2-DIMENSIONAL LATTICE TOPOLOGICAL FIELD THEORIES

ELDEN ELMANTO

Abstract. Fukuma, Hosono and Kawai (FHK) [8] introduced Lattice Topological Field Theories (2D-LTFTs) to understand certain invariants of 3-manifolds. These "toy models" are not only useful for understanding their more "serious" analogs, but also turn out to be interesting in their own right. In particular, they provide an elementary machinery to calculate certain topological invariants. Here we present a reformulation of the FHK construction through diagrammatic methods, following the approach of Baez [5], to prove the main theorem: a 2D-LTFT is a semisimple algebra. As an application, we will prove Mednykh's formula using the elementary machinery of LTFTs, following the approach of Snyder [10].

Contents

The reader is assumed to be familiar with elementary notions from algebra (associative algebras, characteristic polynomials) and category theory (functors, naturality and monoidal categories). Ideas from linear algebra which will form the backbone of this paper will be re-phrased in terms of string diagrams. As this paper was born out of the TQFT course by May and Riehl at the University of Chicago Chicago Summer Research Experience for Undergraduates 2011, perhaps the best background reference is [3] by May et al.

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1. Introduction

In 1994, Fukuma, Hosono and Kawai (FHK) constructed a topological invariant of surfaces via triangulations, termed Lattice Topological Quantum Field Theory (LTFT). The reader is forgiven her skepticism, given that the Euler characteristic has been extremely successful in 2-manifold theory as a topological invariant. In fact, LTFTs were never intended to be "serious" constructions in the context of 2-manifold theory or physics — it is essentially a "toy model" to understand its higher dimensional analogs.

While the LTFT finds its way in numerous "serious" applications of mathematics (see Section 1 of [9] for a good proselytization), this paper finds its construction interesting in itself. Firstly, it provides a preview of the power of "diagrammatic linear algebra" in which categorical arguments (i.e. diagram chasing) are translated into more intuitive one/two-dimensional "topological" arguments, involving strings. Secondly, the punchline of 2D-LTFTs was actually invented very early on in mathematics — we will show that, while the 2D-Topological Quantum Field Theories (TQFTs) are "merely" commutative Frobenius algebras, 2D-LTFTs are "merely" semisimple algebras, due to a classical result by L.E. Dickson. Indeed, the crux of this construction is the translation of Dickson's theorem into the modern language of string diagrams. Thirdly, the construction of 2D-LTFTs turns out to be a powerful computational method for topological invariants, as it provides a very elementary proof of Mednykh's formula, which we will provide at the end of this paper.

This paper will start with an elucidation of the main language of the construction — diagrammatic linear algebra. As opposed to the original paper by FHK, this approach will be index-free. Next, we will provide a concise elucidation of associative algebras, geared towards Dickson's theorem. We will then define what a 2D-LTFT is (which will turn out to be a rather lengthy set of instructions), and prove the main theorem of 2D-LTFTs. Lastly, we will discuss Mednykh's formula and its proof using the elementary machinery of 2D-LTFTs.

2. Diagrammatic Linear Algebra

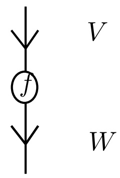
Linear algebra is on the one hand, the study of finite-dimensional vector spaces and linear maps between them, and on the other hand, the study of a particular class of objects (in particular, the dualizable ones) in the monoidal category ($\mathbf{Vect_K}$, $\otimes$, K). We present this perspective using string diagrams, which will form the basic language of this paper.

2.1. Introduction.

We first work in any monoidal category, ($\mathbf{C}$, $\otimes$, I), with the primary example being ($\mathbf{Vect_K}$, $\otimes$, K), then specialize to a *symmetric monoidal category with duals* in the next section.

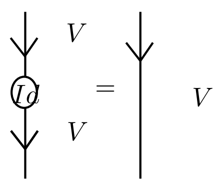
In category theory, objects are typically represented by 0-dimensional data (points) and morphisms are represented by 1-dimensional data (arrows). In diagrammatic linear algebra, we reverse this notation, and represent objects by 1-dimensional arrows and morphisms by 0-dimensional "beads".

Definition 2.1. Given, $V, W \in Ob(\mathbf{C})$, and $f \in Hom(V, W)$, we write:

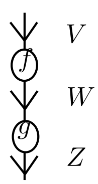


We call a representation of an object a "string," and a representation of a morphisms a "bead." A representation of specific objects and morphism(s) between them is called a "string diagram." Notice that string diagrams are completely determined by a hom-set; stating a particular morphism in our category is sufficient information for us to draw a string diagram. Note also, that the directionality of the string is important in distinguishing $Hom(V, W)$ from $Hom(W, V)$. In particular, if we fix the convention "down-arrow" to represent an object, then we will soon see that an "up-arrow" will refer to its dual object.

In any category we have to define two morphisms — the composition and identity.The identity morphism is represented as:

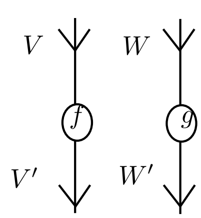


And given two morphisms, $f \in Hom(V, W)$ and $g \in Hom(W, Z)$, we represent $g \circ f \in Hom(V, Z)$ as:



Next, we need a notation to capture the monoidal structure of $\mathbf{C}$. In particular, we need representations for the functor $\otimes : \mathbf{Vect_K} \times \mathbf{Vect_K} \longrightarrow \mathbf{Vect_K}$, and the distinguished object I. Hence, the following:

Definition 2.2. Given, $f \in Hom(V, W)$ and $g \in Hom(V', W')$, we write $f \otimes g : V \otimes V' \longrightarrow W \otimes W'$ as:



Now, let I be the unit object in our monoidal category. Analogous to representing identity morphisms as invisible beads, we represent the unit object as an invisible arrow. Hence, while it makes no sense to draw the unit object the way we draw other objects, it makes sense to think of elements in a unit object as morphisms, say c, between I and I, which is just a bead with invisible strings:

Note why it makes sense to represent this particular object as 0-dimensional. This ambiguous notation allows us to write the distinguished element as both an object and a morphism (a linear mapping from K to K), since an element $c \in I$ can also be thought of as a map from I to I (formally, in any monoidal category, any object V is naturally isomorphic to $Hom(I, V)$ where I is the distinguished object).

The point of this notation is that we would like to manipulate these string diagrams and obtain the results of linear algebra (related to the monoidal structure of $\mathbf{Vect_k}$). The basic idea is that the monoidal structure allows us to define some essential string diagrams; in particular, they are the *dual*, *coevaluation*, *evaluation* and *braid*.

2.2. **Essential String Diagrams.** This subsection and the next are dedicated to the string diagrams and valid moves that define a *symmetric monoidal category with duals*. We will provide a complete definition of what this means at the end of this section, choosing instead to first familiarize the reader with the string diagrams and valid moves that will feature heavily in the main theorem of this paper. For a full treatment of such a category in terms of string diagrams, the reader should refer to [12].

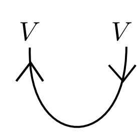
2.2.1. *Evaluation and Coevaluation.* In any monoidal category, there exists a universal property for an object to be dualizable.

Definition 2.3. Fix an object V. Then V is dualizable with W as its dual object when there exist morphisms: $\eta_V : K \longrightarrow V \otimes W$ and $\epsilon_V : W \otimes V \longrightarrow K$ such that the two diagrams commute:

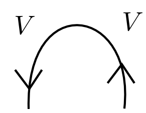
$$\begin{array}{ccc} K \otimes V & \xrightarrow{\eta_V \otimes id_V} & V \otimes W \otimes V \\ & \searrow^{\cong} & \downarrow {\scriptstyle id_V \otimes \epsilon} \\ & & V \otimes K \end{array}$$

$$\begin{array}{ccc} W \otimes K & \xrightarrow{id_W \otimes \eta_V} & W \otimes V \otimes W \\ & \searrow^{\cong} & \downarrow {\scriptstyle \epsilon \otimes id_W} \\ & & K \otimes W \end{array}$$

In string diagrams, the the evaluation map, ϵ_V is represented by:



and the coevaluation map η_V is represented by:

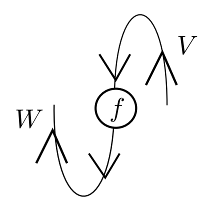


Remark 2.4. We have implicitly defined the string diagram representation of a dual object V. That is, the string diagram with the arrow pointing upwards. Hence, with this convention, there is no need to label this string as V^*, since a string with an arrow pointing upwards labelled V is understood to be V^*. Obviously, this string diagram only exists if the object in our monoidal category is dualizable.

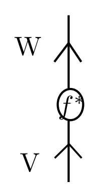
We will later show that this definition coincides with the familiar definition of a dual object in our primary example, $(\mathbf{Vect_k}, \otimes, \mathbf{K})$.

