Spectral Measures and the Spectral Theorem

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We develop properties about Hilbert spaces and spectral measures in order to give a generalization of the spectral theorem to infinite dimensions. We follow the treatment of [Ha] closely, which is one of only a few rare sources that treat this form of the spectral theorem. We then apply our new machinery to representation theory and prove an irreducibility criterion that Professor Sally stated in his summer lectures.

1 Preliminary Facts about Hilbert Spaces

We work in a Hilbert space $\mathcal{H}$ with the inner-product of v and w denoted $(v \mid w)$.

1.1 The Riesz Lemma

We begin by proving an incredibly useful lemma on the existence of operators, but first, we need a standard theorem on Hilbert spaces.

Lemma: If η is a linear functional on $\mathcal{H}$, then $\psi(v) = (v \mid w)$ for suitable choice of $w \in \mathcal{H}$.

Proof: Let $\mathcal{K}$ =Ker(ψ). We may suppose $\mathcal{K}^\perp \neq \{0\}$, else the theorem would be clear with A the zero operator. Then take w to be some non-zero vector in $\mathcal{K}^\perp$. Normalize w so that $\psi(w) = \|w\|^2$. For a given vector v, let $v = v_1 + v_2$, where $v_1 = \dfrac{\psi(v)}{|w|^2}w$ and $v_2 = v - v_1$. We observe that:

$$\psi(v_2) = \psi(v) - \psi(\frac{\psi(v)}{|w|^2}w) = \psi(v) - \frac{\psi(v)}{|w|^2}\psi(w) = \psi(v) - \psi(v) = 0$$

Therefore, $v_2 \in \mathcal{K}$, which implies:

$$(v \mid w) = (v_1 + v_2 \mid w) = (v_1 \mid w) + (v_2 \mid w) = (v_1 \mid w) = (\frac{\psi(v)}{|w|^2} w \mid w) = \psi(v)$$

Definition: A **bounded bilinear functional** is a map $\varphi : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ which is linear in the first term, conjugate linear in the second term, and such that there exists a non-negative constant we denote $|\varphi|$ for which $|\varphi(v,w)| \leq \|\varphi\|\|v\|\|w\|$.

Lemma (Riesz): If φ is a bounded bilinear functional on $\mathcal{H}$, then there exists a unique operator A such that $\phi(v,w) = (Av \mid w)$ for all $v, w \in \mathcal{H}$.

Proof: For fixed v, let $\psi_v(w) = \overline{\phi(v,w)}$. Then, by the above, there exists some vector which we suggestively denote Av such that $\psi_v(w) = (w, Av) \Rightarrow \varphi(v,w) = (Av, w)$. Because φ is a linear functional, it follows easily that A is a linear transformation, so we check that it is bounded:

$$\|Av\|^2 = |\varphi(v, Av)| \leq \|\varphi\|\|v\|\|Av\|$$

Therefore, $\|A\| \leq \|\varphi\|$.

1.2 Adjoints

Theorem: For A an operator, there exists a unique operator A^*, the **adjoint** of A, satisfying the identity $(Av \mid w) = (v \mid A^*w)$ for all $v, w \in \mathcal{H}$.

Proof: Let $\varphi(w,v) = \overline{(w \mid Av)}$. Clearly, φ is a bounded, bilinear functional, so there exists a unique operator A^* such that $\varphi(w,v) = (A^*w \mid v) \Rightarrow (Av \mid w) = (v \mid A^*w)$.

Definition: An operator A is **Hermitian** if $A = A^*$. An operator A is **normal** if $\|Av\| = \|A^*v\|$ for all $v \in \mathcal{H}$.

Proposition: An operator A is normal iff $AA^* = A^*A$.

Proof: We take v an arbitrary vector in $\mathcal{H}$. Then A is normal $\Leftrightarrow$
$(Av \mid Av) = (A^*v \mid A^*v) \Leftrightarrow (A^*Av \mid v) = (AA^*v \mid v) \Leftrightarrow AA^* = A^*A$

Remark: Note that in the Riesz lemma, if φ is symmetric, that is, $\varphi(v,w) = \overline{\varphi(w,v)}$, then the resulting operator will be Hermitian.

