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FIBONACCI ASSOCIATION

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THE THEORY OF SIMPLY PERIODIC NUMERICAL FUNCTIONS*

Edouard Lucas, Lycée Charlemagne, Paris

Translated from the French by Sidney Kravitz, Dover, N. J.
Translation Edited by
Douglas Lind, University of Virginia, Charlottesville, Va.

INTRODUCTION

This is a translation of the first part of Lucas' memoire, "Théorie des Fonctions Numériques Simplement Périodiques," which appeared in the American Journal of Mathematics, Volume 1 (1878), pp. 184–240. The second half of the paper, published in pages 289–321 of the same volume, investigates periodicity of second-order recurring sequences modulo a prime, and applies this to the study of large primes.

The decision to publish a translation of Lucas' memoire was made for two reasons. First, although it is the first lengthy paper containing fundamental results about recurring sequences, the original is difficult to lay one's hands on. Secondly, the relationships and identities in the paper are constantly being rediscovered. Thus in order to help avoid needless duplication, it was decided to publish the first part of the paper in full. It is our hope that this will become a standard reference for such relationships and identities.

This translation diverges from the original only in a few places. Where an error has been detected in an equation, the corrected version of that equation has been used, together with a (*) to the left of the equation. We have also used more modern terminology where appropriate. For example, we have replaced $n(n-1) \cdots \cdot 2 \cdot 1$ by $n!$, $\sqrt{-1}$ by i, Log. nép. by log, and C_n^r by $\binom{n}{r}$.

The purpose of this paper is to study the symmetric functions of the roots of a quadratic equation, and their application to the theory of prime numbers. We indicate, at the start, the complete analogy of these symmetric functions with the circular and hyperbolic functions. We then show the relationship that exists between these symmetric functions and the theories of determinants, combinatorial analysis, continued fractions, divisibility, quadratic divisors, continued radicals, division of the circumference of a circle, quadratic diophantine analysis, quadratic residues, the decomposition of large numbers into prime factors, etc. This presentation is the starting point for a more complete

* First published in the American Journal of Mathematics, Vol. 1 (1878), pp. 184-240, 289–321.

study of the properties of the symmetric functions of the roots of an algebraic equation with rational coefficients of any degree, and their relationship to the theories of Elliptic and Abelian functions, power residues, and the diophantine analysis of higher degrees.

1. DEFINITION OF THE SIMPLY PERIODIC NUMERICAL FUNCTIONS

Let a and b denote the two roots of the equation

$$x^2 = Px - Q \tag{1}$$

whose coefficients P and Q are positive or negative relatively prime integers. We have

$$a + b = P, \quad ab = Q .$$

Denoting by δ the difference $a - b$ of the roots, and by Δ the square of this difference, we also have

$$a = \frac{P + \delta}{2}, \quad b = \frac{P - \delta}{2}, \quad \delta = \sqrt{\Delta} = \sqrt{P^2 - 4Q} .$$

We will consider the two numerical functions U and V defined by the equations

$$U_n = \frac{a^n - b^n}{a - b}, \quad V_n = a^n + b^n . \tag{2}$$

For all positive integer values of n these functions U_n and V_n give rise to three different kinds of series, depending on the nature of the roots a and b of equation (1). This equation may have

1) Real integer roots
2) Real irrational roots
3) Imaginary roots.

The numerical functions of the first kind correspond to all the integer values of a and b and can be directly calculated, for all positive integer values of n, by the use of the formulas (2). If we suppose more particularly $a = 2$ and $b = 1$, we find, on forming the values of U_n and V_n, the recurrent series

n:	0,	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	
U_n:	0,	1,	3,	7,	15,	31,	63,	127,	255,	511,	1023,	2047,	⋯
V_n:	2,	3,	5,	9,	17,	33,	65,	129,	257,	513,	1025,	2049,	⋯

studied for the first time by the illustrious Fermat. We will note, from now on, that the series V_n is contained, for the cases we will consider, in the series U_n, since the formulas (2) give us the general relation

$$U_{2n} = U_n V_n . \tag{3}$$

The numerical functions of the second kind correspond to all the irrational values of a and b whose sum and product are rational. We may calculate them as a function of the sum P and of the discriminant Δ of the given equation by means of the following formulas. The expansion of the binomial gives us

$$2^n a^n = P^n + \frac{n}{1} P^{n-1}\delta + \frac{n(n-1)}{2!} P^{n-2}\delta^2 + \frac{n(n-1)(n-2)}{3!} P^{n-3}\delta^3 + \cdots + \delta^n ,$$

$$2^n b^n = P^n - \frac{n}{1} P^{n-1}\delta + \frac{n(n-1)}{2!} P^{n-2}\delta^2 - \frac{n(n-1)(n-2)}{3!} P^{n-3}\delta^3 + \cdots + (-\delta)^n,$$

and by subtraction and addition,

$$\tag{4} \left\{ \begin{aligned} 2^{n-1}U_n &= \frac{n}{1} P^{n-1} + \frac{n(n-1)(n-2)}{3!} P^{n-3}\Delta + \frac{n(n-1)(n-2)(n-3)(n-4)}{5!} P^{n-5}\Delta^2 + \cdots \\ 2^{n-1}V_n &= P^n + \frac{n(n-1)}{2!} P^{n-2}\Delta + \frac{n(n-1)(n-2)(n-3)}{4!} P^{n-4}\Delta^2 + \cdots \end{aligned} \right.$$

Thus for the first few terms we obtain

$U_0 = 0$, $U_1 = 1$, $U_2 = P$, $U^3 = P^2 - Q$, $U_4 = P^3 - 2PQ$,

$V_0 = 2$, $V_1 = P$, $V_2 = P^2 - 2Q$, $V_3 = P^3 - 3PQ$, $V_4 = P^4 - 4P^2Q + 2Q^2$.

The simplest numerical functions of the second kind correspond to the assumptions

$$P = 1, \quad Q = -1, \quad \Delta = 5 ,$$

or to the equation

$$x^2 = x + 1 .$$

In this case we have

$$a = 2 \sin \frac{3\pi}{10} = \frac{1 + \sqrt{5}}{2}, \quad b = -2 \sin \frac{\pi}{10} = \frac{1 - \sqrt{5}}{2} ,$$

and consequently, denoting by u_n and v_n the resulting functions,

$$u_n = \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}} , \qquad v_n = \frac{(1 + \sqrt{5})^n + (1 - \sqrt{5})^n}{2^n} .$$

For the first few values of positive integers n the series are

n:	0,	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	...
u_n:	0,	1,	1,	2,	3,	5,	8,	13,	21,	34,	55,	89,	...
v_n:	2,	1,	3,	4,	7,	11,	18,	29,	47,	76,	123,	199,	...

The series u_n was considered for the first time by Leonard Fibonacci, of Pisa.[1] It has been studied by Albert Girard,[2] who noted that the three numbers u_n, u_n, u_{n+1} form an isosceles triangle whose apex angle is very nearly equal to the central angle of the regular pentagon. Robert Simson[3]

[1] Il liber Abbaci di Leonardo Pisano, pubblicato secondo la lezione del Codice Magliabechiano, da B. Boncompagni. Roma, 1867, pages 283–284.

[2] L'arithmetique de Simon Stevin, of Bruges, review, correction, and addition of several tracts and annotations by Albert Girard, Leide, 1633, pages 169-170.

[3] Philosophical Transactions of the Royal Society of London, Vol. xlviii, Part I, (1753). An explanation of an obscure passage in Albert Girard's commentary on Simon Stevin's works, pages 368 and those which follow.

remarked in 1753 that this series is obtained by the calculation of the quotients of the convergent fractions of the irrational expressions

$$\frac{\sqrt{5}+1}{2} \quad \text{and} \quad \frac{\sqrt{5}-1}{2} .$$

In 1843 J. Binet[1] gave, by means of this series, the expression for the number of discrete combinations. In 1844, Lamé[2] indicated the application that can be made of this series in determining the upper limit of the number of operations to be performed in finding the greatest common divisor of two integers.

From time to time we will also consider, for example, the series U_n of the second kind given by the assumptions

$$P = 2, \quad Q = -1, \quad \Delta = 2^2 \cdot 2 ,$$

or by the equation

$$x^2 = 2x + 1 .$$

We then have the series

n:	0,	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	....
U_n:	0,	1,	2,	5,	12,	29,	70,	169,	408,	985,	2378,	5741,	...
V_n:	2,	2,	6,	14,	34,	82,	198,	478,	1154,	2786,	6726,	16238,	...

which we call the Pell series in honor of the mathematician who was the first to solve[3] a celebrated problem of diophantine analysis proposed by Fermat concerning the solution in integers of the Diophantine equation

$$x^2 - \Delta y^2 = \pm 1 .$$

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[1]Comptes Rendus de l'Academie des Sciences de Paris, Vol. xvii, page 562; Vol. xix, page 939.

[2]Comptes Rendus de l'Academie des Sciences de Paris, Vol. xix, page 867.

[3]According to David Eugene Smith's A Source Book in Mathematics, Dover Publications, New York, 1959, p. 214, "The name Pell equation originated in a mistaken notion of Leonard Euler that John Pell was the author of the solution which was really the work of Lord Brouncker."—Translator's note.

The numerical functions of the third kind correspond to all the imaginary values of a and b whose sum and product are real and rational. The simplest result from the assumptions

$$P = 1, \quad Q = 1, \quad \Delta = -3.$$

We have, in this case,

$$a = \frac{1 + \sqrt{-3}}{2}, \quad b = \frac{1 - \sqrt{-3}}{2},$$

so that a and b are the imaginary cubic roots of negative unity. In addition

$$U_{3n} = 0, \quad U_{3n+1} = (-1)^n, \quad U_{3n+2} = (-1)^n.$$

Thus the values of U_n recur periodically in the order

$$0, \quad 1, \quad 1, \quad 0, \quad -1, \quad -1, \quad \cdots$$

leading to a large number of simple formulas, derived from the general properties of the functions U_n and V_n, concerning the trisection of the circumference of a circle.

From time to time we will also consider the analogous series derived from the equation

$$x^2 = 2x - 2,$$

for which

$$a = 1 + i, \quad b = 1 - i, \quad \Delta = -2^2,$$

and the series derived from the equation

$$x^2 = 2x - 3.$$

for which

$$a = 1 + \sqrt{-2}, \quad b = 1 - \sqrt{-2}, \quad \Delta = -2 \cdot 2^2 .$$

We will call this last series the conjugate Pell series.

II. RELATIONSHIP OF THE FUNCTIONS U_n AND V_n WITH THE CIRCULAR AND HYPERBOLIC FUNCTIONS

If we substitute

$$z = \frac{n}{2} \log \frac{a}{b}$$

into the formulas

$$\cos iz = \frac{e^z + e^{-z}}{2},$$

$$(\star) \qquad \sin iz = -\frac{e^z - e^{-z}}{2i},$$

we obtain

$$\cos\left(\frac{ni}{2} \log \frac{a}{b}\right) = \frac{1}{2}\left[\frac{a^{n/2}}{b^{n/2}} + \frac{b^{n/2}}{a^{n/2}}\right],$$

$$(\star) \qquad \sin\left(\frac{ni}{2} \log \frac{a}{b}\right) = \frac{-1}{2i}\left[\frac{a^{n/2}}{b^{n/2}} - \frac{b^{n/2}}{a^{n/2}}\right].$$

We then have two relations between the functions U_n and V_n and the circular functions

$$(5) \qquad \begin{cases} V_n = 2Q^{n/2} \cos\left(\frac{ni}{2} \log \frac{a}{b}\right), \\ U_n = \dfrac{2Q^{n/2}}{\sqrt{-\Delta}} \sin\left(\frac{ni}{2} \log \frac{a}{b}\right). \end{cases}$$

This immediately results in a correspondence between each formula in plane trigonometry and analogous formulas for U_n and V_n, and vice versa.

Thus the formula (3)

$$U_{2n} = U_n V_n ,$$

corresponds to the formula

$$\sin 2z = 2 \sin z \cos z .$$

The equations

$$V_n + \delta U_n = 2a^n, \quad V_n - \delta U_n = 2b^n , \tag{6}$$

which can easily be derived from the formulas (2) correspond exactly to the relations

$$\cos z + i \sin z = e^{iz}, \quad \cos z - i \sin z = e^{-iz} ,$$

and the formulas (4) are completely analogous to those which have been given in Actes de Leipzick, in 1701, by Jean Bernoulli, for the expansion of

$$\frac{\sin nz}{\sin z}$$

and of $\cos nx$ in terms of powers of sines and cosines of the angle z. Also the formulas

$$\begin{cases} [V_m + \delta U_m][V_n + \delta U_n] = 2[V_{m+n} + \delta U_{m+n}] , \\ [V_n + \delta U_n]^r = 2^{r-1}[V_{nr} + \delta U_{nr}] , \end{cases} \tag{7}$$

derived from the relations (6) coincide with the formulas

$$(\cos x + i \sin x)(\cos y + i \sin y) = \cos (x + y) + i \sin (x + y) ,$$
$$(\cos x + i \sin x)^r = \cos rx + i \sin rx ,$$

which have been given by de Moivre.