2.2.2. *Dual and adjoint.* From these two strings, we can then define the dual of a morphism $f : V \longrightarrow W$ for dualizable objects V and W. We call this morphism the adjoint.

Definition 2.5. Let V and W be dualizable objects in a monoidal category. Then given, $f \in Hom(V, W)$, its adjoint is the following morphism:

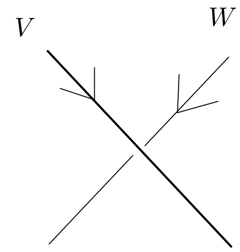


As a shorthand, we denote the above string in this way:



This string is obtained by rotating the string that corresponds to $f : V \longrightarrow W$ by 180 degrees. In this case, the "special" morphism is really the adjoint of f, written as f^*.

2.2.3. *Braid.* Lastly, we have the braid which represents a special morphism from the object $V \otimes W$ to the object $W \otimes V$:

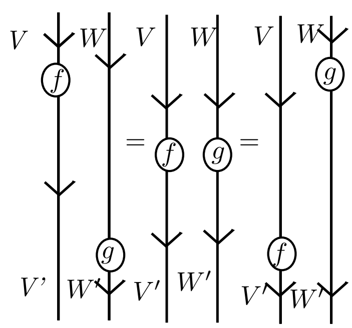


In particular, a braiding turns our monoidal category into a braided monoidal category if is natural and satisfies certain coherence conditions.

2.3. **Manipulation of String Diagrams.** We now turn our interest towards "invariant" manipulations of the string diagrams, i.e manipulations of a string diagram from an initial state to a final state that do not change what it represents. We will call such "invariant" manipulations *valid moves*. The way one should think about these valid moves is that they represent certain universal properties on objects in the category. For example, we can describe the properties of a monoidal object in a category by drawing the valid moves for associativity and unitality.

2.3.1. *Shifting.* Shifting is a move that is valid for all objects in any monoidal category; we claim the following:

Proposition 2.6. *Let $f \in Hom(V, V')$ and $g \in Hom(W, W')$. Then the following is a valid move:*



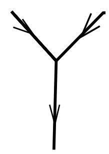
This claim follows immediately from the functoriality of $\otimes$.

Proof. It suffices to show validity for the first move. On the left hand side, the first string represents the composition $id_V \circ f$ and the second string represents $g \circ id'_W$, hence the entire string diagram is determined by the morphism $((id_V \otimes g) \circ (f \otimes id_W))$. By functoriality of $\otimes$, this is just $(id_V \circ f) \otimes (g \circ id_W) = (f \otimes g)$ which determines the string in the middle. □

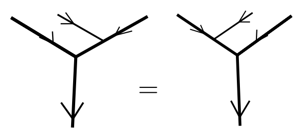
We will soon see that shifting, albeit being almost a triviality, is a particularly useful move in proofs using string diagrams.

Henceforth, we will assume that we are working with the object V unless stated otherwise, hence we do not label any string.

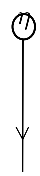
2.3.2. *Associating.* We define multiplication, a linear map $\mu : V \otimes V \longrightarrow V$, generically as this string:



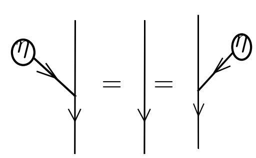
Assuming that all string diagrams of the above form refers to the same multiplication, we can state the associativity axiom of a monoid without writing stating what the bead is.



2.3.3. *Taking Units.* We also need the string diagrams for the valid move of *taking units* (completing the axioms for a monoid). First we define the unital map $\eta : I \longrightarrow V$:

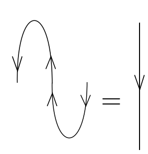


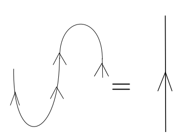
Then the valid move, *taking units*, is the following:



2.3.4. *Straightening I.* Straightening is the condition for an object to be dualizable (stated in (2.3)). Let us state this definition in terms of string diagrams.

Definition 2.7 (Straightening I)**.** Let V be an object in a monoidal category. Then V is dualizable, with V^* as its dual if the evaluation and coevaluation satisfies the following valid moves:



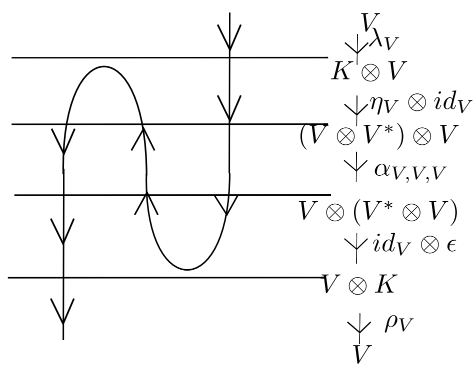


Note why this should be "morally" true: thinking of the diagram as an oriented 1-manifold, we see that both diagrams have an "in-boundary" and an "out-boundary" determined by the directionality of the arrows (this applies to string diagrams in general). In this regard, both diagrams are "topologically" the same. Indeed, this is the way in which most "valid moves" are *a priori* verified. Let us briefly check that this coincides with our usual understanding of duals in linear algebra, i.e $V^* = Hom(V, K)$ for V, a finite-dimensional vector space. Hence we claim:

Proposition 2.8. *Let V be an object in the category of ($\mathbf{Vect_K}$, $\otimes$, K), and let V^* be $Hom(V, K)$. Let the evaluation map be that map that sends $f \otimes v$ to $f(v)$ where f is a linear functional and v is a vector, and let the coevaluation map be that map that sends 1_K to $1 \in End(V) \cong V \otimes V^*$. If V is a finite-dimensional vector space, then V is dualizable, and $Hom(V, K)$ is its dual object.*

Proof. It suffices to show that, with V^* assigned as $Hom(V, K)$, the first move in (2.7) is valid (the second move follows analogously).

To this end, we claim that if we compute an arbitrary vector through the string on the left-hand, then we will obtain the same vector. First, fix a basis $\{e_i\}_1^n$ of V and choose the corresponding basis $\{f_i\}_1^n$ of $Hom(V, K)$, where $f_i(e_j) = \delta_{ij}$. The computation is done in the following way:

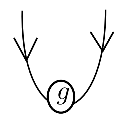


Now, since we have chosen the basis, the coevaluation map η_V is that map that sends the element $1_k \in K$ to the element $1 \in End(V)$ which corresponds to the element $\sum_{i=1}^n e_i \otimes f_i$ of $V \otimes V^*$. Furthermore, by hypothesis, ϵ_V sends an element $f \otimes v \in V^* \otimes V$ into $f(v) \in K$, and λ_V, ρ_V, $\alpha_{V,V,V}$ are natural isomorphisms which exist in any monoidal category. With this information, one can choose an arbitrary element $v = \sum \beta_i e_i$ and check that the coeffiecients are preserved after applications of all the maps above.

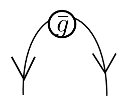
□

2.3.5. *Straightening II.* There is a second form of straightening that we can do with dualizable objects, and it arises from one of the most central concepts in our construction — nondegeneracy. In particular, we discuss nondegeneracy on one object — the ideas and proofs can be easily generalized to isomorphic, and even arbitrary objects. We will state two definitions of nondegeneracy and invite the reader to check that they are equivalent. We also invite the reader to draw connections regarding these two forms of straightening.

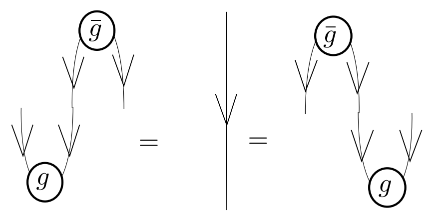
Definition 2.9 (Straightening II)**.** Let V be an object. A *pairing* on V is a morphism $g : V \otimes V \to I$. For convenience, the image under g of the vector $v \otimes w$ is written as (v, w). The string diagram for g is written as:



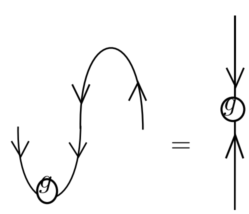
Definition 2.10. Let V be an object, and fix a pairing $g : V \otimes V \to I$. Then g is nondegenerate if there exists a morphism, $\bar{g} : I \to W \otimes V$ called a copairing:



such that the following moves are valid:

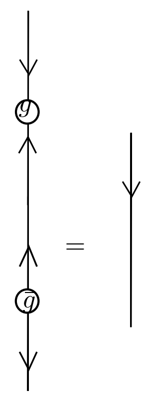


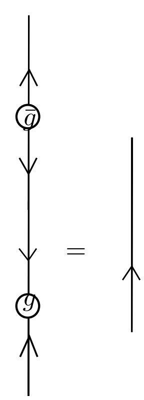
Definition 2.11. Let V be an object, and let g be a pairing on V. Then g is nondegenerate when the composition of g with the coevaluation map is a map, g from V to V^*, i.e the following move is valid:



and the map is an isomorphism, i.e there exists a map,

such that the following moves are valid:





The next proposition should not be surprising, and one can prove it via a string-diagrammatic argument (we omit the proof for brevity, but we will use the two definitions of interchangeably throughout the paper).

