n &= F_{n+1} F_{n-1} & L_n^2 + (-1)^n &= 5F_{n+1} F_{n-1} . \end{aligned}$$

(c) Sums of the terms of each sequence. Let us nominate by Φ_n and Λ_n the sums of the first $n+1$ terms, from the index 0 to the index 1. Using the formulas of the paragraph 7c and taking $z_1 = 1$, we get:

$$\Phi_n = F_{n+2} - 1 \quad \Lambda_n = L_{n+2} - 1 .$$

For $n+1$ multiple of 4, it becomes

$$\Phi_n = \frac{F_{n+1}}{2} \frac{F_{n+5}}{2} + \frac{F_{n+1}}{2} \quad \Lambda_n = \frac{5F_{n+1}}{2} \frac{F_{n+3}}{2}$$

For $n-1$ multiple of 4, one finds

$$\Phi_n = \frac{L_{n+1}}{2} \frac{F_{n+1}}{2} \quad \Lambda_n = \frac{L_{n+1}}{2} \frac{L_{n+3}}{2}$$

(d) Sums of the two-by-two terms. The sum of the first $(n/2) + 1$ terms of even index is

$$S_n(F) = F_{n+1} - 1 \quad S_n(L) = L_{n+1} + 1 .$$

That of the first $(n'+1)/2$ terms of odd index is:

$$S_{n'}(F) = F_{n'+1} \quad S_{n'}(L) = L_{n'+1} - 2 .$$

For the two-by-two sums between the indices p and q , of the same evenness, we find, using the formulas (10):

$$S_q(F) - S_p(F) = \frac{L_{q-p}}{2} \frac{L_{q+p}}{2} + 1 \quad S_q(L) - S_p(L) = \frac{L_{q-p}}{2} \frac{L_{q+p}}{2} + 1$$

when $(q - p)/2$ is odd. When this quantity is even, we have

$$S_q(F) - S_p(F) = \frac{F_{q-p}}{2} \frac{L_{q+p}}{2} + 1 \quad S_q(L) - S_p(L) = \frac{5F_{q-p}}{2} \frac{F_{q+p}}{2} + 1$$

(e) Other relations between the terms of the sequences F and L . Cancelling out the hyperbolic lines between the expressions (3) and (4), we obtain the following relations, in which we can note again the prominent part taken by the factor 5 which is equal to $4ch^2$.

$$\begin{aligned} L_n^2 &= 5F_n^2 + 4(-1)^n & L_{2n} &= 5F_n^2 + 2(-1)^n \\ L_n^2 &= F_n^2 + 4F_{n-1}F_{n+1} & 2L_{2n} &= L_n^2 + 5F_n^2 \end{aligned}$$

According to the first of these relations, we see that no one term of the Lucas sequence can be a multiple of 5, and that L_n draws nearer to $F_n \sqrt{5}$, when n grows indefinitely. One also finds

$$L_{2n} = L_n^2 - 2(-1)^n$$

10. Research of a Linear Sequence. The matter here is to research if a given number can be a term of given rank in a linear sequence. In other words, the values of z and n are given, and those of z_0 and z_1 are unknown. The relation

$$(5) \quad z_n = z_0 F_{n-1} + z_1 F_n$$

contains the solution of the problem. It is a simple equation which must be solved by integers, which is always possible. On the other hand, if z_0 and

z_1 are a solution, there is an infinity of other solutions defined by $z_0 + kF_n$ and $z_1 - kF_{n-1}$, where k is any integer, positive or negative.

As an example, let us search the sequences in which $z_7 = 81$. The equation of the problem is

$$8z_0 + 13z_1 = 81.$$

Few trials show that z_0 and z_1 can be respectively taken equal to 2 and 5. Consequently, the solutions are:

$$\begin{aligned} z_0 &= \dots -24 & -11 & 2 & 15 & 28 & \dots \\ z_1 &= \dots 21 & 13 & 5 & -3 & -11 & \dots \end{aligned}$$

The differences between the terms of two such sequences defined by the values k' and k'' of k are equal to the product by $k' - k''$ of the terms of a Fibonacci sequence.

11. We can generalize the notion of linear sequence if we admit that the parameter n can vary in a continuous way, without being limited to integers, so that z_n is a continuous function $z(n)$ of the parameter n and can consequently take irrational values. This expedient can be used to simplify the records, but it is not of practical value for the applications.

With the notation of paragraph 3, we can write the formulas (2^{bis}), according to the case:

$$(2^{\text{ter}}) \quad z_n = \sqrt{|AB|} F_{n+\phi} \quad \text{or} \quad z_n = \frac{\sqrt{|AB|}}{2 \operatorname{ch} \lambda} L_{n+\phi}$$

The quantities z and n are well integers, but it is not the case for the functions $F_{n+\phi}$ and $L_{n+\phi}$, like for the parameters λ, ϕ , and m (or $\sqrt{|AB|}$).

Thus, any linear sequence can be reduced to a generalized Lucas or Fibonacci sequence by use of an irrational factor.

In the special case of the connected sequences (paragraph 7), the formulas (5^{bis}), (6^{bis}) and (7^{bis}) can be modified like the formula (2^{bis}) and therefore simplified. One replaces to this end:

[Continued on page 298.]