We will also note that if, in equation (1), we let

$$X = x^r, \quad \alpha = a^r, \quad \beta = b^r,$$

then α and β are the roots of the equation

$$X^2 = V_r X - Q^r . \tag{8}$$

Consequently each of the formulas which appears in this theory may be generalized by replacing U_n and V_n by U_{nr}/U_r and V_{nr}, P by V_r, Q by Q^r, and the difference δ of the roots a and b by the difference δU_r of the roots α and β of equation (8).

The formulas (4) thus become

$$2^{n-1}\frac{U_{nr}}{U_r} = \frac{n}{1}V_r^{n-1} + \frac{n(n-1)(n-2)}{3!}\Delta U_r^2 V_r^{n-3} + \frac{n(n-1)(n-2)(n-3)(n-4)}{5!}\Delta^2 U_r^4 V_r^{n-5} + \cdots$$

$$2^{n-1}V_{nr} = V_r^n + \frac{n(n-1)}{2!}\Delta U_r^2 V_r^{n-2} + \frac{n(n-1)(n-2)(n-3)}{4!}\Delta^2 U_r^4 V_r^{n-4} + \cdots$$

However we will bypass, for the moment, the other consequences of the transformation of equation (1) by the substitutions of the variable, as well as the study of the more general functions

$$AU_n + BV_n + C$$

in which A, B, and C are arbitrary positive or negative integers.

III. RECURRENCE RELATIONS FOR THE CALCULATION OF THE VALUES OF THE FUNCTIONS U_n AND V_n

The calculation of the values of U_n and V_n corresponding to consecutive integer values of n may be rapidly accomplished by means of the formulas completely analogous to those of Thomas Simpson:

$$\sin (n+2)z = 2 \cos z \sin (n+1)z - \sin nz$$

$$\cos (n+2)z = 2 \cos z \cos (n+1)z - \cos nz .$$

In effect we multiply two members of equation (1) by x^n and successively replace x by a and b. We obtain

$$a^{n+2} = Pa^{n+1} - Qa^n , \quad b^{n+2} = Pb^{n+1} - Qb^n ,$$

and, by subtraction and addition,

$$\text{(10)} \qquad \begin{cases} U_{n+2} = PU_{n+1} - QU_n \\ V_{n+2} = PV_{n+1} - QV_n \end{cases} .$$

These formulas show us that the functions U and V form, for consecutive integer values of n, two recurrent series of integers. These series have the same rule of formation, but the initial conditions differ. We will generalize these formulas by the use of symbolic calculus. Let F denote an arbitrary function. It is evident from equation (1) that

$$F(x^2) = F(Px - Q) .$$

If we replace x by a and b, we have

$$a^n F(a^2) = a^n F(Pa - Q) ,$$
$$b^n F(b^2) = b^n F(Pb - Q) ,$$

and, by subtraction and addition we obtain the symbolic equalities

$$\text{(11*)} \qquad \begin{aligned} U^n F(U^2) &= U^n F(PU - Q) , \\ V^n F(V^2) &= V^n F(PV - Q) . \end{aligned}$$

After the expansion we replace the exponents of U and V by indices, taking into account the zero exponent. Thus the symbols U^2 and $PU - Q$, V^2 and $PV - Q$ are respectively equivalent and may replace each other in the algebraic transformations.

In the Fibonacci series for example we have the following results

$$(12)\qquad \begin{cases} u^{n+p} = u^{n-p}(u+1)^p \\ u^{n-p} = u^n(u-1)^p \ , \end{cases}$$

which are completely analogous to those which we can obtain from combinatorial analysis or the arithmetic triangle and in particular from the formula for the binomial factorials due to Vandermonde.

Taking, as a starting point, the equation

$$x^2 = x - 1 \ ,$$

we will also find new relations between the coefficients of the same power of the binomial.

Consideration of equation (8) leads to the following relations

$$(13)\qquad \begin{cases} U_{n+2r} = V_r U_{n+r} - Q^r U_n \ , \\ V_{n+2r} = V_r V_{n+r} - Q^r V_n \ , \end{cases}$$

which permit the calculation of the values of the functions U_n and V_n which correspond to values of the argument n in arithmetic progression with difference r.

Inversely we will find, in the theory of circular and hyperbolic functions, formulas analogous to formulas (11) and (13).

IV. RELATIONSHIP OF THE FUNCTIONS U_n AND V_n WITH DETERMINANTS

We may express U_n and U_{nr}, V_n and V_{nr} by means of determinants. We have the formulas

$$\begin{array}{rcl} U_2 - PU_1 & = & 0, \\ U_3 - PU_2 + QU_1 & = & 0, \\ U_4 - PU_3 + QU_2 & = & 0, \\ U_5 - PU_4 + QU_3 & = & 0, \\ \cdots\cdots\cdots\cdots\cdots & & \cdots \\ U_{n+1} - PU_n + QU_{n-1} & = & 0 \ . \end{array}$$

From these we derive

$$(14) \qquad U_{n+1} = (-1)^n \begin{vmatrix} -P & 1 & 0 & 0 & \cdots \\ Q & -P & 1 & 0 & \cdots \\ 0 & Q & -P & 1 & \cdots \\ 0 & 0 & Q & -P & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}_n .$$

We also obtain

$$(15) \qquad V_n = (-1)^n \begin{vmatrix} -P & 2 & 0 & 0 & \cdots \\ Q & -P & 1 & 0 & \cdots \\ 0 & Q & -P & 1 & \cdots \\ 0 & 0 & Q & -P & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}_n .$$

We may verify these results by expanding the determinants using the elements of the last line of the last column. The values of U_{nr}/U_r and of V_{nr} may also be found by determinants by replacing, as we have done elsewhere, P by V_r and Q by Q^r.

Finally we note that these formulas are susceptible to an extensive generalization. In fact, in the formulas (11) which contain an arbitrary function we let n equal successively $1, 2, 3, \cdots, m$. We then obtain m equations from which we can determine the value of one or the other of the functions U or V.

REMARK — We may also develop U_n using the following formula,

$$(16) \qquad U_{n+1} = \begin{vmatrix} P & \sqrt{Q} & 0 & 0 & \cdots \\ \sqrt{Q} & P & \sqrt{Q} & 0 & \cdots \\ 0 & \sqrt{Q} & P & \sqrt{Q} & \cdots \\ 0 & 0 & \sqrt{Q} & P & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}_n .$$

However the use of formula (14) is preferable.

V. RELATIONSHIP OF THE FUNCTIONS U_n AND V_n WITH CONTINUED FRACTIONS

The functions U_n and V_n may be expanded by continued fractions. Let us consider the expression

$$\frac{R_n}{S_n} = a_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots + \cfrac{a_n}{b_n}}}} \; . \tag{17}$$

Let R_n and S_n be the numerator and the denominator of the n^{th} convergent. We know that

$$\left\{ \begin{aligned} R_{n+2} &= b_{n+2} R_{n+1} + a_{n+2} R_n \, , \\ S_{n+2} &= b_{n+2} S_{n+1} + a_{n+2} S_n \, . \end{aligned} \right. \tag{18}$$

In addition

$$R_n S_{n+1} - R_{n+1} S_n = (-1)^{n+1} a_1 a_2 a_3 \cdots a_{n+1} \, . \tag{19*}$$

Consequently if we let

$$a_0 = b_1 = b_2 = \cdots = b_n = P \, ,$$
$$a_1 = a_2 = a_3 = \cdots = a_n = -Q \, ,$$

we obtain the expression

$$\frac{U_{n+1}}{U_n} = P - \cfrac{Q}{P - \cfrac{Q}{P - \cfrac{Q}{P - \ddots}}} \tag{20}$$

in which n denotes the number of quantities equal to P.

Thus for the Fibonacci series we have

$$\text{(21)} \qquad \frac{1}{2}\,\frac{(1+\sqrt{5})^{n+1}-(1-\sqrt{5})^{n+1}}{(1+\sqrt{5})^{n}-(1-\sqrt{5})^{n}} = 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}}$$

For the Fermat series

$$\text{(22)} \qquad \frac{2^{n+1}-1}{2^{n}-1} = 3-\cfrac{2}{3-\cfrac{2}{3-\cfrac{2}{3-\ddots}}}$$

and in the Pell series

$$\frac{(1+\sqrt{2})^{n+1}-(1-\sqrt{2})^{n+1}}{(1+\sqrt{2})^{n}-(1-\sqrt{2})^{n}} = 2-\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2-\ddots}}}}$$

Moreover, in general we have

$$\text{(24)} \qquad \frac{U_{n+1}}{U_n} = a\,\frac{1-\left(\frac{b}{a}\right)^{n+1}}{1-\left(\frac{b}{a}\right)^{n}} \quad .$$

Let a denote the larger of the roots (in absolute value) of equation (1). We have

$$\text{(25)} \qquad \lim_{n\to\infty} \frac{U_{n+1}}{U_n} = a \quad .$$

However we note that this last result is not applicable to the series of the third kind, i. e., when the roots of the given equation (1) are imaginary.

By means of this last formula it is easy to rapidly calculate a term of this series U_n when we know only the preceding term. For example, in the Fibonacci series

$$u_{44} = 7014\ 08733\ ,$$

and

(⋆) $$a = \frac{1+\sqrt{5}}{2} = 1 \cdot 61803\ 39887\ 49894\ 8482 \cdots$$

By the abbreviated method, if we calculate the product $a \cdot u_{44}$ to the nearest whole number we find exactly (since u_n is an integer)

$$u_{45} = 11349\ 03170\ .$$

We can moreover directly determine the last digit of u_n. Thus in this particular case it is easy to see that two terms whose ranks differ by any multiple of 60 end in the same digit. If we suppose then that p is less than 60 we can show that the last digits of u_p and of u_q are complementary when the sum of p and q is equal to 60. We may then now suppose that p is less than (⋆: this was originally "equal to" — Ed. note) 60, and even p less than 15 if we note that the terms u_{15+p} and u_{15-p} have the same units digit when p is odd and that their final digits are complementary when p is even.

We have, more generally, the formula

(26) $$\frac{U_{(n+1)r}}{U_{nr}} = V_r - \cfrac{Q^r}{V_r - \cfrac{Q^r}{V_r - \cfrac{Q^r}{V_r - \ddots}}}$$

in which n is the number of V_r's. When n increases indefinitely,

(27) $$\lim_{n\to\infty} \frac{U_{(n+1)r}}{U_r} = a^r\ .$$

In the theory of circular functions formula (26) corresponds to the formula[1]

$$\frac{\sin (n+1)z}{\sin nz} = 2\cos z - \cfrac{1}{2\cos z - \cfrac{1}{2\cos z - \cfrac{1}{2\cos z - \ddots}}} \tag{28}$$

in which the expression $2 \cos z$ is repeated n times. We likewise have for the V_n the relation

$$\frac{V_{nr}}{V_{(n-1)r}} = V_r - \cfrac{Q^r}{V_r - \cfrac{Q^r}{V_r - \cfrac{Q^r}{V_r - \ddots - \cfrac{Q^r}{\left(\frac{V_r}{2}\right)}}}} \tag{29}$$

in which the quantity V_r is repeated n times.