Proposition 2.12. *The two definitions of nondegeneracy are equivalent.*

In other words, a nondegenerate pairing is a nondegenerate pairing, and *Straightening II* refers to any one of the valid moves above. One proves this proposition by starting with one definition, then constructing the obvious string diagram that "looks like" the string diagram needed in the other definition, and check that the moves are valid.

Specializing to our primary example, the following corollary is immediate since a bijection (a morphism that is both mono and epi) is equivalent to an isomorphism in the category of vector spaces:

Corollary 2.13. *Let V be a finite-dimensional vector space. Fix a pairing $g : V \otimes V \longrightarrow K$. Then the induced map, $g : V \longrightarrow V^*$, $v \longmapsto (v, -)$ is a bijection if and only if the pairing g is nondegenerate.*

Furthermore, we claim the following:

Proposition 2.14. *Suppose that there exists a nondegenerate pairing, g, on V. Then V is finite-dimensional.*

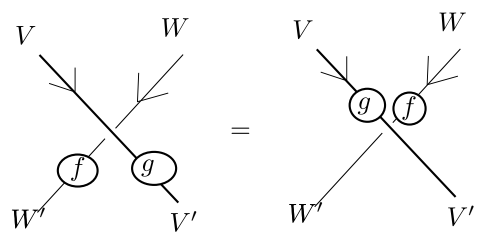
Proof. Without loss of generality, we can define a copairing as that map that sends 1_K to a vector $\sum_{i=1}^n v_i \otimes w_i$. Pick an arbitrary $v \in V$ and "feed" it through the the first string diagram in the valid move defined in 2.10, and we obtain $\sum_{i=1}^n (v, v_i) w_i$. Now since g is nondegenerate, we have the first valid move in 2.10 which literally says that $v = \sum_{i=1}^n (v, v_i) w_i$. Since v was chosen arbitrarily, we have that $\{w_i\}_{i=1}^n$ spans V, and it is therefore finite-dimensional. □

Notice that having a nondegenerate pairing is a powerful statement in string diagrams - if we can identify any object with its dual isomorphically, then there is no reason to put arrows in our string diagrams, and there is only one kind of straightening.

2.3.6. *Braid moves.* We will focus on three kinds of braid moves: *sliding*, *unbraiding I* and *unbraiding II.* Recall that in a braided monoidal category, the braiding is a natural isomorphism between the functors $- \otimes -$ and $- \otimes -$ precomposed with the covariant functor that takes $\mathbf{C} \times \mathbf{C}$ into its opposite category. Denoting the braiding functor as B, the following diagram must commute for every pair of objects V, W, and morphisms $f \in Hom(V, V')$ and $g \in Hom(W, W')$ in our monoidal category.

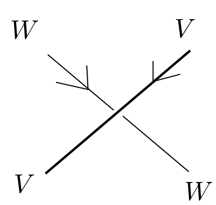
$$\begin{array}{ccc} V \otimes W & \xrightarrow{B_{V,W}} & W \otimes V \\ {\scriptstyle f\otimes g}\downarrow & & \downarrow{\scriptstyle g\otimes f} \\ V' \otimes W' & \xrightarrow[B_{V',W'}]{} & W' \otimes V' \end{array}$$

The above diagram commuting, gives us the valid move which we call *sliding*:

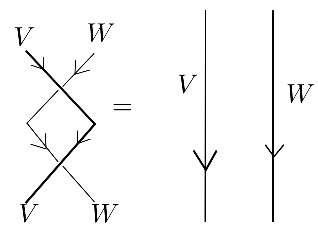


Note, that the above diagram states that the order in which the morphisms $B_{V,W}$, and $g \otimes f$ are applied does not matter - hence we can slide the beads past each other.

Next, the isomorphism condition states that for every braiding morphism $B_{V,W} : V \otimes W \longrightarrow W \otimes V$, then there exists an inverse morphism:



such that the following move *unbraiding I* is valid:



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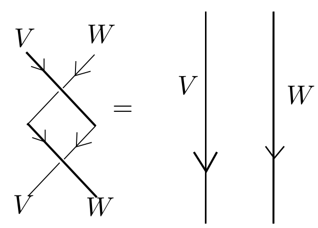
Lastly, if our braided monoidal category is also symmetric, which is the case for our primary example, then if we apply braiding twice we have another valid move called *unbraiding II*:



It is important to note that the *unbraiding I* holds generally, while *unbraiding II* holds only in a symmetric monoidal category.

2.4. **A Final Remark.** We will end off by capturing what we have done so far in a definition. Essentially, we have introduced some interesting morphisms that can be defined for any monoidal category, but *not* all objects in that monoidal category. Hence, it will be useful to actually define a category in which every object has the essential string diagrams, and obeys the valid moves above. Indeed, there exists such a category:

Definition 2.15. Let $(\mathbf{C}, \otimes, I)$ be a (without loss of generality, strict) monoidal category. Then $(\mathbf{C}, \otimes, I)$ is a *symmetric monoidal category with duals* if (1) every object is dualizable and (2) every braiding is self-inverse (i.e *unbraiding II* is a valid move move for any pair of objects).

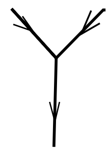
Note that we can assume $(\mathbf{C}, \otimes, I)$ to be a strict monoidal category due to Mac Lane's coherence theorem. The reader can easily verify that the subcategory of $(\mathbf{Vect_K}, \otimes, K)$, whereby the objects are *finite-dimensional* vector spaces, is such a category.

3. Semisimple Algebras and Dickson's Theorem

In this section, we give a modern re-dress of L.E. Dickson's classification theorem of semisimple algebras, stating its conclusion in the language of string diagrams. For completion, we will define an (associative and unital) algebra:

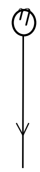
Definition 3.1. An (associative and unital) K-algebra, denoted by A, is an object with two morphisms, the mutiplication

$$\mu : A \otimes A \to A$$

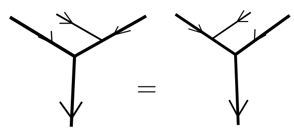


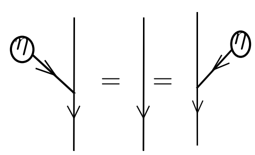
and the unit

$$\eta : K \to A$$



such that the these moves are valid:





We will now define a semisimple algebra

Definition 3.2. Let K be an arbitrary field, and let A, be a K-algebra. A left ideal, $I \subset A$, is nilpotent if there exists an integer n such that $I^n = 0$.

This is to say, that there exists an integer, n, such that if one multiplies an element x of I on the left n times by itself, then one will obtain 0. Obviously, a nilpotent ideal is an ideal.

Definition 3.3. A finite dimensional K-algebra A is semisimple if the only left nilpotent ideal is the $\{0\}$ subspace

This means that there exists no proper left nilpotent ideal of A (the empty ideal and A itself are the only nilpotent ideals).

Definition 3.4. Let K be an arbitrary field, and A a $K - algebra$. The *left-action* of an element $a \in A$ is the linear transformation:

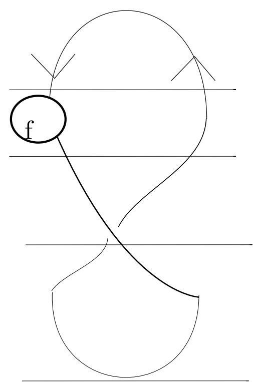
$$L_a : A \longrightarrow A$$
$$b \longmapsto ab$$

It is easy to check that this is indeed a linear operator. Obviously, to each linear operator we can associate with it a matrix. The following definition thus makes sense for any K-algebra:

Definition 3.5. Let K be an arbitrary field, and A a K-algebra. The *trace form* on A is a *pairing* of A with A:

$$Tr : A \otimes A \longrightarrow K$$
$$a \otimes b \longmapsto Tr(L_a L_b)$$

Note that the trace form is just a particular form of morphism in the category $(\mathbf{Vect_K}, \otimes, K)$, hence it determines a string diagram. We claim that the following is the string diagram for the trace form of any linear map $f : V \longrightarrow V$:



Proof. Note that the above diagram is a map from K to K, hence we our computation should yield an element of the ground field (or a complex number if $\mathbb{C}$ is our ground field. We will adopt the Einstein summation notation for convenience (see [7] for details). Fix a basis e_i for A and let e^k be a basis for A^* such that $e^k e_i = \delta^k_i$. The first map sends the identity in our field to the element $e_i \otimes e^i$, while the next map sends this to $f(e_i) \otimes e^i = f^k_i e_k \otimes e^i$. By *sliding*, the braiding map sends this element to $f^k_i e^i \otimes e_k$ and taking the evaluation map, we have $f^k_i \delta^i_k$ which evaluates the trace of the matrix of f. □