1.3 Projections

Definition: If $\mathcal{M}$ is a closed subspace of $\mathcal{H}$, then elementary Hilbert space theory tells us that every vector $v \in \mathcal{H}$ has a unique decomposition $v = v_1 + v_2$, where $v_1 \in \mathcal{M}$ and $v_2 \in \mathcal{M}^{\perp}$. We define the projection onto $\mathcal{M}$ to be the map $P : v \mapsto v_1$. Note that P is necessarily an operator. If $\mathcal{M} = \mathbb{C}$, we denote P by 1, and if $\mathcal{M} = \{0\}$, we denote P by 0.

Definition: Let $\{P_i\}_{i\in I}$ be projections onto $\mathcal{M}_i$. We partially order these by $P_i \leq P_j$ if $\mathcal{M}_i \subset \mathcal{M}_j$. We further define $\sum_{i\in I} P_i$ to be the projection onto $\cup_{i\in I}\mathcal{M}_i$.

Theorem: An operator P is a projection if and only if it is Hermitian and idempotent ($P^2 = P$).

Proof: It is clear that if P is a projection, then $P^2 = P$. For two vectors v and w, let $v = v_1 + v_2$ and $w = w_1 + w_2$, where $Pv = v_1$ and $Pw = w_1$. Then:

$$(Pv \mid w) = (v_1 \mid w_1) + (v_1 \mid w_2) = (v_1 \mid w_1) + 0 = (v_1 \mid w_1) + (v_2 \mid w_1) = (v \mid Pw)$$

Therefore, P is Hermitian.

Now suppose that P is Hermitian and idempotent. Let $\mathcal{M} = \{w \in \mathcal{H} \mid Pw = w\}$. We claim that P is a projection onto $\mathcal{M}$. It suffices to prove that for all $v \in \mathcal{H}$, $(Pv \mid v - Pv) = 0$.

$$(Pv \mid v - Pv) = (Pv \mid v) - (Pv \mid Pv) = (Pv \mid v) - (P^2v \mid v) = (Pv \mid v) - (Pv \mid v) = 0$$

Corollary: If P is a projection, then for all $v \in \mathcal{H}$, $\|Pv\|^2 = (Pv \mid v)$.

Proof: $\|Pv\|^2 = (Pv \mid Pv) = (P^2v \mid v) = (Pv \mid v)$

2 Spectral Measures

2.1 Definition and Basic Properties

Let $\mathcal{B}(\mathbb{C})$ be the set of Borel sets in $\mathbb{C}$ and $P(\mathcal{H})$ the set of projections on $\mathcal{H}$. **Definition:** A (complex) **spectral measure** is a function $E : \mathcal{B}(\mathbb{C}) \to P(\mathcal{H})$ satisfying the following properties:

1. $E(\emptyset) = 0$ and $E(\mathbb{C}) = 1$
2. If $\{B_n\}_{n\in\mathbb{N}}$ is a family of disjoint Borel sets, then $E(\bigcup B_n) = \sum E(B_n)$

Remark: The standard techniques of the theory of complex-valued measures can be used to show many of the basic facts about spectral measures. For one, the requirement that $E(\emptyset) = 0$ is superfluous given the second property. Furthermore, we can see that if $B_0 \subset B_1$, then $E(B_0) \leq E(B_1)$. Another interesting property that spectral measures share with their complex counterparts is modularity, that is:

$$E(B_0 \cup B_1) + E(B_0 \cap B_1) = E(B_0) + E(B_1).$$

From this and the observation that $E(B_0)E(B_0 \cup B_1) = E(B_0)$, we can see that $E(B_0 \cap B_1) = E(B_0)E(B_1)$.

Proposition: Suppose $E : \mathcal{B}(\mathbb{C}) \to P(\mathcal{H})$ is any function such that $\forall v, w \in \mathcal{H}$, the function $E^*(B) = (E(B)v \mid w)$ satisfies $E^*(\bigcup B_n) = \sum E^*(B_n)$ and that $E(\mathbb{C}) = 1$. Then E is a spectral measure.

Proof: The remark shows that we needn't worry about the empty set, so it suffices to show only the second property. Suppose $\{B_n\}$ a disjoint family of Borel sets in $\mathbb{C}$. Then:

$$\sum \|E(B_n)v\|^2 = \sum (E(B_n)v \mid v) = \sum (E(\bigcup B_n)v \mid v) = \|E(\bigcup B_n)v\|^2$$

Thus, the sequence $v_n = E(B_n)v$ is summable. We note now that for all disjoint Borel sets B, C, we have

$$(E(B)v + E(C)v \mid w) = (E(B)v \mid w) + (E(C)v \mid w) = (E(B \cup C)v \mid w)$$

so that our claim is clear simply by examining the partial sums of $\sum E(B_n)$.