The numerous properties of determinants and of continued fractions give rise to analogous properties for the functions U_n and V_n. Thus the well-known property of two consecutive convergents in formula (19) gives

$$\begin{cases} U_n^2 - U_{n-1}U_{n+1} = Q^{n-1} \\ V_n^2 - V_{n-1}V_{n+1} = -Q^{n-1}\Delta \end{cases} \tag{30}$$

and, more generally,

$$\begin{cases} U_{nr}^2 - U_{(n-1)r}U_{(n+1)r} = Q^{(n-1)r}U_r^2 \\ V_{nr}^2 - V_{(n-1)r}V_{(n+1)r} = -Q^{(n-1)r}\Delta\, U_r^2 \,. \end{cases} \tag{31}$$

In the theory of circular functions we have the analogous formulas

[1]Journal de Crelle, volume xvi, page 95, 1837.

$$\sin^2 x - \sin(x = y)\sin(x + y) = \sin^2 y$$

$$\cos^2 x - \cos(x - y)\cos(x + y) = \sin^2 y \quad .$$

It is moreover easy to directly verify the formula (31) by replacing U, V and Q as functions of a and b. Thus we also have

$$\begin{aligned} \Delta U^2_{n+r} &= a^{2n+2r} + b^{2n+2r} - 2Q^{n+r} \\ \Delta U^2_n &= a^{2n} + b^{2n} - 2Q^n \quad . \end{aligned}$$

Hence by subtraction

$$\Delta(U^2_{n+r} - Q^r U^2_n) = (a^{2n+r} - b^{2n+r})(a^r - b^r) \ ,$$

and consequently

(32) $$U^2_{n+r} - Q^r U^2_n = U_r U_{2n+r} \ .$$

We will have in the same manner the relation

(33) $$V^2_{n+r} - Q^r V^2_n = \Delta U_r U_{2n+r} \ .$$

For $r = 1$ formula (32) gives more particularly the relation

(34) $$U^2_{n+1} - QU^2_n = U_{2n+1} \ .$$

This last formulas has been applied by M. Gunther to show the Diophantine equation

$$y^2 - Qx^2 = Kz$$

in whole numbers[1]. It is easy to see that a large number of formulas from this and the following sections lead to analogous consequences, but with much more generality.

[1]Journal de Mathématiques pures et appliquées, de M. Resal, pages 331-341, October, 1876.

VI. EXPANSION OF THE FUNCTIONS U_n AND V_n AS A SERIES OF FRACTIONS

The formulas (30) lead to the expansions of U_{n+1}/U_n and V_{n+1}/V_n in series whose terms have for denominators the product of two consecutive terms of the series U and V. We have, in fact,

$$(\star)\qquad \frac{U_{n+1}}{U_n} = \frac{U_2}{U_1} + \left(\frac{U_3}{U_2} - \frac{U_2}{U_1}\right) + \left(\frac{U_4}{U_3} - \frac{U_3}{U_2}\right) + \cdots + \left(\frac{U_{n+1}}{U_n} - \frac{U_n}{U_{n-1}}\right).$$

By combining the fractions contained within each parenthesis

$$(35^\star)\qquad \frac{U_{n+1}}{U_n} = \frac{U_2}{U_1} - \frac{Q}{U_1U_2} - \frac{Q^2}{U_2U_3} - \frac{Q^3}{U_3U_4} - \cdots - \frac{Q^{n-1}}{U_{n-1}U_n}$$

We thus have for the Fibonacci series, by letting n increase indefinitely,

$$(36)\qquad \frac{1+\sqrt{5}}{2} = 1 + \frac{1}{1\cdot 1} - \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 8} - \frac{1}{8\cdot 13} + \cdots$$

Following the same procedure we obtain the more general formulas

$$(37)\qquad \frac{U_{(n+1)r}}{U_{nr}} = \frac{U_{2r}}{U_r} - \left[\frac{Q^r}{U_rU_{2r}} + \frac{Q^{2r}}{U_{2r}U_{3r}} + \cdots + \frac{Q^{(n-1)r}}{U_{(n-1)r}U_{nr}}\right]U_r^2$$

and

$$(38^\star)\qquad \frac{V_{(n+1)r}}{V_{nr}} = \frac{V_r}{V_0} + \left[\frac{1}{V_0V_r} + \frac{Q^r}{V_rV_{2r}} + \cdots + \frac{Q^{(n-1)r}}{V_{(n+1)r}V_{nr}}\right]\Delta U_r^2 .$$

We also derive the two relations

$$(39)\qquad \begin{aligned} U_{n+r}V_n - U_nV_{n+r} &= 2Q^nU_r \\ V_{n+r}V_n - \Delta U_nU_{n+r} &= 2Q^nV_r \end{aligned}$$

from which we will later derive the expansions

$$(40^\star)\qquad \begin{aligned} \frac{U_{n+kr}}{V_{n+kr}} &= \frac{U_n}{V_n} + 2Q^n U_r\left[\frac{1}{V_n V_{n+r}} + \frac{Q^r}{V_{n+r}V_{n+2r}} + \cdots + \frac{Q^{(k-1)r}}{V_{n+(k-1)r}V_{n+kr}}\right] \\ \frac{V_{n+kr}}{U_{n+kr}} &= \frac{V_n}{U_n} - 2Q^n U_r\left[\frac{1}{U_n U_{n+r}} + \frac{Q^r}{U_{n+r}U_{n+2r}} + \cdots + \frac{Q^{(k-1)r}}{U_{n+(k-1)r}U_{n+kr}}\right] \end{aligned}$$

When k increases indefinitely the first members of the preceding equalities have limits of $1/\sqrt{\Delta}$ and $\sqrt{\Delta}$, respectively. We will take account of the conditions for convergence in the second member later.

We may thus develop the square root of an integer in a series of fractions having unity as the numerator. This was a familiar usage to the scholars of Greece and of Egypt. Thus, for example, this approximate value

$$\frac{\sqrt{3}}{4} = \frac{1}{3} + \frac{1}{10} + \epsilon$$

is reported by Columelle in Chapter V of his work, "de Re Rustica." Further, this approximate value of

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{12 \cdot 34} + \epsilon$$

is given by the Indial atithors Baudhayana and Apastamba[1]. This approximate value is equal to the ratio of the terms $V_8/2 = 577$ ($\star$: this was originally "$V_8 = 577$" — Ed. note) and $U_8 = 408$ of the Pell series.

VII. RELATIONSHIP OF THE FUNCTIONS U_n AND V_n WITH THE THEORY OF DIVISIBILITY

If we let $\alpha = a^r$ and $\beta = b^r$ so that $\alpha\beta = Q^r$, we obtain the following results from the formulaswhich gives the quotient of $\alpha^n - \beta^n$ by $\alpha - \beta$.

[1]The Culvasûtras by G. Thibaut, Pages 13-15, Journal of the Asiatic Society of Bengal, 1875.

1) When n denotes an even number

$$(41) \qquad \frac{U_{nr}}{U_r} = V_{(n-1)r} + Q^r V_{(n-3)r} + Q^{2r} V_{(n-5)r} + \cdots + Q^{\left(\frac{n}{2}-1\right)r} V_r \;;$$

2) When n denotes an odd number

$$(42) \qquad \frac{U_{nr}}{U_r} = V_{(n=1)r} + Q^r V_{(n-3)r} + Q^{2r} V_{(n-5)r} + \cdots + Q^{\frac{n-1}{2}r} \;.$$

When n denotes an even number the quotient of $\alpha^n - \beta^n$ by $\alpha + \beta$ also gives

$$(43) \qquad \frac{U_{nr}}{V_r} = U_{(n-1)r} - Q^r U_{(n-3)r} + Q^{2r} U_{(n-5)r} - \cdots + (-Q^r)^{\frac{n}{2}-1} U_r \;.$$

Finally, when n denotes an odd number the quotient of $\alpha^n + \beta^n$ by $\alpha + \beta$ gives

$$(44) \qquad \frac{V_{nr}}{V_r} = V_{(n-1)r} - Q^r V_{(n-3)r} + Q^{2r} V_{(n-5)r} - \cdots + (-Q^r)^{\frac{n-1}{2}} \;.$$

For $n = 2$ we rederive the formula

$$(3) \qquad U_{2r} = U_r V_r \;,$$

and for $n = 3$ we have

$$(45) \qquad \begin{cases} U_{3r} = U_r(V_{2r} + Q^r) \;, \\ V_{3r} = V_r(V_{2r} - Q^r) \;. \end{cases}$$

The preceding relations show us that U_m is always divisible by U_n when m is divisible by n. Likewise V_m is always divisible by V_n when

m is odd and divisible by n. As a consequence, U_m and V_m can be prime numbers only if m is prime (*: there are some trivial exceptions, such as $u_4 = 3$ in the Fibonacci sequence — Ed. note), but the converse of this theorem is not correct.

In the Fibonacci series u_3 is divisible by 2, u_4 is divisible by 3, u_5 is divisible by 5, consequently u_{3n}, u_{4n}, and u_{5n} are respectively divisible by 2, 3, and 5. However, even though 53 is prime, we have $u_{53} = 953 \times 559\ 45741$.

Consider again the equalities

$$(6) \qquad V_n + \delta U_n = 2a^n, \quad V_n - \delta U_n = 2b^n .$$

By multiplying member by member, we obtain the relation

$$(46) \qquad V_n^2 - \Delta U_n^2 = 4Q^n ,$$

which corresponds in trigonometry to the formula

$$\cos^2 z + \sin^2 z = 1 .$$

This relation shows us that if U_n and V_n admit a common divisor θ this divisor will be a factor of $2Q$ (*: this was originally just "Q" — Ed. note). On the other hand

$$V_n = \left(\frac{P + \delta}{2}\right)^n + \left(\frac{P - \delta}{2}\right)^n .$$

By cancelling the multiples of Q which eventually recur when δ is replaced by Q we have the congruence

$$(47) \qquad V_n \equiv P^n \pmod{Q} .$$

Therefore every divisor θ of U_n and V_n would divide $2P$ and $2Q$ (*: this was originally "P and Q" — Ed. note). We have supposed that P and Q are relatively prime. This proposition then results.

Theorem: The numbers U_n and V_n are relatively prime. (*: the numbers U_n and V_n are actually either relatively prime or have greatest common divisor 2 — Ed. note).

If we let P belong to the exponent μ modulo Q we know that the congruence

$$P^n \equiv 1 \pmod{Q}$$

is true for all values of n equal to any multiple of μ, μ itself being a divisor of Euler's function $\phi(Q)$, the number of integers less than and relatively prime to Q. Consequently, because of equation (47), we may solve the congruence

$$V_n \equiv 1 \pmod{Q} \tag{48}$$

for all values of n equal to any multiple of μ.

VIII. LINEAR AND QUADRATIC FORMS OF THE DIVISORS OF U_n AND V_n WHICH CORRESPOND TO EVEN AND ODD VALUES OF THE ARGUMENT n

The formula (46) also leads to other important consequences on the form of the divisors of U_n and V_n because we directly derive from it the following propositions depending on whether n is an even or odd number.

Theorem: The terms of odd rank of the series U_n have divisors of the quadratic form $x^2 - Qy^2$.

Taking account of well-known results of the theory of divisors of quadratic forms we have, in particular, for the linear forms corresponding to the odd prime divisors of U_{2r+1}

For the Fibonacci series: $4q + 1$;

For the Fermat series: $8q + 1, 7$;

For the Pell series: $4q + 1$.

Thus, for odd n, the Fibonacci series or the Pell series cannot contain prime factors of the form $4q + 3$.

Theorem: The terms of even rank of the series V_n have divisors of the quadratic form $x^2 + \Delta y^2$.

In particular, the linear forms corresponding to the odd prime divisors of V_{2r} are

For the Fibonacci series: $20q + 1, 3, 7, 9$;

For the Fermat series: $4q + 1$;

For the Pell series: $8q + 1, 3$.

Theorem: The terms of odd rank of the series V_n have divisors of the quadratic form $x^2 + Q\Delta y^2$.

In particular, the linear forms corresponding to the odd prime divisors of V_{2r+1} are

For the Fibonacci series: $20q + 1, 9, 11, 19$;

For the Fermat series: $8q + 1, 3$;

For the Pell series: $8q + 1, 7$.

IX. FORMULAS CONCERNING THE ADDITION OF THE NUMERICAL FUNCTIONS

Multiplying member by member the relations

$$V_m + \delta U_m = 2a^m, \quad V_n + \delta U_n = 2a^n ,$$

we obtain

$$V_m V_n + \Delta U_m U_n + \delta[U_m V_n + U_n V_m] = 4a^{m+n} .$$

If we change a to b, and δ to $-\delta$ we further derive by addition and subtraction the formulas

$$(49) \quad \begin{cases} 2U_{m+n} = U_m V_n + U_n V_m , \\ 2V_{m+n} = V_m V_n + \Delta U_m U_n , \end{cases}$$

which correspond in trigonometry to the formulas for the addition of angles:

$$\sin(x+y) = \sin x \cos y + \sin y \cos x$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y .$$

If we change n to $-n$ in the formulas (49), taking account of the relations

(50) $$U_{-n} = -\frac{U_n}{Q^n}, \quad V_{-n} = \frac{V_n}{Q^n},$$

we obtain

(51) $$\begin{cases} 2Q^n U_{m-n} = U_m V_n - U_n V_m , \\ 2Q^n V_{m-n} = V_m V_n - \Delta U_m U_n . \end{cases}$$

Letting $m = n + r$ we obtain the formulas (39) given earlier.