We will now state Dickson's theorem:

Theorem 3.6. (*Dickson*) *Let K be an algebraically closed field of characteristic zero, and A a finite-dimensional K-algebra. An element x of A is zero or properly left nilpotent if and only if $Tr(L_a L_x) = 0$ for any $a \in A$.*

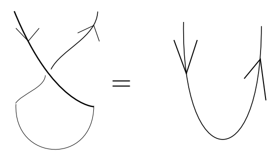
Proof. See [4] for proof. The proof utilizes elementary techniques in linear algebra involving characteristic polynomials

□

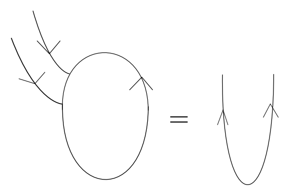
The corollary follows immediately:

Corollary 3.7. *A necessary and sufficient condition for a finite-dimensional K-algebra over an algebraically closed field of characteristic zero to be semisimple is for its trace form to be nondegenerate.*

To formulate the conclusion diagrammatically, note that $(\mathbf{Vect_K}, \otimes, K)$ is a symmetric monoidal category, which means that a natural braiding exists. Hence, we can define the following string diagrams in terms of the strings diagrams that we already have, namely the evaluation and the braid (the reader should convince herself that such an assignment is well-defined):



We know that we can always equip any algebra with a pairing defined by the trace form:



Hence, if we are given an algebra (over an algebraically closed field of characteristic zero) with a pairing (the string diagram on the right) such that the above move is declared to be valid (i.e the pairing is given to us as the trace form) and, furthermore, that the pairing is nondegenerate, then the given algebra is indeed semisimple!

Note that the above arguments work as well for two-sided or right ideals — the definition of semisimplicity can be applied to any sidedness of the ideal. Hence, from hereon in, we will not distinguish the sidedness of our ideals.

4. Triangulated Constructions on 2-Cobordisms

We are now ready to define the notion of an FHK construction with the goal of defining a 2D-LTFT. Essentially, the scheme for constructing a 2D-LTFT is as follows:

$$\begin{array}{c} \textit{Nondegenerate Triangulated Construction} \\ \downarrow \\ \textit{FHK Construction} \\ \downarrow \\ 2D-LTFT \end{array}$$

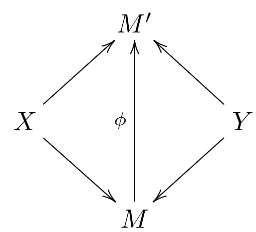
Roughly speaking, a nondegenerate triangulated construction is a map that takes (1) a 2-cobordism along with (2) a pre-determined set of instructions, and returns us a string diagram in the category of ($\mathbf{Vect_K}$, $\otimes$, K). An FHK construction is a triangulated construction with some restrictive conditions on the pre-determined set of instructions. Out of an FHK construction, we can construct a functor (i.e. specify where objects and morphisms go to) from the category of $\mathbf{2-Cob}$ to the category of $\mathbf{Vect_K}$. It turns out that this functor is symmetric monoidal, i.e. a

2D-TQFT (for a treatment of the subject of 2D-TQFTs, we recommend [2] and [3]).

First, some topological preliminaries from the theory of 2D-TQFTs:

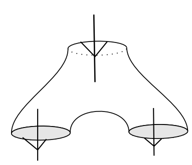
Definition 4.1 (Oriented 2-Cobordism)**.** Let X and Y be oriented 1-manifolds. An *oriented 2-cobordism* from X to Y is a compact, oriented 2-manifold M together with two smooth maps $X \longrightarrow M \longleftarrow Y$ such that X maps diffeomorphically onto the in-boundary M and Y maps diffeomorphically onto the out-boundary of M.

Definition 4.2 (Cobordism Class)**.** Given two oriented 2-cobordism from X to Y, call them M and M', we say that they are equivalent if there is an orientation-preserving diffeomorphism ϕ making this diagram commute:



A *2-cobordism class* is an equivalence class of cobordisms under the above equivalence relation.

Taking the convention that the in-boundary will be denoted with an arrow going into the 2-cobordism and the out-boundary will be denoted with an arrow going out of the 2-cobordism, we can represent a cobordism class by drawing a representative element. For example, we draw the following to represent the cobordism class that takes the oriented 1-manifold S^1 to the oriented one 1-manifold $S^1 \amalg S^1$.

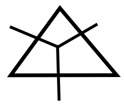


For further details, one should consult [2] and [3]. Henceforth, when we say a 2-cobordism, we really mean a 2-cobordism class.

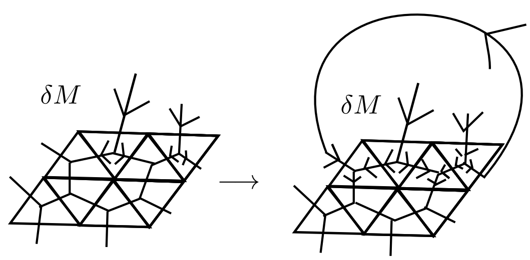
4.1. **Nondegenerate Triangulated Construction.**

Definition 4.3 (Nondegenerate Triangulated Construction)**.** Fix a vector space A, g a nondegenerate bilinear pairing, and μ a multiplication. A *nondegenerate triangulated construction* associated with the 3-tuple (A, g, μ) is a map that takes a triangulated 2-cobordism $\triangle M$ to a string diagram in the category $(\mathbf{Vect_K}, \otimes, K)$ constructed in the following way:

1. (1) We do not assume that $\triangle M$ is path connected. Let us carry out the construction on each connected component.
2. (2) We draw the dual of the triangulation of $\triangle M$. To do this, for each edge of a triangle, draw a string intersecting it; for each face, draw a vertex that connects the three lines together:



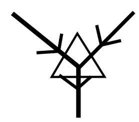
(3) To a dual diagram obtained from a triangulated 2-cobordism, we wish to associate the string diagram of a morphism in the category ($\mathbf{Vect_K}$, $\otimes$, K). To do this, we first have to specify directionality of arrows on the string. There are three cases to consider.
   (a) For a 2-cobordism with an in-boundary, we (1) assign an orientation to the string diagram associated with each *triangle with an edge at the boundary* in this way: all string diagrams should have arrows pointing towards the out-boundary. Then, (2) demand that all arrows in a single string (not separated by a vertex) face the same direction and (3) all arrows should be oriented towards the"out-boundary." We see that this is sufficient to construct a well-defined string diagram (with orientation of arrows), by an "inductive argument" : without loss of generality, assume that the triangles are assigned "row by row" and "column by column." Then, if we assign an orientation to the dual diagram asscociated with a *triangle with an edge at the boundary* in the manner stated above, our second demand forces only one possible definition on the entire "row" (in particular, all triangles with vertices at the boundary):



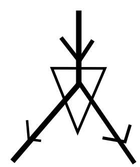
Looking at the diagram above, we see that each "row" is induced with an orientation by the assignment of arrows at the boundary vertex. By the third requirement, we induce the orientation of our string down the "column," i.e the interior of the surface.
   (b) For a 2-cobordism without boundary, we fix the direction of the dual diagram of one triangle, and demand that all arrows in a single string (not separated by a vertex) face the same direction. It is easy to see that this is sufficient to assign a string diagram.
   (c) For a 2-cobordism with an out-boundary but no in-boundary, we (1) assign an orientation to the string diagram associated with each triangle with an edge at the out-boundary in this way: all string diagrams should have arrows pointing towards the out-boundary. Then, (2) demand that all arrows in a single string (not separated by a vertex) face the same direction and (3) all arrows should be oriented towards

the "out-boundary." Applying a similar argument as the case with of a 2-cobordism with an in-boundary, we see that such an assignment induces an orientation to the dual string diagram "upwards." Again, this assignment is well-defined.
Defined this way, all arrows in a our string diagram are oriented towards the out-boundary, hence associate all arrows with the vector space A.

(4) With this construction, we see that there are only two possibilities for the string diagram associated with a triangle, either:



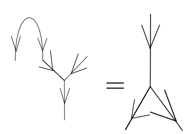
or



Hence, we need to specify the linear maps associated with the first and second diagram.

(5) The linear map corresponding to the first string diagram is associated with the fixed multiplication μ.

(6) Given μ and a nondegenerate pairing g we can define a δ, a comultiplication to deal with the second diagram since we have a copairing $\bar{g}$:



Similarly, this map is well-defined and linear.

(7) Carry out the steps 2–6 for each connected component and we obtain a string diagram. The associated string diagram with $\triangle M$ is obtained by taking tensor products of the string diagrams obtained from each connected component.