2.2 Spectral Integrals and their Associated Operators

We will find frequent occasion to denote $E(\lambda)$ by E_λ.

Defintion: Given a spectral measure E, we define the **spectral integral** with respect to $v, w \in \mathcal{H}$ of the measurable function f to be the Lebesgue-Stieltjes integral $\int f(\lambda) d(E_\lambda v \mid w)$, which we will sometimes abbreviate $\int f(\lambda) dE$.

Definition: The **spectrum** of a spectral measure E is $\Lambda(E) = \mathbb{C} \setminus \cup U_i$, where the union is taken over all open sets U_i for which $E(U_i) = 0$. We say that E is **compact** if $\Lambda(E)$ is compact.

Theorem: For E a compact spectral measure, there is a unique normal operator A such that $\forall v, w \in \mathcal{H}$, $\int \lambda d(E_\lambda v \mid w) = (Av \mid w)$. For the sake of brevity, we will find occasion to write $A = \int \lambda dE$.

Proof: Let $\varphi(v, w) = \int \lambda d(E_\lambda v \mid w)$, which is finite for all (v, w) because $\Lambda(E)$ is compact. Clearly, φ is a bilinear functional. Furthermore, $\varphi(v, v)$ is bounded because $|\varphi(v, v)| \leq \int |\lambda| d(\|E_\lambda v\|^2) \leq \sup\{|\lambda| \mid \lambda \in \Lambda(E)\} \cdot \|v\|^2 = M\|v\|^2$. But by applying the parallelogram law, we see that:

$$|\varphi(v, w)| \leq \tfrac{1}{4}M(\|v + w\|^2 + \|v - w\|^2 + \|v + iw\|^2 + \|v - iw\|^2) \leq M(\|v\|^2 + \|w\|^2)$$

If we set $\|v\| = \|w\| = 1$, we see that $\|\varphi\| \leq 2M$, so φ is bounded. Thus, by the Riesz lemma such a unique operator A must exist.

We now wish to show that A is a normal operator. By the same process, construct an operator A' such that $A' = \int \bar{\lambda} dE$. Then:

$$(v \mid A'w) = \overline{(A'w \mid v)} = \overline{\int \bar{\lambda} d(E_\lambda w \mid v)} = \int \lambda d(v \mid E_\lambda w) = \int \lambda d(E_\lambda v \mid w) = (Av \mid w).$$

Therefore, by the uniqueness of the adjoint, $A' = A^*$.

Let B be a Borel set. We then have:

$$(A^* v \mid E(B)w) = \int \bar{\lambda} d(E_\lambda v \mid E(B)w) = \int \bar{\lambda} d(E(B)E_\lambda v \mid w) = \\ \int \bar{\lambda} d(E(B \cap \lambda)v \mid w) = \int_B \bar{\lambda} d(E_\lambda v \mid w).$$

This means that:

$$(AA^* v \mid w) = \overline{(A^* w \mid A^* v)} = \overline{\int \bar{\lambda} d(E_\lambda w \mid A^* v)} = \int \lambda d(E_\lambda A^* v \mid w) = \int \lambda \cdot \bar{\lambda} d(E_\lambda v \mid w) = \\ \int |\lambda|^2 dE$$

But a parallel argument and the symmetry here shows that this is also equal to $(A^* Av \mid w)$. Because v and w were arbitrary, we must have that A is a normal operator.

2.3 The Spectrum of an Operator

We hope to generalize the notion of the eigenvalue, ubiquitous to the study of finite dimensional vector spaces, to infinite dimensions.

Definition: The spectrum of an operator A is the set $\Lambda(A) = \{\lambda \in \mathbb{C} \mid A - \lambda I$ is not invertible}.

In order to use this definition effectively, we'll need a characterization of invertible and non-invertible operators.

Theorem: An operator A on $\mathcal{H}$ is invertible iff its image is dense in $\mathcal{H}$ and $\exists \alpha > 0$ s.t. $\forall v \in \mathcal{H}$, $\|Av\| \geq \alpha \|v\|$ (we call this second property being **bounded from below**).

Proof: We suppose first that A is invertible. Then the image of A is all of $\mathcal{H}$, which is obviously dense in $\mathcal{H}$. Now let $\alpha = \dfrac{1}{\|A^{-1}\|}$. Then $\forall v \in \mathcal{H}$, $\|v\| = \|A^{-1}Av\| \leq \|A^{-1}\|\|Av\|$.