A comparison of the equalities (49) and (51) immediately gives us

$$U_{m+n} + Q^n U_{m-n} = U_m V_n ,$$
$$U_{m+n} - Q^n U_{m-n} = U_n V_m ,$$

Now let

$$m + n = r, \quad m - n = s .$$

We get

(53) $$\begin{cases} U_r + Q^{\frac{r-s}{2}} U_s = U_{\frac{r+s}{2}} V_{\frac{r-s}{2}} \\ U_r - Q^{\frac{r-s}{2}} U_s = U_{\frac{r-s}{2}} V_{\frac{r+s}{2}} \end{cases} .$$

These relations resemble those which permit the transformation of the sum or the difference of two trigonometric terms into a product. Likewise we have

(53)
$$\begin{cases} V_r + Q^{\frac{r-s}{2}} V_s = V_{\frac{r+s}{2}} V_{\frac{r-s}{2}} \\ V_r - Q^{\frac{r-s}{2}} V_s = \Delta U_{\frac{r+s}{2}} U_{\frac{r-s}{2}} \end{cases}$$

We also have, as with the sum of the sines or the cosines of angles in arithmetic progression,

(54)
$$\begin{cases} U_m + Q^{-r/2} U_{m+r} + Q^{-2r/2} U_{m+2r} + \cdots + Q^{-nr/2} U_{m+nr} \\ \qquad = Q_{\frac{1}{2}(2m+nr)} \dfrac{U_{\frac{1}{2}(n+1)r} Q^{m/4}}{U_{r/2} Q^{nr/2}} \\ V_m + Q^{-r/2} V_{m+r} + Q^{-2r/2} V_{m+2r} + \cdots + Q^{-nr/2} V_{m+nr} \\ \qquad = V_{\frac{1}{2}(2m+nr)} \dfrac{U_{\frac{1}{2}(n+1)r} Q^{m/4}}{U_{r/2} Q^{nr/2}}, \end{cases}$$

and consequently

(55)
$$\frac{U_m + Q^{-r/2} U_{m+r} + Q^{-2r/2} U_{m+2r} + \cdots + Q^{-nr/2} U_{m+nr}}{V_m + Q^{r/2} V_{m+r} + Q^{-2r/2} V_{m+2r} + \cdots + Q^{-nr/2} V_{m+nr}} = \frac{U_{m+\frac{1}{2}nr}}{V_{m+\frac{1}{2}nr}} .$$

We find many simple formulas starting with the relations

(13*)
$$\begin{cases} U_{n+2r} = V_r U_{n+r} - Q^r U_n , \\ V_{n+2r} = V_r V_{n+r} - Q^r V_n . \end{cases}$$

If we successively replace n by $0, r, 2r, \cdots, (n-1)r$, and if we add, we obtain

$$(56^\star)\qquad \begin{aligned} U_r + U_{2r} + \cdots + U_{nr} &= \frac{U_r + Q^r U_{nr} - U_{(n+1)r}}{1 + Q^r - V_r} \\ V_r + V_{2r} + \cdots + V_{nr} &= \frac{V_r + Q^r V_{nr} - V_{(n+1)r}}{1 + Q^r - V_r} \end{aligned} .$$

These formulas are in an indeterminant form when the denominator vanishes, i.e., for

$$1 + a^r b^r - a^r - b^r = 0 ,$$

or

$$(1 - a^r)(1 - b^r) = 0 ,$$

i.e., for values of a or b equal to unity; in this case we use the method of summation of the geometric progression. We have, moreover, in the Fibonacci series, for $r = 1$ and $r = 2$,

$$\begin{aligned} u_1 + u_2 + u_3 + \cdots + u_n &= u_{n+2} - 1 , \\ u_2 + u_4 + u_6 + \cdots + u_{2n} &= u_{2n+1} - 1 , \\ v_1 + v_2 + v_3 + \cdots + v_n &= v_{n+2} - 3 , \\ v_2 + v_4 + v_6 + \cdots + v_{2n} &= v_{2n+1} - 1 . \end{aligned}$$

We also find, more generally,

$$(57)\qquad \begin{aligned} &U_{m+r} + U_{m+2r} + U_{m+3r} + \cdots + U_{m+nr} \\ &= \frac{U_{m+r} + Q^r U_{m+nr} - U_{m+(n+1)r} - Q^r U_m}{1 + Q^r - V_r} \end{aligned}$$

and an analogous result on changing U to V.

The formula for addition enables us to also write

$$2\,\frac{U_{m+n}}{U_n} = \frac{U_m}{U_n}\,V_n + V_m \;.$$

We consequently have

$$(58)\qquad 2\,\frac{U_{m+n}U_{m+n-1}\cdots U_{m+1}}{U_nU_{n-1}\cdots U_1} = \frac{U_{m+n-1}U_{m+n-2}\cdots U_m}{U_nU_{n-1}\cdots U_1}V_n + \frac{U_{m+n-1}U_{m+n-2}\cdots U_{m+1}}{U_{n-1}U_{n-2}\cdots U_1}\cdot V_m.$$

We immediately derive from it this proposition:

Theorem: The product of n consecutive terms of the series U_n is divisible by the product of the first n terms.

We will complete this section with the proof of formulas of extreme importance, because these will serve later as a basis for the theory of the numerical functions of double period, derived from the consideration of the symmetric functions of the roots of the third and fourth degree equations with rational coefficients. The formulas (30) give us

$$(\star)\qquad \begin{aligned} U_{m-1}U_{m+1} &= U_m^2 - Q^{m-1} \\ U_{n-1}U_{n+1} &= U_n^2 - Q^{n-1} \;. \end{aligned}$$

We derive from these

$$U_n^2U_{m-1}U_{m+1} = U_m^2U_{n-1}U_{n+1} = Q^{n-1}\left[U_m^2 - Q^{m-n}U_n^2\right].$$

From the formulas (32)

$$(A)\qquad U_n^2U_{m-1}U_{m+1} - U_m^2U_{n-1}U_{n+1} = Q^{n-1}U_{m-n}U_{m+n} \;;$$

we likewise have

(A') $$V_n^2 V_{m-1} V_{m+1} - V_m^2 V_{n-1} V_{n+1} = -\Delta Q^{n-1} V_{m-n} V_{m+n}.$$

In particular, for $m = n+1$ and for $m = n+2$ we have

(B) $$\begin{cases} U_n^3 U_{n+2} - U_{n+1}^3 U_{n-1} = Q^{n-1} U_1 U_{2n+1} \\ U_n^2 U_{n+1} U_{n+3} - U_{n+2}^2 U_{n-1} U_{n+1} = Q^{n-1} U_2 U_{2n+2} \end{cases}$$

and analogous formulas for the V_n.

The formulas (A) and (B) belong to the theory of elliptic functions and more especially to the functions which Jacobi denoted by the symbols θ and H.

X. ON THE SUM OF THE SQUARES OF THE NUMERICAL FUNCTIONS U_n AND V_n

If in the following relation

(59) $$\Delta U_{r+2k\rho} U_{s+2k\sigma} = V_{r+s+2k(\rho+\sigma)} - Q^{s+2k\sigma} V_{r-s+2k(\rho-\sigma)},$$

we let k successively equal 0, 1, 2, $\cdots$, n, and if we add member by member the equalities obtained, after having divided respectively by

$$1,\ Q^{\rho+\sigma},\ Q^{2(\rho+\sigma)},\ \cdots,\ Q^{n(\rho+\sigma)},$$

we obtain the formula

(60) $$\sum_{k=0}^{n} U_{r+2k\rho} U_{s+2k\sigma} = \frac{U_{r+2n\rho} U_{s+(2n+1)\sigma} - Q^{\sigma-\rho} U_{r+(2n+1)\rho} U_{s+2n\sigma}}{\Delta Q^{n(\sigma+\rho)} U_{\sigma-\rho} U_{\sigma+\rho}}$$

$$+ \frac{U_r U_{s-2\sigma} - Q^{\sigma-\rho} U_{r-2\rho} U_s}{\Delta U_{\sigma-\rho} U_{\sigma+\rho}}$$

We have, in particular, for $2\rho = r$ and $2\sigma = s$,

$$
(61)\quad \begin{cases} U_rU_s + \dfrac{U_{2r}U_{2s}}{Q^{\frac{1}{2}(r+s)}} + \dfrac{U_{3r}U_{3s}}{Q^{r+s}} + \cdots + \dfrac{U_{(n+1)r}U_{(n+1)s}}{Q^{\frac{1}{2}n(r+s)}} \\[2ex] \qquad = \dfrac{U_{(n+1)r}U_{2(n+1)s} - Q^{\frac{1}{2}(s-r)}U_{(n+1)s}U_{2(n+1)r}}{\Delta Q^{\frac{1}{2}n(r+s)}U_{\frac{1}{2}(s-r)}U_{\frac{1}{2}(s+r)}}, \end{cases}
$$

and more particularly,

$$
(62)\quad \begin{cases} \dfrac{U_r^2}{Q^r} + \dfrac{U_{2r}^2}{Q^{2r}} + \dfrac{U_{3r}^2}{Q^{3r}} + \cdots + \dfrac{U_{(n+1)r}^2}{Q^{(n+1)r}} = \dfrac{1}{\Delta}\left[\dfrac{U_{(2n+3)r}}{U_rQ^{(n+1)r}} - 2n - 3\right], \\[3ex] \dfrac{U_r^2}{Q^r} + \dfrac{U_{3r}^2}{Q^{3r}} + \dfrac{U_{5r}^2}{Q^{5r}} + \cdots + \dfrac{U_{(2n+1)r}^2}{Q^{(2n+1)r}} = \dfrac{1}{\Delta}\left[\dfrac{U_{4(n+1)r}}{U_{2r}Q^{(2n+1)r}} - 2n - 2\right]. \end{cases}
$$

By an analogous procedure we may find the values of

$$\sum_{k=0}^{n} \frac{V_{r+2k\rho}V_{s+2k\sigma}}{Q^{k(\rho+\sigma)}}$$

and

$$\sum_{k=0}^{n} \frac{U_{r+2k\rho}V_{s+2k\sigma}}{Q^{k(\rho+\sigma)}} .$$

In particular,

$$
(63^{\star})\quad \begin{aligned} &\frac{V_r^2}{Q^r} + \frac{V_{2r}^2}{Q^{2r}} + \frac{V_{3r}^2}{Q^{3r}} + \cdots + \frac{V_{nr}^2}{Q^{nr}} = 2n - 1 + \frac{U_{(2n+1)r}}{U_rQ^{nr}}, \\ &\frac{V_r^2}{Q^r} + \frac{V_{3r}^2}{Q^{3r}} + \frac{V_{5r}^2}{Q^{5r}} + \cdots + \frac{V_{(2n+1)r}^2}{Q^{(2n+1)r}} = 2n + 2 + \frac{U_{4(n+1)r}}{U_rQ^{(2n+1)r}}. \end{aligned}
$$

We also have, in the general case,

$$(64^{\star})\begin{cases}\displaystyle\sum_{k=0}^{n} U^2 = \frac{V_{2m+2(n+1)r} - V_{2m} - Q^{2r}\left[V_{2m+2nr} - V_{2m-2r}\right]}{\Delta(V_{2r} - Q^{2r} - 1)} \\ \qquad - 2Q^m \dfrac{Q^{(n+1)r} - 1}{\Delta(Q^r - 1)} \\ \displaystyle\sum_{k=0}^{n} V^2_{m+kr} = \frac{V_{2m+2(n+1)r} - V_{2m} - Q^{2r}\left[V_{2m+2nr} - V_{2m-2r}\right]}{V_{2r} - Q^{2r} - 1} \\ \qquad + 2Q^m \dfrac{Q^{(n+1)r} - 1}{Q^r - 1} \quad .\end{cases}$$

We have, for example, in the Fibonacci series

$$(65^{\star})\begin{cases}(-1)^r u_r^2 + u_{2r}^2 + (-1)^r u_{3r}^2 + \cdots + (-1)^{nr} u_{nr}^2 = \frac{1}{5}\left((-1)^{nr}\dfrac{u_{(2n+1)r}}{u_r} - (2n+1)\right), \\ u_r^2 + u_{3r}^2 + u_{5r}^2 + \cdots + u_{(2n+1)r}^2 = \frac{1}{5}\left(\dfrac{u_{4(n+1)r}}{u_{2r}} - (-1)^r(2n+2)\right), \\ (-1)^r v_r^2 + v_{2r}^2 + (-1)^r v_{3r}^2 + \cdots + (-1)^n v_{nr}^2 = 2n - 1 + (-1)^{nr}\dfrac{u_{(2n+1)r}}{u_r}, \\ v_r^2 + v_{3r}^2 + v_{5r}^2 + \cdots + v_{(2n+1)r}^2 = \dfrac{u_{4(n+1)r}}{u_r} + (-1)^r(2n+2) \quad .\end{cases}$$

The simplest formula,

$$(66) \qquad u_1^2 + u_2^2 + u_3^2 + \cdots + u_n^2 = u_n u_{n+1} \; ,$$

gives for the side of the regular star decagon the expression

$$(67) \qquad \frac{1+\sqrt{5}}{2} = 1 + \frac{1}{1^2} - \frac{1}{1^2+1^2} + \frac{1}{1^2+1^2+2^2} + \frac{1}{1^2+1^2+2^2+3^2} + \frac{1}{1^2+1^2+2^2+3^2+5^2} - \cdots$$

We also have

$$(68) \qquad \begin{cases} u_1u_2 + u_2u_3 + u_3u_4 + \cdots + u_{2n-1}u_{2n} = u_{2n}^2 , \\ u_1u_2 + u_2u_3 + u_3u_4 + \cdots + u_{2n}u_{2n+1} = u_{2n+1}^2 - 1 . \end{cases}$$

XI. RELATION OF THE FUNCTIONS U_n AND V_n WITH THE THEORY OF THE GREATEST COMMON DIVISOR

We have found the formula

$$2U_{m+n} = U_mV_n + U_nV_m ;$$

consequently if any odd number whatever θ divides U_{m+n} and U_m, it divides U_nV_m; but we have shown (§17) that U_m and V_m are relatively prime, hence θ divides U_n. Inversely, every odd number that divides U_n and U_m divides U_{m+n}. Therefore, not counting the factor 2, we have this fundamental proposition.