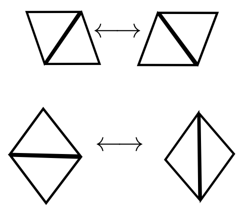
(8) Completing this construction, we see that we have obtained a string diagram. We read it as morphism in $(\mathbf{Vect_K}, \otimes, K)$.

Essentially, the construction associates with every triangulated 2-cobordism, a linear map. The associated linear map with $\triangle M$ is defined to be that map with domain $A^{\otimes n}$ and codomain $A^{\otimes m}$ where n and m are the number of edges at the boundary of $\triangle M$ (if any) obtained by reading the resultant diagram above as a string diagram as a morphism in $(\mathbf{Vect_K}, \otimes, K)$. Such a map is obviously linear as it is the composition of linear maps.

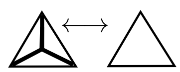
4.2. **FHK Construction.** We can now define the notion of an FHK construction as a particular type of a triangulated construction. The motivation behind the FHK construction is that it should, after some adjustment, produce a functor that is invariant under triangulation.

Theorem 4.4. *Let T_1 and T_2 be two triangulations of M, compact surface with $\partial M = \emptyset$. Then there exists a finite sequence of Pachner moves that transforms T_1 into T_2*

The Pachner moves come in two flavors:
(2-2) move:

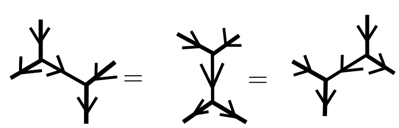


(1-3) move

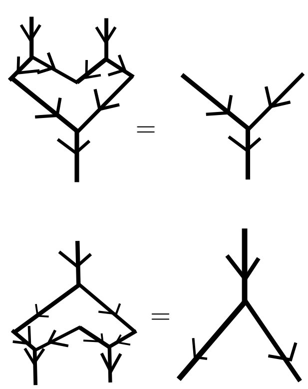


Since the triangulated construction associates a linear map to a triangulated 2-cobordism by considering its dual as a string diagram, the moves on triangulations will also affect the moves on duals. Thus, the definition of an FHK construction should not be surprising.

Definition 4.5 (FHK construction)**.** An *FHK 3-tuple* consists of a vector space A, a nondegenerate form g and a multiplication μ such that, after the construction of comultiplication above, the multiplication μ and the comultiplication δ satisfies the valid moves for associativity and coassociativity, along with these valid moves:
(2-2) move



(1-3) move



An *FHK construction* is any triangulated construction associated with an FHK-tuple.

Note that these valid moves are just the string diagrams corresponding to the duals of the two Pachner moves defined above. In particular, the 2-2 move results in four valid moves (because, by construction, there are 4 possibilities for the string diagram associated with every pair of triangles) and the 1-3 move results in 2 valid moves (each for multiplication and comultiplication). We can prove a theorem that essentially defines an LTFT.

Proposition 4.6. *Given an FHK construction, we can construct a 2D-TQFT, i.e. a strong symmetric monoidal functor, $F : \mathbf{2-Cob} \longrightarrow \mathbf{Vect_K}$ that is triangulation-independent.*

Proof. First and foremost, fix an FHK construction, i.e fix the FHK 3-tuple (A, g, c).

We begin by noting that any triangulated 2-cobordism, determines a triangulation of its in and out-boundary components; in particular they are diffeomorphic to some 1-manifold with triangulation. We say that two triangulated 2-cobordisms, $\triangle M : \triangle X \longrightarrow \triangle Y$, $\triangle' M' : \triangle' Y \longrightarrow, \triangle' Z$ are composable if there exists $\triangle'' M'M : \triangle X \longrightarrow \triangle' Z$ such that $\triangle M'M$ is a representative of the 2-cobordism class obtained by composing the (untriangulated) 2-cobordisms M and M'.

Furthermore, $\triangle'' M'M$ is diffeomorphic to $\triangle' M' \amalg_{\triangle Y} \triangle M$. Hence, it is immediate that two 2-cobordisms are composable if and only if $\triangle Y$ is diffeomorphic to $\triangle' Y$ as 1-manifolds.

Therefore, denoting the FHK construction on any triangulated 2-cobordism, $\triangle N$, by $\bar{F}(\triangle N)$ we see that, up to diffeomorphism,

$$\bar{F}(\triangle'' M'M) = \bar{F}(\triangle' M' \circ \triangle M) = \bar{F}(\triangle M) \circ \bar{F}(\triangle M') \tag{4.7}$$

From hereon in, we will consider all "=" relations up to diffeomorphism. The functor construction goes through a sequence of steps.

(1) We claim that $\bar{F}((\triangle((S^1)^n \times [0,1]))$ is idempotent for $n = 0, 1, 2, 3...$

Proof. The $n = 0$ case is trivial. Otherwise, fix a triangulation of $(S^1)^n \times [0,1]$ and denote it as $\triangle_1((S^1)^n \times [0,1])$. We want to show that the linear operator obtained by applying the functor $\bar{F}$ idempotent, i.e. $\bar{F}(\triangle_1((S^1)^n \times [0,1]))^2 = \bar{F}(\triangle_1((S^1)^n \times [0,1])) \circ \bar{F}(\triangle_1((S^1)^n \times [0,1])) = \bar{F}(\triangle_1((S^1)^n \times [0,1]))$.

Note that $\triangle_1((S^1)^n \times [0,1])$ may not be composable with itself, but we can always re-triangulate $\triangle_1((S^1)^n \times [0,1])$ using a finite sequence of Pachner moves to get $\triangle_2((S^1)^n \times [0,1])$, such that $\triangle_1((S^1)^n \times [0,1])$ and $\triangle_2((S^1)^n \times [0,1])$ are composable. Hence,

$$\begin{aligned}\bar{F}(\triangle_1((S^1)^n \times [0,1]))^2 &= \bar{F}(\triangle_1((S^1)^n \times [0,1])) \circ \bar{F}(\triangle_2((S^1)^n \times [0,1])) \\ &= \bar{F}(\triangle_1((S^1)^n \times [0,1]) \circ \triangle_2((S^1)^n \times [0,1])) \\ &= \bar{F}(\triangle_1((S^1)^n \times [0,1]))\end{aligned}$$

□

(2) We construct F by first stating where it sends *objects* of $\mathbf{2-Cob}$; denote $Range(G)$ by $RanG$, where G is any morphism in $\mathbf{Vect_K}$. Then define:

$$F(\underline{n}) = Ran\bar{F}(\triangle((S^1)^n \times [0,1])) \tag{4.8}$$

Since $\bar{F}$ is an FHK construction, we have that $F(\underline{n})$ depends only on the value of n.

(3) Let us state where the *morphisms* of $\mathbf{2-Cob}$ is sent to. Let $M \in Hom(\underline{n}, \underline{m})$, and $\triangle M$ be a triangulation. then define:

$$F(M) = \bar{F}(\triangle M)\mid_{F(\underline{n})} \tag{4.9}$$

(4) Finally, we claim that this is a strong symmetric monoidal functor from the category $\mathbf{2-Cob}$ to $\mathbf{Vect_K}$:

Proof. We check the necessary axioms.

(a) We claim that given a 2-cobordism, M, F *preserves domain and codomain.* To see this, fix $M \in Hom(\underline{n}, \underline{m})$, and fix a triangulation of M, $\triangle M$. Then,

$$\begin{aligned} F(M)(F(\underline{n})) &= \bar{F}(\triangle M)Ran\bar{F}(\triangle((S^1)^n \times [0,1])) \\ &= Ran\bar{F}(\triangle M \circ \triangle((S^1)^n \times [0,1])) \end{aligned}$$

where triangulation of $((S^1)^n \times [0,1])$ is induced by the triangluation of M.

Now, $\triangle M \circ \triangle(S^1)^n \times [0,1]$ is diffeomorphic to some triangulation of M, $\triangle' M$, hence $Ran\bar{F}(\triangle M \circ \triangle((S^1)^n \times [0,1])) = Ran\bar{F}(\triangle' M) = Ran\bar{F}(\triangle M)$ by triangulation invariance. Analogously, $\triangle((S^1)^m \times [0,1]) \circ \triangle M$ is diffeomorphic to some triangulation of M, $\triangle'' M$; hence, by triangulation invariance,

$$\begin{aligned} Ran\bar{F}(\triangle M) &= Ran\bar{F}(\triangle'' M) \\ &= Ran\bar{F}(\triangle((S^1)^m \times [0,1]) \circ \triangle M) \end{aligned}$$

which is a subset of $Ran\bar{F}(\triangle((S^1)^m \times [0,1])) = F(\underline{m})$.