Now suppose that our two properties hold. We claim first that the range of our operator A is all of $\mathcal{H}$. It suffices to show that it is closed. Suppose that v_n is a Cauchy sequence in the image. For all n, choose w_n so that $Aw_n = v_n$. Then $\|w_n - w_m\| \leq \dfrac{1}{\alpha}\|v_n - v_m\|$, which means that w_n is Cauchy, and thus converges to a point $w \in \mathcal{H}$. But by continuity, $v_n \to Aw$, which implies that the range is closed.

To prove injectivity, we note that if $Av_1 = Av_2$, then $0 = \|Av_1 - Av_2\| \geq \alpha\|v_1 - v_2\|$, so that $v_1 = v_2$. Thus, A is bijective. It is clear that its inverse is also linear. It suffices now to show that A^{-1} is bounded, which we see by:

$$\|w\| = \|Av\| \geq \alpha\|v\| = \alpha\|A^{-1}w\| \Rightarrow \|A^{-1}w\| \leq \tfrac{1}{\alpha}\|w\|$$

Proposition: If A is any operator such that $\|A - I\| < 1$, then A is invertible.

Proof: Let $\alpha = 1 - \|A - I\|$, so that $0 < \alpha$. Then for any $v \in \mathcal{H}$:

$$\|Av\| = \|v - (v - Av)\| \geq \|v\| - \|Av - v\| \geq (1 - \|I - A\|)\|v\| = \alpha\|v\|$$

Therefore, A is bounded from below. Let $\mathcal{M}$ be the range of A in $\mathcal{H}$. Let $\delta = \inf\{\|v - w\| \mid v \in \mathcal{H}, w \in \mathcal{M}\}$. It suffices to prove that $\delta = 0$. Suppose otherwise. Then, because $1 - \alpha < 1$, there exist $v \in \mathcal{H}$ and $w \in \mathcal{M}$ such that $\|v - w\| < \frac{\delta}{1-\alpha}$. Then we must have:

$$\delta \leq \|(v - w) - A(v - w)\| \leq (1 - \alpha)\|v - w\| < \delta$$

Thus, assuming δ not equal to zero gives a contradiction.

Theorem: If A is an operator, then $\Lambda(A)$ is compact. In particular, if $\lambda \in \Lambda(A)$, then $\|\lambda\| \leq \|A\|$.

Proof: We first prove that $\Lambda(A)$ is closed, in which case the first statement will follow from the second. Suppose the λ_0 is not an element of $\Lambda(A)$. Take $\delta < \|A - \lambda_0\|$ and let $\|\lambda - \lambda_0\| < \delta$. Then:

$$\|I - (A - \lambda_0)^{-1}(A - \lambda)\| = \|(A - \lambda_0)^{-1}((A - \lambda_0) - (A - \lambda))\| \le \|A - \lambda_0\|^{-1}|\lambda - \lambda_0\| < 1$$

Thus, $A - \lambda$ is invertible in a ball of radius δ about λ_0, so $\mathbb{C} \setminus \Lambda(A)$ is open, which proves our claim.

Now suppose that $|\lambda| > \|A\|$. Then $\|\frac{A}{\lambda}\| < 1$, which implies that $I - \frac{A}{\lambda}$ is invertible. Multiplying this operator by a scalar, we see that $A - \lambda$ is invertible. Thus, λ is not in $\Lambda(A)$.

Theorem: If E is a compact spectral measure and $A = \int \lambda dE$, then $\Lambda(E) = \Lambda(A)$.

Proof: Suppose that $\lambda_0 \in \mathbb{C} \setminus \Lambda(E)$. By definition, $\Lambda(E)$ is open, so there exists $\delta > 0$ such that $B = B(\lambda_0, \delta) \subset \mathbb{C} \setminus \Lambda(E)$ and $E(B) = 0$. Using that $E(\mathbb{C}) = 1$, one may see that:

$$\|Av - \lambda v\|^2 = \int_{\mathbb{C}} |\lambda - \lambda_0|^2 d(E_\lambda v \mid v) = \int_{\mathbb{C} \setminus B} |\lambda_0 - \lambda|^2 d(E_\lambda v \mid v) \ge \int_{\mathbb{C} \setminus B} \delta^2 d(E_\lambda v \mid v) = \delta^2 \|v\|^2$$

which means that $A - \lambda_0 I$ is bounded from below. We now need only to prove that the image of $A - \lambda_0 I = A_0$ is dense in $\mathcal{H}$.