Theorem: The largest common divisor of U_m and U_n is equal to U_D, where D denotes the largest common divisor of m and of n.

In particular, U_m and U_n are relatively prime when m and n are relatively prime because U_1 is equal to unity. Moreover, we derive from the fundamental theorem a large number of propositions closely resembling those which we obtain in the theory of the largest common divisor and of the smallest common multiple of several given numbers.

A further result from the preceding is that, in the search for the largest common divisor of two terms U_m and U_n, the successive remainders also form terms of the series. In particular the successive remainders of two consecutive terms give, in the case of negative Q, all the terms of the reduced series starting with the smallest of them. Lamé[1] noted that, in the search for the largest common divisor of any two numbers, the number of remainders is at most equal to the number of terms of the Fibonacci series less than the smallest of the given numbers. He derived from it this theorem:

The number of divisions to be performed in the search for the largest common divisor of two given numbers is at most equal, in the ordinary number

[1]Comptes rendus de l'Acadamie des Sciences de Paris, Vol. xix, p. 863. Paris, 1844.

system, to five times the number of digits in the smallest of the given numbers.

We would find a closer limit by calculating by logarithms the rank of the term of the Fibonacci series immediately less than the smallest of the given numbers. Denoting this smallest number by A we easily see that it would suffice to take the smallest integer contained in the fraction

$$\frac{\log A + \log \sqrt{5}}{\log \frac{1 + \sqrt{5}}{2}} = \frac{\log A + 0.349}{0.209}$$

But it is preferable to be content with Lamé's elegant theorem.

XII. ON THE MULTIPLICATION OF THE NUMERICAL FUNCTIONS

We may express the values of U_n and V_n which correspond to all positive integer values of n as functions of the initial values; in fact, for U for example we have successively

$$(69) \quad \left\{ \begin{aligned} U_2 &= PU_1 - QU_0 , \\ U_3 &= (P^2 - Q)U_1 - QPU_0 , \\ U_4 &= (P^3 - 2PQ)U_1 - Q(P^2 - Q)U_0 , \\ U_5 &= (P^4 - 3P^2Q + Q^2)U_1 - Q(P^3 - 2PQ)U_0 , \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \right.$$

We will note first of all that if ϕ_n denotes the coefficient of U_1 in U_{n+1} we have in general

$$U_{n+1} = \phi_n U_1 - Q\phi_{n-1}U_0 .$$

The coefficient ϕ_n is a homogeneous function of degree n of P and Q, considering P of the first degree and Q of the second. If we form a table of the coefficients of ϕ_n, we easily find again the arithmetic triangle, but in a special arrangement. We have moreover thus been able to verify á posteriori

$$(70)\quad \phi_n = P^n - \binom{n-1}{1} P^{n-2}Q + \binom{n-2}{2} P^{n-4}Q^2 - \binom{n-3}{3} P^{n-6}Q^3 + \cdots ,$$

and at the same time

$$(71)\quad \begin{cases} U_{n+1} = \phi_n U_1 - Q\phi_{n-1}U_0 , \\ V_{n+1} = \phi_n V_1 - Q\phi_{n-1}V_0 , \end{cases}$$

with the initial conditions

$$U_0 = 0,\ U_1 = 1,\ V_0 = 2,\ V_1 = P .$$

Consequently we also have

$$\phi_n = U_{n+1} .$$

We have, in particular, for the Fibonacci series, for $P = 1$ and $Q = -1$

$$(72)\quad 1 + \binom{n-1}{1} + \binom{n-2}{2} + \binom{n-3}{3} + \cdots = u_n$$

and for $P = 1$, $Q = 1$

$$(73)\quad 1 - \binom{n-1}{1} + \binom{n-2}{2} - \binom{n-3}{3} + \cdots = \frac{2}{\sqrt{3}} \sin \frac{n\pi}{3}$$

The preceding formulas easily generalize by the consideration of equation (8). In fact if we let

$$(74)\quad \psi_n = V_r^n - \binom{n-1}{1} V_r^{n-2}Q^r + \binom{n-2}{2} V_r^{n-4}Q^{2r} - \binom{n-3}{3} V_r^{n-6}Q^{3r} + \cdots ,$$

we obtain from the above

$$(75)\quad \begin{cases} U_{m+2nr} = \psi_{n-1}U_{m+r} - Q^r\psi_{n-2}U_m , \\ V_{m+2nr} = \psi_{n-1}V_{m+r} - Q^r\psi_{n-2}V_m . \end{cases}$$

For $m = 0$ we also have the relation

$$(76) \qquad \psi_{n-1} = \frac{U_{2nr}}{U_r}$$

which permits the inverse calculation of the function ψ with the aid of the values of U. Moreover this relation, in which n denotes an integer, holds for any value of r. For $r = 0$ we thus have the formula

$$(77) \qquad n = 2^{n-1} - \binom{n-2}{1}2^{n-3} + \binom{n-3}{2}2^{n-5} - \binom{n-4}{3}2^{n-7} + \cdots .$$

We will note that the preceding results correspond to the well known expansions of $\sin nz/\sin z$ and of $\cos nz$ in terms of powers of $\cos z$ first obtained by Viète.[1]

XIII. ON THE RULE FOR THE RECURRENCE OF PRIME NUMBERS IN THE SIMPLY PERIODIC RECURRENT SERIES

We will express the functions U_{np} and V_{np} as functions of U_n and V_n exclusively by formulas analogous to those which have been given by Moivre and by Lagrange[2].

In fact, if we denote by $\binom{m}{n}$ the number of combinations of m objects taken n at a time we have the following relation:

$$(78) \qquad \begin{aligned} \alpha^p + \beta^p = (\alpha+\beta)^p - \frac{p}{1}\alpha\beta(\alpha+\beta)^{p-2} + \frac{p}{2}\binom{p-3}{1}\alpha^2\beta^2(\alpha+\beta)^{p-4} + \cdots \\ + (-1)^r \frac{p}{r}\binom{p-r-1}{r-1}\alpha^r\beta^r(\alpha+\beta)^{p-2r} + \cdots . \end{aligned}$$

which we may verify à posteriori and in which all the coefficients are integers, since we have

$$(\star) \qquad \frac{p}{r}\binom{p-r-1}{r-1} = \binom{p-r-1}{r} + 2\binom{p-r-1}{r-1}$$

[1]Opera, Leyde, 1646, p. 295-299.

[2]Commentarii Acad. Petrop., Vol. xiii, 1741-43, p. 29. Lectures on the Calculus of Functions, p. 119.

Assuming that p is odd, let

$$\alpha = a^n, \quad \beta = -b^n.$$

We obtain

$$(79)\qquad \begin{aligned} U_{np} &= \delta^{p-1}U_n^p + \frac{p}{1}Q^n\delta^{p-3}U_n^{p-2} + \frac{p}{2}\binom{p-3}{1}Q^{2n}\delta^{p-5}U_n^{p-4} + \cdots \\ &\quad + \frac{p}{r}\binom{p-r-1}{r-1}Q^{nr}\delta^{p-2r-1}U_n^{p-2r} + \cdots . \end{aligned}$$

The preceding formula leads to the rule for the repetition of prime numbers in the recurrent series which we are considering here. In the natural series of integers a prime number p appears for the first time at its rank and to the first power; it appears to the second power at the rank p^2, the third time at the rank p^3, and so forth. In addition, all the terms divisible by p^α occupy a rank equal to any multiple of p^α. But in the simply periodic recurrent series it is not quite like this. We will later show that the terms of these contain, at known ranks, all the prime numbers. However if these prime numbers p do not appear for the first time in the series at rank p they will nevertheless recur at intervals equal to p, as in the ordinary series, and the appearance of their successive powers will occur as in the natural series. Thus, in general, in the arithmetic study of the series, two rules must be considered; the rule of the appearance of the prime numbers, and the rule of their repetition.

We will show, in a moment, that the rule of repetition is the same as in the natural series and in the U_n series. In fact, if p denotes a prime number and U_n is the first term of the series divisible by p^λ, we will note that the last term of the preceding formula is divisible by $p^{\lambda+1}$ and not by a higher power of p. Hence we have the following fundamental proposition.

Theorem: If λ denotes the largest exponent of the prime number p contained in U_n, the exponent of the largest power of p that divides U_{pn} is equal to $\lambda + 1$.

Thus, for example, in the Fibonacci series, u_8 is divisible by 7; hence u_{56} is divisible by 7^2 but not by 7^3; in the Pell series, U_7 and U_{30} (*these were originally "u_7" and "u_{30}" — Ed. note) are respectively divisible by 13^2 and 31^2; hence u_{91} and u_{930} are divisible by 13^3 and 31^3 and not by higher powers.

Inversely if $a^p \pm b^p$ is divisible by p^λ, $a \pm b$ is divisible by $p^{\lambda-1}$. This result gives important consequences in the theory of the diophantine equation

$$x^p + y^p + z^p = 0 .$$

This equation, which constitutes Fermat's Last Theorem, has not been solved to the present day.

XIV. NEW LINEAR AND QUADRATIC FORMS OF THE DIVISORS OF U_n AND V_n

Formula (79) gives the following formulas for p successively equal to 3, 5, 7, 9, $\cdots$

$$(80)\quad \begin{cases} U_{3n} = \Delta U_n^3 + 3Q^n U_n , \\ U_{5n} = \Delta^2 U_n^5 + 5Q^n \Delta U_n^3 + 5Q^{2n} U_n , \\ U_{7n} = \Delta^3 U_n^7 + 7Q^n \Delta^2 U_n^5 + 14Q^{2n} \Delta U_n^3 + 7Q^{3n} U_n , \\ U_{9n} = \Delta^4 U_n^9 + 9Q^n \Delta^3 U_n^7 + 27Q^{2n} \Delta^2 U_n^5 + 30Q^{3n} \Delta U_n^3 + 9Q^{4n} U_n , \\ U_{11n} = \Delta^5 U_n^{11} + 11Q^n \Delta^4 U_n^9 + 44Q^{2n} \Delta^3 U_n^7 + 77Q^{3n} \Delta^2 U_n^5 + 55Q^{4n} \Delta U_n^3 + 11Q^{5n} U_n , \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

We thus have

$$(81)\qquad \frac{U_{3n}}{U_n} = \Delta U_n^2 + 3Q^n ,$$

and consequently the following proposition

Theorem: The divisors of U_{3n}/U_n are divisors of the quadratic form $\Delta x^2 + 3Q^n y^2$. In particular the linear forms of the odd prime divisors of U_{6n}/U_{2n} are

For the Fibonacci series: $30q + 1, 17, 19, 23$;

For the Fermat series: $6q + 1$;

For the Pell series: $24q + 1, 5, 7, 11$;

and the linear forms of the odd prime divisors of $U_{3(2n+1)}/U_{2n+1}$ are

For the Fibonacci series: $60q + 1, 7, 11, 17, 43, 49, 53, 59$;

For the Fermat series: $24q + 1, 5, 7, 11$;

For the Pell series: $24q + 1, 15, 19, 23$;

We likewise have

$$(82) \qquad 4\frac{U_{5n}}{U_n} = (2\Delta U_n^2 + 5Q^n)^2 - 5Q^{2n}$$

and consequently

<u>Theorem:</u> The divisors of U_{5n}/U_n are divisors of the quadratic form $x^2 - 5y^2$.