(b) We check that *identities are preserved.* Fix, $\underline{n}$. We claim that $F(Id_{\underline{n}}) = Id_{F(\underline{n})}$. This follows from the fact that $F(Id_{\underline{n}}) = \bar{F}(\triangle((S^1)^n \times [0,1]))$ (up to diffeomorphism) and that $\bar{F}(\triangle((S^1)^n \times [0,1]))$ is an idempotent operator.

(c) We check that *composition is preserved.* Given two composable 2-cobordisms, M and M', the property that $F(M \circ M') = F(M) \circ F(M')$ is inherited from (5.9).

(d) By construction of $\bar{F}$, we note that F is obviously a strong monoidal functor. Lastly, F is symmetric due to the auxiliary theorem (5.1).

□

Hence, we conclude that we have constructed a 2D-TQFT from an FHK construction.

□

Furthermore, one can easily check the following corollary as a consequence of the construction above:

Corollary 4.10. *Fix an FHK construction. Then it determines a unique 2D-TQFT up to natural isomorphism.*

Proof. To see this, for any nondegenerate triangulated construction (A, g, c), the resulting linear map associated with each 2-cobordism depends only on triangulation since we fixed g and c and constructed μ and δ by hand. If, furthermore, our nondegenerate triangulated construction is an FHK construction, then the linear map associated with each 2-cobordism is independent of triangulation. □

We finally have the definition:

Definition 4.11. A 2-dimensional Lattice Topological Field Theory (LTFT) is a 2D-TQFT constructed from an FHK construction by (4.9) up to natural isomorphism.

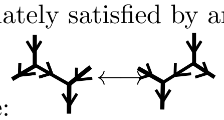
5. The Main Theorem

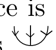
We will now prove the main theorem of 2D-LTFTs, which is in similar vein as the main theorem of 2D-TQFTs. Our proof differs from the original one, and follows the general approach of Baez.

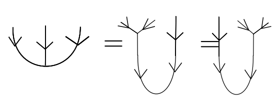
Note that our construction above does not necessitate that a 2D-LTFT exists. The problem is that to have an FHK construction, we need to associate the machinery defined above with a 3-tuple that works. The main theorem first shows that, by choosing A to be a finite-dimensional semisimple algebra, with a multiplication μ and g to be its trace form, we can define an FHK 3-tuple. And, conversely, an FHK 3-tuple must confer a semisimple structure on A.

Theorem 5.1 (Sufficiency)**.** *Let A be a semisimple algebra with multiplication μ and g as its trace form. Then, (A, g, μ) is an FHK 3-tuple*

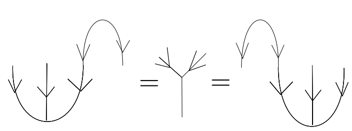
Proof. Let (A, μ) be a finite-dimensional semisimple algebra. By Dickson's theorem [1.3], we have a nondegenerate pairing, the trace form. Call this pairing g. Hence, there exists a nondegenerate triangulated construction associated with A. The valid move for associativity is immediately satisfied by any algebra.

We first prove the valid move: 

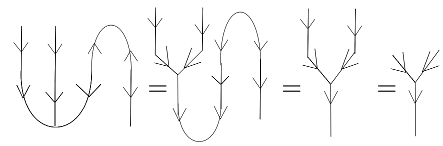
(1) First, define an function: $c : A \otimes A \otimes A \longrightarrow K$ by $c = g(\mu \otimes id_A)$. We note that the the trace is an associative pairing, hence $\bar{c} = g(id_A \otimes \mu)$. Hence, representing c as , we have the following valid moves:



(2) We claim that we can recover our multiplication this way:

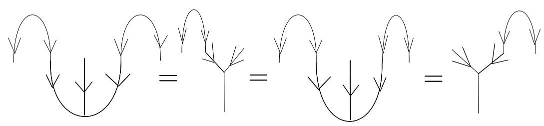


Proof. We prove the first relation, since the second follows immediately from (1). First, apply shifting on all the incoming and outgoing arrows, then use the valid move in 1. Using the fact that g is nondegenerate, we can use *Straightening II* as a valid move and the proof is complete.



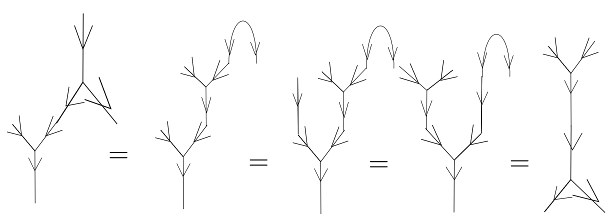
□

(3) It follows immediately, from the definition of comultiplication from a non-degenerate triangulated construction that:



(4) We are now ready to prove the claim:

Proof. The claim follows by showing that can we apply a sequence of valid moves to go from the right hand side to an intermediary step. Going from the left hand side to the intermediary step is completely analagous. First, we apply the definition of a comultiplication, then apply *shifting* and *associating*. Lastly, we re-apply the definition of a comultiplication.



□

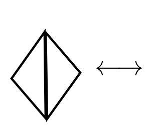
We omit the proof for coassociativity since the argument is lengthy, but the crux of the proof is noting that c is invariant under cyclic permutations of its domain, and applying the definition of comultiplication, the valid move follows.

We now show the valid move associated with the 1-3 move.

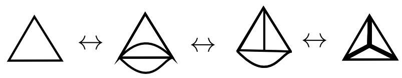
(5) We start with the following lemma:

Lemma 5.2. *The 1-3 move is equivalent to the 2-2 move and the bubble move.*

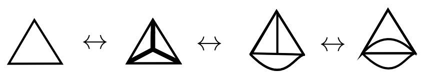
Proof. The following is a bubble move defined by FHK:



Assuming that we have the 2-2 move and the bubble move, we can first apply the bubble move, then the 2-2 move:

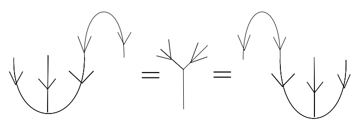


Assuming that we have the 1-3 move and the 2-2 move, we can first apply the 1-3 move, then the 2-2 move:



□

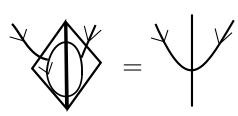
(6) Hence, it suffices to show that the move arising out of the dual of the bubble move is valid. Recall that the multiplication is recovered in in this way:



Since comultiplication is defined by "gluing" another copairing, we can think of our triangulated construction as associating every triangle with the morphism $c : A \otimes A \otimes A :\longrightarrow K$.

Now, the requirement that no two arrows on an edge of the dual diagram face each other, ensures that all copairings composed with c is associated with the pairing g such that their composition straightens. Therefore, every edge of the original triangulation is associated with both a pairing and a copairing. Hence we can choose to associate each edge with either the pairing g or the copairing $\bar{g}$. Without loss of generality, we choose g.

Therefore, the dual of an edge of a triangle is just g that we started with. Hence the following is the valid move that we obtain from the bubble move (with the triangles drawn in):



The validity of this diagram is true by the computation for the trace form in Section 3, and the fact that we started the construction by using the trace form as g.

□

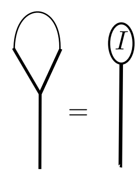
Theorem 5.3 (Necessity)**.** *If (A, g, μ) is an FHK 3-tuple, then necessarily g is its trace form, and (A, μ) is a semisimple algebra.*

Proof. By the previous theorem, we know that a such a 3-tuple exists. A is obviously an algebra (not necessarily associative or unital) and the condition that g is nondegenerate necessitates that A is finite-dimensional by [2.14].

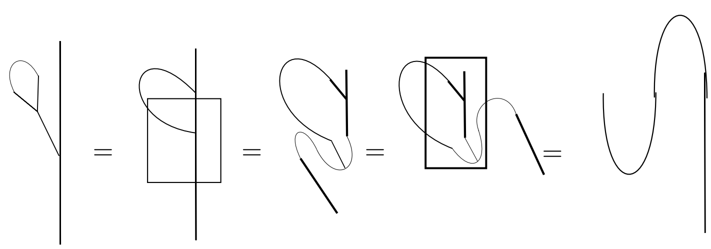
Now, by lemma 5.2, we could have defined an FHK 3-tuple in terms of the valid move arising out dual of the bubble move instead of the valid moves arising out of the dual of the 1-3 move. Now, the valid moves in the definition of an FHK 3-tuple tells us that A, μ and g must be chosen such that μ and δ satisfies the valid moves arising out of the 2-2 move and the bubble move. In particular, A is necessarily an associative algebra. We need only show two more things: that A is unital, and that A is semisimple.

(1) We claim that A is unital.