Suppose that A_0 is any normal operator which is bounded from below. Let R be the image of A_0. It suffices to prove that $R^\perp = 0$. Suppose that $w \in R^\perp$. Then for all $v \in \mathcal{H}$, $0 = (A_0 v \mid w) = (v \mid A_0^* w)$, which means that $A_0^* w = 0$. But

$$0 = \|A_0^* w\| = \|A_0 w\| \ge \alpha \|w\|,$$

which means that $w = 0$.

On the other hand, suppose that $\lambda_0 \in \Lambda(E)$. Take $\delta > 0$. Then $E(B(\lambda_0, \delta)) \neq 0$. As $E(B)$ must contain some unit vector v, we have, by the argument above:

$$\|Av - \lambda_0 v\|^2 = \int_B |\lambda - \lambda_0|^2 d(E_\lambda v \mid v) \le \delta^2 \|v\|^2$$

Thus, because δ was arbitrary, $A - \lambda_0$ cannot be bounded from below, and thus, is not invertible.

3 The Spectral Theorem

3.1 Useful Theorems of Analysis

For the proof of the spectral theorem, we will need two standard theorems of analysis. The curious reader is referred to [R1] and [R2] respectively. We denote by $C(X)$ the set of complex-valued continuous functions on X and by $C_0(X)$ the subset of $C(X)$ of functions which go to 0 at infinity (formally: for all $\epsilon > 0$, there exists a compact set K for which $|f(x)| < \epsilon$ for all $x \notin K$.

Theorem (Stone-Weierstrass): For a given compact set K, suppose that $\mathcal{A}$ is an algebra of continuous functions $f : K \to C$ which is closed under conjugation, separates points of K and there are no elements of K which all functions in $\mathcal{A}$ vanish on. Then A is dense in $\mathcal{C}(K)$.

Theorem (Riesz): If X is a locally compact Hausdorff space and φ is a bounded linear functional on $C_0(X)$, then there is a unique regular complex Borel measure μ such that $\varphi(f) = \int f d\mu$ and such that $\|\varphi\| = |\mu(X)|$.

3.2 Proof of the Spectral Theorem

Theorem: Let A be a Hermitian operator. Then there exists a spectral measure E such that for all v, w in $\mathcal{H}$, $(Av \mid w) = \int_{\Lambda(A)} \lambda d(E_\lambda v \mid w)$.

Proof: Fix vectors v and w. For a given polynomial p, let $L(p) = (p(A)v \mid w)$. Note that $|L(p)| \leq \sup\{|p(\lambda)| \mid \lambda \in \Lambda(A)\} \cdot \|v\|\|w\|$. Because $\Lambda(A)$ is compact, we can apply Stone-Weierstrass to see that the polynomials are dense in $C(\Lambda(A))$. Therefore, L defines a bounded linear functional on all of $C(\Lambda(A)) = C_0(\Lambda(A))$. Thus, there is a unique bounded complex measure $\mu_{(v,w)}$ such that $(p(A)v \mid w) = \int p(\lambda) d\mu_{(v,w)}(\lambda)$.

Fix a Borel set B. We may now define $\mu_B(v, w) = \mu_{(v,w)}(B)$. An obvious check reveals that μ_B is a bounded symmetric bilinear form. Therefore, by the Riesz lemma, there exists a unique Hermitian operator $E(B)$ such that $\mu_B(v \mid w) = (E(B)v \mid w)$. Setting $p(\lambda) = 1$, we see that $E(\Lambda(A)) = E((C)) = 1$, and setting $p(\lambda) = \lambda$, we see that $\int \lambda d(E_\lambda v \mid w) = (Av \mid w)$.