The linear forms of the odd prime divisors are, for the three series taken as examples

$$20q + 1, 9, 11, 19.$$

We likewise have

$$(83) \qquad 4\frac{U_{7n}}{U_n} = \Delta\left[2\Delta U_n^3 + 7Q^nU_n\right]^2 + 7Q^{2n}V_n^2 ,$$

and consequently

<u>Theorem:</u> The divisors of U_{7n}/U_n are divisors of the quadratic form $\Delta x^2 + 7y^2$.

Let us now suppose that p denotes an even number and further, in formula (78) let

$$\alpha = a^n, \ \beta = -b^n ,$$

we obtain

$$
(84) \qquad \begin{aligned} V_{np} &= \delta^p U_n^p + \frac{p}{1} Q^n \delta^{p-2} U_n^{p-2} + \frac{p}{2}\binom{p-3}{1} Q^{2n} \delta^{p-4} U_n^{p-4} + \cdots \\ &\quad + \frac{p}{r}\binom{p-r-1}{r-1} Q^{nr} \delta^{p-2r} U_n^{p-2r} + \cdots . \end{aligned}
$$

We have, in particular, for $p = 2$, the formula

$$
(85) \qquad V_{2n} = \Delta U_n^2 + 2Q^n ,
$$

and consequently the following proposition

<u>Theorem</u>: The divisors of V_{2n} are divisors of the quadratic form $\Delta x^2 + 2Q^n y^2$.

The linear forms corresponding to the odd prime divisors are, for n even

In the Fibonacci series: $40q + 1, 7, 9, 11, 13, 19, 23, 37$;

In the Fermat series: $8q + 1, 3$;

In the Pell series: $4q + 1$;

and for odd n

In the Fibonacci series: $40q + 1, 3, 9, 13, 27, 31, 37, 39$;

In the Fermat series: $4q \pm 1$;

In the Pell series: $4q + 1$;

In the applications we shall combine these results with those which have been given in Section VIII.

Finally, in formula (78) (*: this was originally (79) — Ed. note), let

$$
\alpha = a^n , \qquad \beta = b^n .
$$

Regardless of whether p is an even or odd number we obtain

$$
(86) \qquad V_{np} = V_n^p - \frac{p}{1} Q^n V_n^{p-2} + \frac{p}{2}\binom{p-3}{1} Q^{2n} V_n^{p-4} - \cdots + (-1)^r \frac{p}{r}\binom{p-r-1}{r-1} Q^{nr} V_n^{p-2r} + \cdots
$$

Letting p successively equal $2, 3, 4, 5, 6, \cdots$ we thus have the following results which also lead to formulas resembling the preceding ones.

$$
(87) \qquad \begin{cases} V_{2n} = V_n^2 - 2Q^n\,, \\ V_{3n} = V_n^3 - 3Q^n V_n\,, \\ V_{4n} = V_n^4 - 4Q^n V_n^2 + 2Q^{2n}\,, \\ V_{5n} = V_n^5 - 5Q^n V_n^3 + 5Q^{2n} V_n\,, \\ V_{6n} = V_n^6 - 6Q^n V_n^4 + 9Q^{2n} V_n^2 - 2Q^{3n}\,. \end{cases}
$$

XV. RELATION OF THE FUNCTIONS U_n AND V_n WITH CONTINUED RADICALS

From the equation

$$x^2 = Px - Q\,,$$

we get the formula

$$x = \sqrt{-Q + Px}$$

and successively

$$x = \sqrt{-Q + P\sqrt{-Q + Px}}$$

$$x = \sqrt{-Q + P\sqrt{-Q + P\sqrt{-Q + Px}}}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Since we may suppose that P is positive we consequently have, for negative Q

$$(88) \qquad a = \sqrt{-Q + P\sqrt{-Q + P\sqrt{-Q + \cdots}}}$$

a denoting the positive root of the given equation. Thus, in the Fibonacci series

$$\frac{1 + \sqrt{5}}{2} = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}$$

in the Pell series,

$$1 + \sqrt{2} = \sqrt{1 + 2\sqrt{1 + 2\sqrt{1 + \cdots}}}$$

and in the Fermat series,

$$2 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}}$$

We know that this last radical occurs in the calculation of π by the method of perimeters devised by Archimedes.

But the results obtained in the preceding section lead to more important formulas which will find their use in the search for large prime numbers. For example, from the first of the formulas (87) we get

$$V_n = \sqrt{2Q^n + V_{2n}} \quad ;$$

and likewise, changing n to 2n, 4n, 8n, $\cdots$

$$V_{2n} = \sqrt{2Q^{2n} + V_{4n}} \, ,$$

$$V_{4n} = \sqrt{2Q^{4n} + V_{8n}} \, ,$$

$$V_{8n} = \sqrt{2Q^{8n} + V_{16n}} \, ;$$

and consequently

$$(89) \qquad \begin{cases} V_n = \sqrt{2Q^n + V_{2n}} \, , \\ V_n = \sqrt{2Q^n + \sqrt{2Q^{2n} + V_{4n}}} \\ V_n = \sqrt{2Q^n + \sqrt{2Q^{2n} + \sqrt{2Q^{4n} + V_{8n}}}} \, , \\ V_n = \sqrt{2Q^n + \sqrt{2Q^{2n} + \sqrt{2Q^{4n} + \sqrt{2Q^{8n} + V_{16n}}}}} \, , \end{cases}$$

and so forth indefinitely. These formulas are analogous to those for $\cos \pi/4$, $\cos \pi/8$, $\cos \pi/16$, $\cos \pi/32$, $\cdots$, $\cos \pi/2^r$.

In the same manner the second of the relations (87) gives

$$
(90) \qquad \left\{ \begin{aligned}
V_n &= \sqrt{3Q^n + V_{3n}/V_n}\ , \\
V_n &= \sqrt{2Q^n + \sqrt{3Q^{2n} + V_{6n}/V_{2n}}}\ , \\
V_n &= \sqrt{2Q^n + \sqrt{2Q^{2n} + \sqrt{3Q^{4n} + V_{12n}/V_{4n}}}}
\end{aligned} \right.
$$

These formulas resemble those for $\cos \pi/6$, $\cos \pi/12$, $\cos \pi/24, \cdots$, $\cos \pi/3 \cdot 2^r$.

The third of the relations (87) further leads to formulas which correspond to those given for $\cos \pi/10$, $\cos \pi/20$, $\cos \pi/40, \cdots$, $\cos \pi/5 \cdot 2^r$; and likewise for any others.

XVI. EXPANSIONS OF POWERS OF U_n AND V_n AS LINEAR FUNCTIONS OF TERMS WHOSE ARGUMENTS ARE MULTIPLES OF n

We may express the powers of U_n and V_n as linear functions of terms whose ranks are multiples of n, using formulas analogous to those which give the powers of $\sin z$ and of $\cos z$, expansions based on the sines and cosines of multiples of the angle z. Letting, first of all, p be an odd number, the expansion of $(\alpha - \beta)^p$ gives

$$
(\alpha - \beta)^p = (\alpha^n - \beta^p) - \binom{p}{1}\alpha\beta\,(\alpha^{p-2} - \beta^{p-2}) + \binom{p}{2}\alpha^2\beta^2\,(\alpha^{p-4} - \beta^{p-4}) - \cdots
$$

Consequently, letting

$$
\alpha = a^n, \qquad \beta = b^n ,
$$

we obtain the formula

$$
(91) \qquad \begin{aligned}
\Delta^{\frac{p-1}{2}} U_n^p = U_{np} &- \binom{p}{1} Q^n U_{(p-2)n} + \binom{p}{2} Q^{2n} U_{(p-4)n} \\
&- \binom{p}{3} Q^{3n} U_{(p-6)n} + \cdots \\
&+ (-1)^{(p-1)/2} \binom{p}{(p-1)/2} Q^{(p-1)n/2} U_n \ .
\end{aligned}
$$

For p successively equal to 3, 5, 7, 9, $\cdots$ we have

$$
(92)\quad \begin{cases}
\Delta U_n^3 = U_{3n} - 3Q^n U_n \,, \\
\Delta^2 U_n^5 = U_{5n} - 5Q^n U_{3n} + 10Q^{2n} U_n \,, \\
\Delta^3 U_n^7 = U_{7n} - 7Q^n U_{5n} + 21Q^{2n} U_{3n} - 35Q^{4n} U_n \,, \\
\Delta^4 U_n^9 = U_{9n} - 9Q^n U_{7n} + 36Q^{2n} U_{5n} - 84Q^{4n} U_{3n} + 126Q^{6n} U_n \,. \\
\cdots\cdots\cdots\cdots\cdots\cdots
\end{cases}
$$

Supposing now that p denotes an even number the expansion of $(\alpha - \beta)^p$ also gives

$$
(93)\quad \begin{aligned}
\Delta^{n/2} U_n^p = V_{pn} - \binom{p}{1} Q^n V_{(p-2)n} + \binom{p}{2} Q^{2n} V_{(p-4)n} - \binom{p}{3} Q^{3n} V_{(p-6)n} + \cdots \\
+ (-1)^{p/2} \binom{p}{p/2} Q^{pn/2} \,.
\end{aligned}
$$

For p successively equal to 2, 4, 6, 8, $\cdots$

$$
(94)\quad \begin{cases}
\Delta U_n^2 = V_{2n} - 2Q^n \,, \\
\Delta^2 U_n^4 = V_{4n} - 4Q^n V_{2n} + 6Q^{2n} \,, \\
\Delta^3 U_n^6 = V_{6n} - 6Q^n V_{4n} + 15Q^{2n} V_{2n} - 20Q^{3n} \,, \\
\Delta^4 U_n^8 = V_{8n} - 8Q^n V_{6n} + 28Q^{2n} V_{4n} - 56Q^{3n} V_{2n} + 70Q^{4n} \,, \\
\cdots\cdots\cdots\cdots\cdots\cdots
\end{cases}
$$

Assuming that p is an odd number the expansion of $(\alpha + \beta)^p$ gives

$$
(95)\quad \begin{aligned}
V_n^p = V_{pn} + \binom{p}{1} Q^n V_{(p-2)n} + \binom{p}{2} Q^{2n} V_{(p-4)n} + \binom{p}{3} Q^{3n} V_{(p-6)n} + \cdots \\
+ \binom{p}{(p-1)/2} Q^{(p-1)n/2} V_n \,.
\end{aligned}
$$

More particularly

$$
(96)\quad \begin{cases}
V_n^3 = V_{3n} + 3Q V_n \,, \\
V_n^5 = V_{5n} + 5Q^n V_{3n} + 10Q^{2n} V_n \,, \\
V_n^7 = V_{7n} + 7Q^n V_{5n} + 21Q^{2n} V_{5n} + 35Q^{3n} V_n \,, \\
V_n^9 = V_{9n} + 9Q^n V_{7n} + 36Q^{2n} V_{5n} + 84Q^{3n} V_{3n} + 126Q^{4n} V_n \,, \\
\cdots\cdots\cdots\cdots\cdots\cdots
\end{cases}
$$

Likewise, when p denotes an even number

$$(97) \qquad V_n^p = V_{pn} + \binom{p}{1} Q^n V_{(p-2)n} + \binom{p}{2} Q^{2n} V_{(p-4)n} + \binom{p}{3} Q^{3n} V_{(p-6)n} + \cdots + \binom{p}{p/2} Q^{pn/2} .$$

More particularly we have

$$(98) \qquad \begin{cases} V_n^2 = V_{2n} + 2Q^n , \\ V_n^4 = V_{4n} + 4Q^n V_{2n} + 6Q^{2n} , \\ V_n^6 = V_{6n} + 6Q^n V_{4n} + 15Q^{2n} V_{2n} + 20Q^{3n} , \\ V_n^8 = V_{8n} + 8Q^n V_{6n} + 28Q^{2n} V_{4n} + 56Q^{3n} V_{2n} + 70Q^{4n} . \end{cases}$$