Proof. Note that since g is nondegenerate, we can fix an identification of A with A^* as vector spaces - namely, via g itself. Therefore, there is no need to draw arrows, and there is only one kind of straightening - in particular, we can always straighten strings whenever there are two curves. Now, define the unit as the map $\mu \circ \bar{g} : K \longrightarrow A$:



With this definition, we check that the valid move for unitality is satisfied via the diagram below. We first, apply associativity, then note that the boxed region of the string diagram is a multiplication and we apply the valid moves in (5.1.1). Apply the valid move arising out of the bubble move in the next boxed region to see that we get two curves which straightens.



□

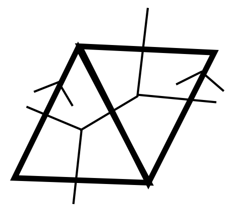
(2) Lastly, we claim that A is semisimple. By the argument in (5.1.6), we see that g is necessarily the trace form since the 1-3 move is defined to be valid. Now g is assumed to be nondegenerate, hence A is a semisimple algebra.

□

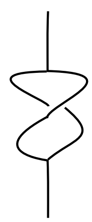
5.1. **The Auxiliary Theorem: Symmetry Requirements.** In this section, we complete the proof that the functor we constructed out of an FHK construction (4.6) is indeed a 2D-TQFT by showing that the functor is symmetric. In particular, we are done if our functor maps objects of $\mathbf{2}-\mathbf{Cob}$ into the centre of the algebra, $Z(A)$. We need only show this on the generator for objects, $\underline{1}$.

Theorem 5.4. $F(\underline{1}) = Z(A)$

Proof. Let us compute the linear map $\bar{F}(\triangle(S^1 \times [0,1]))$. Pick the simplest triangulation of $S^1 \times [0,1]$ and then draw the dual diagram, noting that, by (5.3.1) we need not draw arrows.

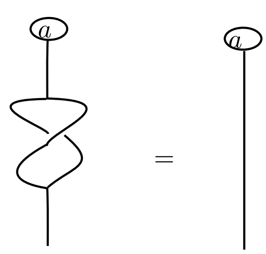


Now, identifying the edges of the triangle (and hence the dual diagram) the linear map is represented by this string:

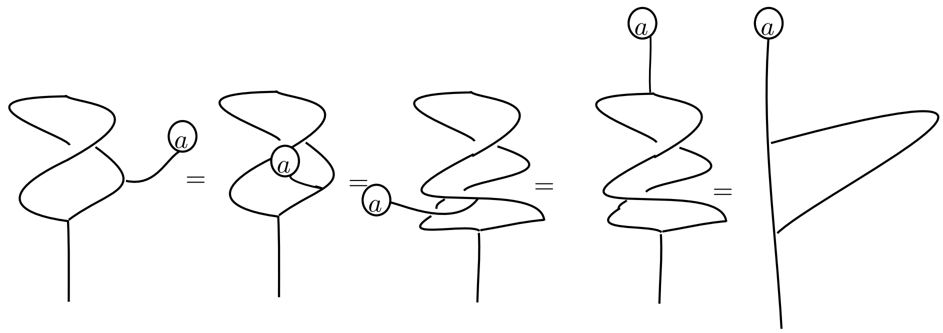


which is the map: $\mu \circ B_{A,A} \circ \delta : A \longrightarrow A$.

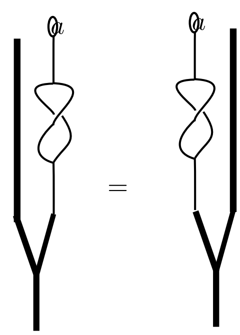
We first claim that if $a \in Z(A)$ then $a \in F(\underline{1}) = Ran\bar{F}(\triangle(S^1 \times [0,1]))$. For convenience set ϕ as the linear map $\bar{F}(\triangle(S^1 \times [0,1]))$. Since ϕ is idempotent, it suffices to show that for any $a \in Z(A)$, $\phi(a) = a$. Fixing a, it suffices to show that the following move is valid:



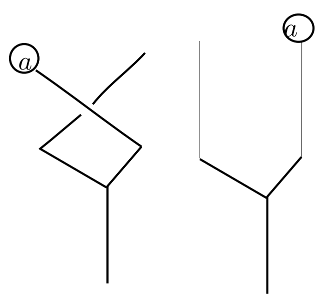
First, use *sliding* to a position where we can apply associativity. However, instead of applying associativity immediately, we use the fact that a is in $Z(A)$ to move it inside the "bubble." In this way applying associativity is the same as twisting two strings past one (this can easily be checked by (5.1.1)). Now, move slide a back to the top and use *untwist I* to see that we obtain a string diagram which we can validly turn into the string diagram for a by the argument in (5.3.1):



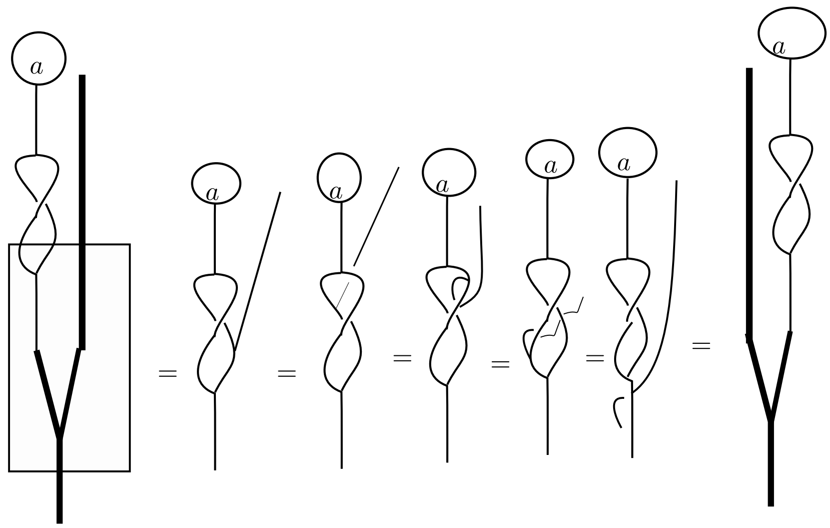
On the other hand, it suffices to show that for any $a \in A$, $\mu(\phi(a)\otimes b) = \mu(b\otimes\phi(a))$ for all $b \in A$:



First, we claim that the following move is valid:



This is true since we have a natural isomorphism between $V \otimes K$ and $K \otimes V$. Hence, if we had labelled K by a, the same move will be valid. Now to prove our claim, simply use associativity on the region marked by the box, then use *sliding* to get the appropriate diagram for application of the above move:



∎

6. An Elementary Proof of Mednykh's Formula

As an application of 2D-LTFTs, we provide an elementary proof of the following theorem due to Mednykh:

Theorem 6.1 (Mednykh, 78)**.** *Let G be a finite group, and S be a closed, orientable surface. Then the following formula is true:*

$$\sum_{V\in\hat{G}} dim(V)^{\chi(S)} = \#G^{\chi(S)-1}\#Hom(\pi_1(S),G) \tag{6.2}$$

Here, $\hat{G}$ refers to the set of isomorphism classes of irreducible representations of G, $dim(V)$ refers to the dimension of V, $\chi(S)$ is the Euler characteristic of S and $\pi_1(S)$ is the fundamental group of S. It is noteworthy that this formula traverses many areas of mathematics! First, recall three basic theorems from representation theory of finite groups.

Theorem 6.3 (Maschke)**.** *If K is a field of characteristic zero, then the group algebra $K[G]$ is semisimple.*

Theorem 6.4 (Artin-Wedderburn)**.** *Let K be a field of characteristic zero. Let $\rho_i : G \longrightarrow GL(W_i)$ be a class of irreducible representations of G (there are finitely many of them for a finite group). Extend ρ_i to an algebra homomorphism ρ_i : $K[G] \longrightarrow End(W_i) \cong M_{n_i}(K)$, where n_i is the dimension of W_i Then we have the isomorphism of algebras:*

$$K[G] \cong \Pi_{i=1}^{h} M_{n_i}(K) \tag{6.5}$$

Theorem 6.6. *Let G be a finite group, and let 1 the identity element of its group algebra over the complex numbers. Then $Tr(1) = \#G$ and $Tr(s) = 0$ for $g \neq 1$.*

Proof. Follows from the fact that this is true for the regular representation of G which uniquely extends to the $\mathbb{C}$[G] module $\mathbb{C}[G]$. ∎

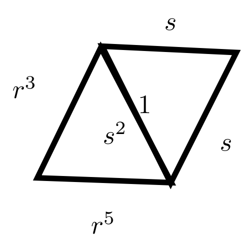
Hence, as shown above, the semisimple algebra, $\mathbb{C}$[G], defines an FHK construction. We will denote by $F_{\mathbb{C}[G]}(\triangle M)$, the linear map associated with a triangulated 2-cobordism $\triangle M$ via FHK construction (suppressing the choice of g and μ). In this case, we are interested in 2-cobordisms without boundary, i.e closed orientable surfaces. Note that the linear map associated with the 2-cobordism via an FHK construction has $\mathbb{C}$ as domain and codomain — it is simply a complex number.