By an earlier theorem, it's enough to check that E is projection-valued to see that it is actually a spectral measure. Because E is Hermitian, we need only to check that it is an idempotent operator. Take p, q to be arbitrary polynomials. We define a new measure

$\nu(B) = \int_B p(\lambda) d(E_\lambda v \mid w)$. Then, by the construction of E:

$$\int q(\lambda) d\nu(\lambda) = \int q(\lambda)p(\lambda) dE = (q(A)p(A)v \mid w) = (p(A)q(A)v \mid w) = \\ (q(A)v \mid p(A)w) = \int q(\lambda) d(E_\lambda v \mid w)$$

Thus, by the arbitrary choice of q, Stone-Weierstrass and the fact that compactly-supported, continuous functions are dense in $\mathcal{L}^1$, we have $\int \chi_B(\lambda)p(\lambda)d(E_\lambda v \mid w) = \nu(B) = (E(B)v \mid p(A)w)$ for all Borel sets B. But applying Stone-Weierstrass again, this time on p, we now see that:

$$(E(B)v \mid w) = \int \chi_B(\lambda) dE = \int \chi_B(\lambda) \cdot \chi_B(\lambda) dE = (E(B)v \mid E(B)w) = (E(B)^2 v \mid w)$$

Thus, as v and w were arbitrary, this was exactly what we intended to prove.

3.3 Applications to Representation Theory

Definition: Let G be a group. A **unitary representation** of G into $\mathcal{H}$ is a homomorphism $T : G \to \mathcal{U}(\mathcal{H})$, where $\mathcal{U}(\mathcal{H})$ is the set of unitary (inner-product preserving) operators on $\mathcal{H}$. We denote $T(g)$ by T_g to emphasize that $T(g)$ is itself a function on $\mathcal{H}$. We say a representation T of G is **irreducible** if the only closed invariant subspaces of $\mathcal{H}$ under $T(G)$ are 0 and $\mathcal{H}$.

Definition: The **commuting algebra** of a representation T of G is the set of operators A such that $T_g A = A T_g$ for all $g \in G$. We denote the commuting algebra of T by $\mathcal{C}(T)$. Note that I is always in $\mathcal{C}(T)$.

Theorem: A representation T of G is irreducible iff $\mathcal{C}(T)$ is one-dimensional.

Proof: Suppose first that T is reducible. Then there is a proper closed subspace $\mathcal{M} \subset \mathcal{H}$ which is invariant under the action of T. Let P be the projection onto $\mathcal{M}$. Then for any $v \in \mathcal{H}$, if $v = v_1 + v_2$ where $v_1 = Pv$ and if $w_i = T_g v_i$, we have

$$PT_g v = PT_g v_1 + PT_g v_2 = Pw_1 + Pw_2 = w_1 = Tv_1 = TPv$$

Thus, because P is not a scalar multiple of the identity, the commuting algebra is not one-dimensional.

Suppose now that T is irreducible and A is in $\mathcal{C}(T)$. We first assume that A is Hermitian. Choose E so $A = \int \lambda dE_\lambda$. Fix v and w vectors in $\mathcal{H}$ as well as some polynomial p. We define two measures on Borel sets B: $\mu(B) = \int_B d(E_\lambda T_g v \mid w)$ and $\nu(B) = \int_B d(T_g E_\lambda v \mid w)$. Then:

$$\int p(\lambda)d\nu = \int p(\lambda)d(E_\lambda v \mid T_g^* w) = (p(A)v \mid T_g^* w) = (T_g p(A)v \mid w) = (p(A)T_g v \mid w) = \int p(\lambda)d\mu$$

Thus, applying Stone-Weierstrass, $(E_B T_g v \mid w) = (T_g E_B v \mid w)$. As v and w were arbitrary, this tells us that T_g commutes with E_B for all $g \in G$ and B a Borel set. But, because T is irreducible and E is projection-valued, this implies that $E_B = 0$ or 1 for all B. Therefore:

$$(Av \mid w) = \int \lambda d(E_\lambda v \mid w) = \lambda_0 (v \mid w) = (\lambda_0 v \mid w)$$

so that $A = \lambda_0 I$, as was claimed.

Now we loosen the restriction that A is self-adjoint. Let $A_1 = \frac{A+A^*}{2}$ and $A_2 = \frac{A-A^*}{2i}$. We observe that both A_1 and A_2 are self-adjoint and in the commuting algebra of T. Thus, by the above, there are constants λ_1 and λ_2 so that $A_1 = \lambda_1 I$ and $A_2 = \lambda_2 I$. Then $A = (\lambda_1 + i\lambda_2)I$.

4 Bibliography

[Ha] Halmos, Paul R. *Introduction to Hilbert Space*, 1st Ed., Chelsea Publishing Company, New York, 1951.

[R1] Rudin, Walter. *Principles of Mathematical Analysis*, 3rd Ed., McGraw-Hill, New York, 1976.

[R2] Rudin, Walter. *Real and Complex Analysis*, 3rd Ed., McGraw-Hill, New York, 1987.