The relations (91), (93), (95), and (97) are themselves particular cases of the following formulas:

$$(99) \qquad \begin{cases} V_r^n U_{(m-n)r} = U_{mr} + \binom{n}{1} Q^r U_{(m-2)r} + \binom{n}{2} Q^{2r} U_{(m-4)r} + \binom{n}{3} Q^{3r} U_{(m-6)r} + \cdots \\ V_r^n V_{(m-n)r} = V_{mr} + \binom{n}{1} Q^r V_{(m-2)r} + \binom{n}{2} Q^{2r} V_{(m-4)r} + \binom{n}{3} Q^{3r} V_{(m-6)r} + \cdots \\ \Delta^n U_r^{2n} U_{(m-2n)r} = U_{mr} - \binom{2n}{1} Q^r U_{(m-2)r} + \binom{2n}{2} Q^{2r} U_{(m-4)r} - \cdots \\ \Delta^n U_r^{2n} U_{(m-2n)r} = V_{mr} - \binom{2n}{1} Q^r V_{(m-2)r} + \binom{2n}{2} Q^{2r} V_{(m-4)r} - \cdots \\ \Delta^n U_r^{2n+1} U_{(m-2n-1)r} = U_{mr} - \binom{2n+1}{1} Q^r U_{(m-2r)} + \binom{2n+1}{2} Q^{2r} U_{(m-4)r} - \cdots \\ \Delta^n U_r^{2n+1} V_{(m-2n-1)r} = V_{mr} - \binom{2n+1}{1} Q^r V_{(m-2r)} + \binom{2n+1}{2} Q^{2r} V_{(m-4)r} - \cdots \end{cases}$$

These relations find their principal use in the summation of like powers of the functions U_n and V_n. The expansion of the powers of a binomial also gives rise to a certain number of others. Thus we have, for example

$$\alpha = (\alpha + \beta) - \beta, \quad \beta = (\beta + \alpha) - \alpha .$$

Hence, for p equal to an odd number

$$\alpha^p + \beta^p = (\alpha+\beta)^p - \binom{p}{1}\beta(\alpha+\beta)^{p-1} + \binom{p}{2}\beta^2(\alpha+\beta)^{p-2} + \cdots$$
$$+ \binom{p}{p-1}\beta^{p-1}(\alpha+\beta)\,,$$
$$\alpha^p + \beta^p = (\alpha+\beta)^p - \binom{p}{1}\alpha(\alpha+\beta)^{p-1} + \binom{p}{2}\alpha^2(\alpha+\beta)^{p-2} + \cdots$$
$$+ \binom{p}{p-1}\alpha^{p-1}(\alpha+\beta)\ .$$

We thus have, by adding and subtracting and letting $\alpha = a^n$, $\beta = b^n$, the following formulas

$$(100^\star)\quad \begin{cases} 2V_{np} = V_0 V_n^p - \binom{p}{1}V_n V_n^{p-1} + \binom{p}{2}V_{2n}V_n^{p-2} + \cdots + \binom{p}{p-1}V_{(p-1)n}V_n \\ 0 = \binom{p}{1}U_n V_n^{p-1} - \binom{p}{2}U_{2n}V_n^{p-2} + \cdots - \binom{p}{p-1}U_{(p-1)n}V_n\ . \end{cases}$$

Analogous expansions can be found for p equal to an even number and still others with the aid of the identities

$$\alpha = (\alpha - \beta) + \beta, \quad \beta = (\beta - \alpha) + \alpha\ .$$

The following formula which may be derived from combinatorial considerations

$$(\star)\quad \begin{aligned}(\alpha+\beta)^{p+q-1} = \alpha^p\Big[&(\alpha+\beta)^{q-1} + \binom{p}{1}(\alpha+\beta)^{q-2}\beta + \binom{p+1}{2}(\alpha+\beta)^{q-3}\beta^2 + \cdots \\ &+ \binom{p+q-2}{q-1}\beta^{q-1}\Big] \\ + \beta^q\Big[&(\alpha+\beta)^{p-1} + \binom{q}{1}(\alpha+\beta)^{p-2}\alpha + \binom{q+1}{2}(\alpha+\beta)^{p-3}\alpha^2 + \cdots \\ &+ \binom{p+q-2}{p-1}\alpha^{q-1}\Big],\end{aligned}$$

gives, on changing α to β and by addition and subtraction

$$
(101^{*})\quad \left\{
\begin{aligned}
2V_n^{p+q-1} &= V_{pn}V_n^{q-1} + \binom{p}{1}Q^n V_{(p-1)n}V_n^{q-2} + \binom{p+1}{2}Q^{2n}V_{(p-2)n}V_n^{q-3} + \cdots \\
&\quad + \binom{p+q-2}{q-1}Q^{(q-1)n}V_{(p-q+1)n} + V_{qn}V_n^{p-1} + \binom{q}{1}Q^n V_{(q-1)n}V_n^{p-2} \\
&\quad + \binom{q+1}{2}Q^{2n}V_{(q-2)n}V_n^{p-3} + \cdots + \binom{p+q-2}{p-1}Q^{(p-1)n}V_{(q-p+1)n}\,, \\
0 &= U_{pn}V_n^{q-1} + \binom{p}{1}Q^n U_{(p-1)n}V_n^{q-2} + \binom{p+1}{2}Q^{2n}U_{(p-2)n}V_n^{q-3} + \cdots \\
&\quad + \binom{p+q-2}{q-1}Q^{(q-1)n}U_{(p-q+1)n} - U_{qn}V_n^{p-1} - \binom{q}{1}Q^n U_{(q-1)n}V_n^{p-2} \\
&\quad - \binom{q+1}{2}Q^{2n}U_{(q-2)n}V_n^{p-3} - \cdots - \binom{p+q-2}{p-1}Q^{(p-1)n}U_{(q-p+1)n}\,.
\end{aligned}
\right.
$$

We would obtain two other formulas resembling the preceding ones by letting $\alpha = a^n$ and $\beta = b^n$; we simplify these formulas by letting $p = q$.

XVII. OTHER FORMULAS CONCERNING THE EXPANSION OF THE NUMERICAL FUNCTIONS U_n AND V_n

Let us consider the functions α and β of z,

$$\alpha = \left(\frac{z + \sqrt{z^2 - 4h}}{2}\right)^n, \quad \beta = \left(\frac{z - \sqrt{z^2 - 4h}}{2}\right)^n .$$

Upon differentiation we get

$$\frac{d\alpha}{\alpha\, dz} = \frac{n}{\sqrt{z^2 - 4h}}\,,$$

and making the radical vanish

$$(z^2 - 4h)\frac{d\alpha^2}{dz^2} - n^2\alpha^2 = 0\,.$$

A second differentiation gives us

$$(z^2 - 4h)\frac{d^2\alpha}{dz^2} + z\frac{d\alpha}{dz} - n^2\alpha = 0\,.$$

Moreover, it is easy to see that the function β, $\alpha+\beta$, and $\alpha-\beta$ satisfy the same differential equation. Denoting any one of them by $f(z)$ we therefore have by an application of Leibniz's Theorem

$$(z^2-4h)\frac{d^{p+2}f(z)}{dz^{p+2}}+(2p+1)z\frac{d^{p+1}f(z)}{dz^{p+1}}+(p^2-n^2)\frac{d^pf(z)}{dz^p}=0\ ,$$

and for $z=0$,

$$4h\frac{d^{p+2}f(0)}{dz^{p+2}}=(p^2-n^2)\frac{d^pf(0)}{dz^p}\quad .$$

If we assume $z=V_r$ and $h=Q^r$ then Maclaurin's Formula gives us, for even n, the two expansions

$$(102)\begin{cases}\dfrac{V_{nr}}{2(-Q^r)^{n/2}}=1-\dfrac{n^2}{1\cdot 2}\dfrac{V_r^2}{2^2Q^r}+\dfrac{n^2(n^2-2^2)}{4!}\dfrac{V_r^4}{2^4Q^{2r}}-\dfrac{n^2(n^2-2^2)(n^2-4^2)}{6!}\cdot\dfrac{V_r^6}{2^6Q^{3r}}+\cdots\\[2ex] \dfrac{-U_{nr}}{2(-Q^r)^{n/2}U_r}=\dfrac{n}{1}\dfrac{V_r}{2Q^r}-\dfrac{n(n^2-2^2)}{3!}\cdot\dfrac{V_r^3}{2^3Q^{3r}}+\dfrac{n(n^2-2^2)(n^2-4^2)}{5!}\cdot\dfrac{V_r^5}{2^5Q^{5r}}-\cdots\end{cases}$$

and for n odd,

$$(103)\begin{cases}\dfrac{U_{nr}}{(-Q^r)^{(n-1)/2}}=U_r\left[1-\dfrac{n^2-1^2}{2!}\dfrac{V_r^2}{2^2Q^r}+\dfrac{(n^2-1^2)(n^2-3^2)}{4!}\dfrac{V_r^4}{2^4Q^{2r}}-\cdots\right]\\[2ex] \dfrac{V_{nr}}{(-Q^r)^{(n-1)/2}}=V_r\left[n-\dfrac{n(n^2-1^2)}{3!}\dfrac{V_r^2}{2^2Q^r}+\dfrac{n(n^2-1^2)(n^2-3^2)}{5!}\dfrac{V_r^4}{2^4Q^{2r}}-\cdots\right]\end{cases}$$

We can moreover verify these formulas and the following ones, á posteriori, by noting that if we let

$$G_{m,k} = (m^2 - 2^2)(m^2 - 4^2) \cdots (m^2 - 4k^2) ,$$
$$H_{m,k} = (m^2 - 1^2)(m^2 - 3^2) \cdots (m^2 - (2k - 1)^2)$$

we have the relations

$$mG_{m,k} = (m - 2k)H_{m+1,k} = (m + 2k)H_{m-1,k} .$$

Instead of expanding the functions U_{nr} and V_{nr} in powers of V_r we may also expand them in powers of U_r. For even n we thus find

$$(104^{*}) \quad \begin{cases} \dfrac{V_{nr}}{2Q^{nr/2}} = 1 + \dfrac{n^2}{2!}\dfrac{\Delta U_r^2}{2^2Q^r} + \dfrac{n^2(n^2 - 2^2)}{4!}\dfrac{\Delta^2 U_r^4}{2^4Q^{2r}} + \dfrac{n^2(n^2 - 2^2)(n^2 - 4^2)}{6!} \cdot \dfrac{\Delta^3 U_r^6}{2^6Q^{3r}} + \cdots \\[2ex] \dfrac{U_{nr}}{Q^{nr/2-r}} = \dfrac{U_{2r}}{2}\left[n + \dfrac{n(n^2 - 2^2)}{3!}\dfrac{\Delta U_r^2}{2^2Q^r} + \dfrac{n(n^2 - 2^2)(n^2 - 4^2)}{5!} \cdot \dfrac{\Delta^2 U_r^4}{2^4Q^{2r}} + \cdots \right] \end{cases}$$

and for odd n

$$(105) \quad \begin{cases} \dfrac{V_{nr}}{Q^{(n-1)r/2}} = V_r\left[1 + \dfrac{n^2 - 1^2}{2!}\dfrac{\Delta U_r^2}{2^2Q^r} + \dfrac{(n^2 - 1^2)(n^2 - 3^2)}{4!}\dfrac{\Delta^2 U_r^4}{2^2Q^{2r}} + \dfrac{(n^2 - 1^2)(n^2 - 3^2)(n^2 - 5^2)}{6!}\dfrac{\Delta^3 U_r^6}{2^6Q^{3r}} + \cdots \right], \\[2ex] \dfrac{U_{nr}}{Q^{(n-1)r/2}} = U_r\left[n + \dfrac{n(n^2 - 1^2)}{3!}\dfrac{\Delta U_r^2}{2^2Q^r} + \dfrac{n(n^2 - 1^2)(n^2 - 3^2)}{5!}\dfrac{\Delta^2 U_r^4}{2^4Q^{2r}} + \dfrac{n(n^2 - 1^2)(n^2 - 3^2)(n^2 - 5^2)}{7!}\dfrac{\Delta^3 U_r^6}{2^6Q^{3r}} + \cdots \right]. \end{cases}$$

Taking account of one or the other of the relations

$$V_{nr}^2 = V_{2nr} + 2Q^{nr}, \quad \Delta U_{nr}^2 = V_{2nr} - 2Q^{nr} ,$$

we will obtain new formulas. Thus for example

$$(106^*) \qquad \frac{U_{nr}^2}{Q^{(n-2)r}U_r^2} = n^2 - \frac{n^2(n^2-1^2)}{3\cdot 4}\,\frac{\Delta U_r^2}{Q^r} + \frac{n^2(n^2-1^2)(n^2-2^2)}{3\cdot 4\cdot 5\cdot 6}\,\frac{\Delta^2 U_r^4}{Q^{2r}} - \cdots .$$