Now, by a *flag*, we mean a pair (face, edge) such that the edge is contained in the face. A pair of triangles thus have 6 flags. We note by (5.1.5) that in the FHK construction, we associate each flag with A, and we can choose to associate with each edge, a copairing of g denoted by $\bar{g}$ (this is a more convenient choice for our current purposes), and each edge with a linear map $Tr \circ (\mu \otimes id_A) = Tr \circ (\mu \otimes id_A) = c : A \otimes A \otimes A \longrightarrow K$.

The next definition requires us to be more careful:

Definition 6.7. Let G be a finite group and $\triangle S$ a triangulated closed surface. By a *consistent labelling* of $\triangle S$ by G we mean a way of labeling *all* flags of $\triangle S$ such that (1) the product around each triangle is 1 and (2) an edge shared by two triangles (hence, two flags) are assigned the group element and its inverse.

This is an example of a consistent labelling of two triangles using the generators of the dihedral group, D_8:



Note We call the number of consistent labelings on $\triangle S$ by G as $Z(G, \triangle S)$.

Proposition 6.8. *Let S be a closed, orientable surface and G a finite group, $\bar{g}$ a nondegenerate copairing that sends the identity to $\frac{\sum g \otimes g^{-1}}{\#G}$, and $c = Tr(\mu \otimes id_{\mathbb{C}[G]}$. Fix a triangulation on S and call the triangulated closed surface $\triangle S$. Then, $F_{\mathbb{C}[G]}(\triangle S) = \#G^{\#F - \#E} Z(G, \triangle S)$. Moreover, for any other triangulation of S, $\triangle' S$, $F_{\mathbb{C}[G]}(\triangle S) = F_{\mathbb{C}[G]}(\triangle' S)$. Therefore, we can write $F_{\mathbb{C}[G]}(S) = \#G^{\#F - \#E} Z(G, S)$*

Note that it suffices to show for one triangulation of S since $F_{\mathbb{C}[G]}(S)$ is triangulation-invariant.

Proof. We note that, by construction, $F_{\mathbb{C}[G]}(\triangle S) : K = K^{\otimes \#edges} \longrightarrow \mathbb{C}[G]^{\otimes \#flags} \longrightarrow K^{\otimes \#faces} = K$. by first applying the map $\bar{g} : K \longrightarrow \mathbb{C}[G] \otimes \mathbb{C}[G]$ and $Tr : \mathbb{C}[G] \otimes \mathbb{C}[G] \otimes \mathbb{C}[G] \longrightarrow K$. Note that we can write a labeling as an ordered array: $\bigotimes g \otimes g^{-1}$. Moreover, by theorem 6.6, it follows that

$$Tr(\bigotimes g \otimes g^{-}1) = \begin{cases} \#G & \text{if } \bigotimes g \otimes g^{-1} \textit{ consistent} \\ 0 & \textit{otherwise} \end{cases} \tag{6.9}$$

WIth this, we can carry out the computation:

$$\begin{aligned} 1 &\mapsto \bigotimes_{edges}(1/\#G\sum g\otimes g^{-1}) \\ &= \#G^{-\#edges}(\sum_{flag\,labelings}(\bigotimes_{edges} g\otimes g^{-1})) \\ &\mapsto \#G^{-\#edges}\#G^{\#faces}Z(G,\triangle S) \end{aligned}$$

□

Theorem 6.10. $F_{\mathbb{C}[G]}(S) = \#G^{\chi(S)-1}\#Hom(\pi_1(S), G)$

Proof. Fix a base vertex of S, v_0, and fix oriented paths P_v from v_0 to every other vertex v. To see this, we construct a bijection between the consistent labellings of $\triangle S$ and the set $G^{V-\{v_0\}} \times Hom(\pi_1(S), G)$, $\Phi : \{Consistent\ Labelings\} \longrightarrow G^{V-\{v_0\}} \times Hom(\pi_1(S), G)$

Consider a consistent labelling f as a map from the oriented edges of $\triangle S$ to G. We first compute the group element in $G^{V-\{v_0\}}$ associated with f. For a vertex $v \neq v_0$, we can compute its associated group element with respect to v_0 by multiplying the group elements on the edges traversed by the path, $\Pi_{edge\in P_v} f(edge)$. Compute this for every vertex and we obtain a $\#(V-\{v_0\})$-tuple, which is an element in $G^{V-\{v_0\}}$. Since we have fixed the oriented paths, we see that this computation is well-defined. Now, for every loop, we assign the group homomorphism that takes a loop class $[L] \in \pi_1(S, v_0)$ to the group element obtained by multiplying all group elements associated with the edges traversed in this loop, $\Pi_{edge\in[L]} f(edge)$. By the consistency condition, we see that this assignment only depends on the loop class of L. Hence, this map is injective.

Conversely, if we have a set of cardinality $V - \{v_0\}$, $\{f(P_v)\}_{v\neq v_0}$, consisting of group elements which tells us which group elements correspond to each P_v for all v, and a fixed homomorphism that assigns a loop class $[L]$ to a group element, we can recover the consistent labelings on $\triangle S$. We compute the labeling on an arbitrary edge, E from v to v': Let L be the loop $P_{v'^{-1}} \circ E \circ P_v$. Then $f(E) = f(P_v')f(L)f(P_v^-1)$. Obviously, the value of $f(E)$ depends only the the choice of $\{f(P_v)\}_{v\neq v_0}$ and the chosen homomorphism. Therefore, $F_{\mathbb{C}[G]}(S) = \#G^{\#F-\#E}Z(G,S) = \#G^{\#F-\#E-\#V-1}\#Hom(\pi_1(S), G)$. Since $\chi(S) = \#F - \#E - \#V - 1$, the result follows.

□

Analogously, we can define $Z(M_n, \triangle S)$ as the number of consistent flag labelings of $\triangle S$ by the basis of the matrix algebra $M_n(\mathbb{C})$ which we write as e_j. We say that a flag labeling is consistent when two flags sharing the same edge is labelled by (i,j) and (j,i), and a flag adjacent to (i,j) is labelled (j,k). With this, we claim an analogous result:

Proposition 6.11. *Let S be a closed, orientable surface, $M_n(\mathbb{C})$ the matrix algebra over a complex numbers, $\bar{g}$ a nondegenerate copairing that sends the identity to $\frac{\sum_{i,j} e_{i,j}\otimes e_{j,i}}{n}$, and c sends all triples of the form $e_{i,j}\otimes e_{j,k}\otimes e_{k,i}$ to n and 0 otherwise. Fix a triangulation on S and call the triangulated closed surface $\triangle S$. Then, $F_{M_n(\mathbb{C})}(\triangle S) = n^{\#F-\#E}Z(M_n, \triangle S)$.*

Moreover, for any other triangulation of S, $\triangle' S$, $F_{M_n(\mathbb{C})}(\triangle S) = F_{M_n(\mathbb{C})}(\triangle' S)$. Therefore, we can write $F_{M_n(\mathbb{C})}(S) = \#n^{\#F-\#E}Z(M_n, S)$

Proof. Completely analogous to [6.8]. □

Now, we note that $Z(M_n, S) = n^{\#V}$. To see this, simply note that a consistent label is equivalent to a way in which we associate each vertex with a number k, $1 \leq k \leq n$. Therefore we have $F_{M_n(\mathbb{C})}(S) = \#n^{\chi(S)}$. Hence, we can finally prove Mednykh's formula:

$$\begin{aligned} \sum_{V \in \hat{G}} dim(V)^{\chi(S)} &= \sum F_{M_{dim(V)}(\mathbb{C})}(S) \\ &= F_{\mathbb{C}[G]}(S) \\ &= \#G^{\chi(S)-1} \#Hom(\pi_1(S), G) \end{aligned}$$

The first equality is given by by $F_{M_n(\mathbb{C})}(S) = \#n^{\chi(S)}$, the second equality is given by Theorem 6.4 and the last equality is given by Theorem 6.10.

7. Acknowledgements

First, and certainly foremost, I would like to thank my mentor Evan Jenkins for his guidance throughout the REU, and for introducing me to this fascinating topic. His lucid and engaging explanations on this rather abstract topic are simply first class. Secondly, I would like to thank Professor J. P. May for two excellent courses this summer - the first is the inspiration for this paper (hopefully, traces of that course are evident in this paper), while the second (although it is beyond me at times) will no doubt be the highlight of this summer's mathematics! Thirdly, I would like to thank Lichen for sharing (starting?) my enthusiasm for mathematics (especially this summer), and her infinite encouragement. Last, but certainly not least, I would like to thank Professor Paul J. Sally Jr. to whom I owe immeasurable (uncountable) debt for having truly educated me this past academic year, and providing me with the means to live this summer.

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