We can moreover put this last formula and some others into a rather remarkable form by noting that we have, for any positive integers m and n the identity

$$\frac{m^2(m^2-1^2)(m^2-3^2)\cdots(m^2-(n-1)^2)}{(2n)!/2} = \frac{(m-n)(m-n+1)(m-n+2)\cdots(m+n-1)}{(2n)!} + \frac{(m-n+1)(m-n+2)\cdots(m+n)}{(2n)!} .$$

Consequently the coefficients of the formula (106) are integers and we have[1]

$$\frac{n^2(n^2-1^2)(n^2-2^2)\cdots(n^2-(r-1)^2)}{(2r)!/2} = \binom{n+r-1}{2r} + \binom{n+r}{2r}$$

We will note that formulas (104) and (105) refer to any value of n. We then have expansions in convergent series when $\Delta U_r^2/2^2Q^r$ is not greater than unity. In fact, if we assume

$$\frac{\Delta U_r^2}{2^2Q^r} \le 1 ,$$

the ratio of a term to its precedent finally becomes negative (for positive Δ) and less than unity in absolute value. This condition is satisfied for $r = 1$ in the Pell series. Whatever the value of n we thus have

[1]Denoting by a the residue of n^2 modulo p, relatively prime to n, we derive from this identity an immediate demonstration of a proposition contained in No. 128 of the Disquisitiones Arithmeticae.

$$
(107)\quad \begin{cases} \dfrac{1}{2}\left[(\sqrt{2}+1)^n + (\sqrt{2}-1)^n\right] = 1 + \dfrac{n^2}{2!} + \dfrac{n^2(n^2-2^2)}{4!} + \dfrac{n^2(n^2-2^2)(n^2-4^2)}{6!} + \cdots \\[2ex] \dfrac{1}{2}\left[(\sqrt{2}+1)^n + (\sqrt{2}-1)^n\right] = n + \dfrac{n(n^2-1^2)}{3!} + \dfrac{n(n^2-1^2)(n^2-3^2)}{5!} + \cdots \end{cases}
$$

XVIII. EXPANSIONS IN SERIES OF IRRATIONALS AND THEIR NATURAL LOGARITHMS

The expansions of the functions in series, by MacLaurin's formula, lead to a large number of new formulas for the expansion of the numerical functions which we are considering here, and consequently, for those of the circular and hyperbolic functions. When the corresponding series does not converge except for values of the variable whose modulus is less than a given limit, we may always assume that this variable x is chosen in such a manner that the series represents the function for all values of x whose modulus is less than unity. Consider the series

$$
F(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4 + \cdots .
$$

Assuming that z is positive we will have

$$
F\left(\frac{z}{1+z}\right) = A_0 + A_1 \frac{z}{1+z} + A_2 \frac{z^2}{(1+z)^2} + A_3 \frac{z^3}{(1+z)^3} + \cdots
$$

$$
F\left(\frac{1}{1+z}\right) = A_0 + A_1 \frac{1}{1+z} + A_2 \frac{1}{(1+z)^2} + A_3 \frac{1}{(1+z)^3} + \cdots
$$

Consequently

$$
F\left(\frac{1}{1+z}\right) + F\left(\frac{z}{1+z}\right) = 2A_0 + A_1 \frac{1+z}{1+z} + A_2 \frac{1+z^2}{(1+z)^2} + A_3 \frac{1+z^3}{(1+z)^3} + \cdots ,
$$

$$
F\left(\frac{1}{1+z}\right) - F\left(\frac{1}{1+z}\right) = A_1 \frac{1-z}{1+z} + A_2 \frac{1-z^2}{(1+z)^2} + A_3 \frac{1-z^3}{(1+z)^3} + \cdots
$$

If we denote by a the largest of the roots, assumed positive, of the fundamental equation (1), with r an even or arbitrary integer, depending on whether the root b is negative or positive, and if we let $a = b^r/a^r$, we obtain

$$(108)\quad \begin{cases} F\left(\dfrac{a^r}{a^r+b^r}\right) + F\left(\dfrac{a^r b^r}{a^r+b^r}\right) = 2A_0 + A_1\dfrac{V_r}{V_r} + A_2\dfrac{V_{2r}}{V_r^2} + A_3\dfrac{V_{3r}}{V_r^3} + \cdots \\[2ex] F\left(\dfrac{a^r}{a^r+b^r}\right) - F\left(\dfrac{a^r b^r}{a^r+b^r}\right) = \sqrt{\Delta}\left[A_1\dfrac{U_r}{V_r} + A_2\dfrac{U_{2r}}{V_r^2} + A_3\dfrac{U_{3r}}{V_r^3} + \cdots\right] \end{cases}$$

If we assume $z = -b^r/a^r$ we obtain two expansions analogous to the preceding ones. These expansions are sometimes very slowly convergent but their study leads to important properties in the theory of prime numbers.

The expansion of the binomial $(1-x)^m$ thus gives, for any m, the series

$$(109)\quad \begin{cases} \dfrac{V_{mr}}{V_r^m} = V_0 - \dbinom{m}{1}\dfrac{V_r}{V_r} + \dbinom{m}{2}\dfrac{V_{2r}}{V_r^2} - \dbinom{m}{3}\dfrac{V_{3r}}{V_r^3} + \cdots \\[2ex] \dfrac{U_{mr}}{V_r^m} = \dbinom{m}{1}\dfrac{U_r}{V_r} - \dbinom{m}{2}\dfrac{U_{2r}}{V_r^2} + \dbinom{m}{3}\dfrac{U_{3r}}{V_r^3} - \cdots \end{cases}$$

which we could have derived from the Bernoulli series. For $m = -1$ we have

$$(110)\quad \begin{cases} \dfrac{V_r^2}{Q^r} = V_0 + \dfrac{V_r}{V_r} + \dfrac{V_{2r}}{V_r^2} + \dfrac{V_{3r}}{V_r^3} + \cdots, \\[2ex] \dfrac{U_{2r}}{Q^r} = \dfrac{U_r}{V_r} + \dfrac{U_{2r}}{V_r^2} + \dfrac{U_{3r}}{V_r^3} + \dfrac{U_{4r}}{V_r^4} + \cdots, \end{cases}$$

For example, in the Fibonacci series

$$(111)\quad \begin{cases} 9 = 2 + \dfrac{3}{3} + \dfrac{7}{9} + \dfrac{18}{27} + \dfrac{47}{81} + \cdots, \\[2ex] 3 = \dfrac{1}{3} + \dfrac{3}{9} + \dfrac{8}{27} + \dfrac{21}{81} + \dfrac{55}{243} + \cdots, \end{cases}$$

The numerators of these two series of fractions are given by the recurrence relation

$$N_{n+2} = 3N_{n+1} - N_n \; .$$

We will obtain similar formulas for $m = \pm\frac{1}{2}$. The expansion of

$$(1 + x)^m \pm (1 - x)^m$$

gives formulas analogous to the relations (109).

The expansion of $\log (1 - x)$ gives the formulas

$$(112) \quad \begin{cases} \log \dfrac{V_r^2}{Q^r} = 1 + \dfrac{1}{2} \dfrac{V_{2r}}{V_r^2} + \dfrac{1}{3} \dfrac{V_{3r}}{V_r^3} + \dfrac{1}{4} \dfrac{V_{4r}}{V_r^4} + \cdots \\[2ex] \log \dfrac{b^{2r}}{Q^r} = 2\sqrt{\Delta}\left[\dfrac{U_r}{V_r} + \dfrac{1}{2} \dfrac{U_{2r}}{V_r^2} + \dfrac{1}{3} \dfrac{U_{3r}}{V_r^3} + \dfrac{1}{4} \dfrac{U_{4r}}{V_r^4} + \cdots\right] ; \end{cases}$$

That of $\log (1 - x)/(1 + x)$ gives

$$(113) \quad \log \frac{b^{2r}}{Q^r} = 2\sqrt{\Delta}\left[\frac{U_r}{V_r} + \frac{1}{3} \frac{\Delta U_{3r}}{V_r^3} + \frac{1}{5} \frac{\Delta^2 U_{5r}}{V_r^5} + \frac{1}{7} \frac{\Delta^2 U_{7r}}{V_r^7} + \cdots\right].$$

and in the Pell series

$$(114) \quad \sqrt{2} \log (1 + \sqrt{2}) = 1 + \frac{1}{2 \cdot 3} + \frac{1}{2^2 \cdot 5} + \frac{1}{2^3 \cdot 7} + \frac{1}{2^5 \cdot 9} + \cdots \; .$$

The formula

$$\frac{1}{2} \log \frac{z + h}{z - h} = hz\left[\frac{1}{z^2 - h^2} - \frac{2}{3} \frac{h^2}{(z^2 - h^2)^2} + \frac{2 \cdot 4}{3 \cdot 5} \frac{h^4}{(z^2 - h^2)^3} - \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7} \frac{h^6}{(z^2 = h^2)^4} + \cdots\right] ,$$

in which we assume

$$z + h = a^r, \quad z - h = b^r, \quad z^2 - h^2 = Q^r, \quad h^2 = \frac{U_r^2}{4} \ ,$$

also gives

$$\text{(115)} \qquad \log \frac{a^{2r}}{Q^r} = \frac{\sqrt{\Delta}U_r}{2}\left[\frac{1}{Q^r} - \frac{2}{3}\,\frac{\Delta U_r^2}{2^2Q^{2r}} + \frac{2\cdot 4}{3\cdot 5}\,\frac{\Delta^2 U_r^4}{2^3Q^{3r}} - \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\,\frac{\Delta^3 U_r^6}{2^4Q^{4r}} + \cdots\right] ;$$

For this series to converge we must have $\Delta U_r^2 \leq 4Q^r$. At the limit of convergence we thus find

$$\text{(116)} \qquad \log\,(1 + \sqrt{2}) = \sqrt{2}\left[1 - \frac{2}{3} + \frac{2\cdot 4}{3\cdot 5} - \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7} + \frac{2\cdot 4\cdot 6\cdot 8}{3\cdot 5\cdot 7\cdot 9} - \cdots\right] .$$

Likewise the expansions of arcsin z and of $(\arcsin z)^2$ give

$$\text{(117)} \qquad \begin{cases} \log \dfrac{a^{2r}}{Q^r} = \dfrac{\sqrt{\Delta}U_r}{Q^{r/2}}\left[1 - \dfrac{1}{1\cdot 2\cdot 3}\,\dfrac{\Delta U_r^2}{2^2Q^r} + \dfrac{(1,3)^2}{1\cdot 2\cdot 3\cdot 4\cdot 5}\,\dfrac{\Delta^2U_r^4}{2^4Q^{2r}} - \cdots\right], \\[2ex] \dfrac{1}{4}\log^2 \dfrac{a^{2r}}{Q^r} = \dfrac{\Delta U_r^2}{2^2Q^r} - \dfrac{1}{2}\cdot\dfrac{2}{3}\,\dfrac{\Delta^2U_r^4}{2^4Q^{2r}} + \dfrac{1}{3}\cdot\dfrac{2\cdot 4}{3\cdot 5}\,\dfrac{\Delta^3U_r^6}{2^6Q^{3r}} - \cdots \ , \end{cases}$$

and at the limit of convergence

$$\text{(118)} \qquad \begin{cases} \log\,(1 + \sqrt{2}) = 1 - \dfrac{1}{1\cdot 2\cdot 3} + \dfrac{(1,3)^2}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \dfrac{(1\cdot 3\cdot 5)^2}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} + \cdots , \\[2ex] \log^2(1 + \sqrt{2}) = 1 - \dfrac{1}{2}\cdot\dfrac{2}{3} + \dfrac{1}{3}\cdot\dfrac{2\cdot 4}{3\cdot 5} - \dfrac{1}{5}\cdot\dfrac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7} + \cdots \end{cases}$$

The remarkable formula of M. Scholtz leads to the expansion

$$\text{(119)} \qquad \begin{aligned} \log^3 \frac{a^{2r}}{Q^r} = \frac{\Delta^{\frac{3}{2}}U_r^3}{Q^{\frac{3r}{2}}}\Bigg[1 &- \frac{3\cdot 3}{4\cdot 5}\left(1 + \frac{1}{3^2}\right)\frac{\Delta U_r^2}{2^2Q^r} + \frac{3\cdot 5\cdot 3}{4\cdot 6\cdot 7}\left(1 + \frac{1}{3^2} + \frac{1}{5^2}\right)\frac{\Delta^2U_r^4}{2^4Q^{2r}} - \cdots \\ &\pm \frac{3\cdot 5\cdot 7\cdots(2n-1)3}{4\cdot 6\cdot 8\cdots 2n(2n+1)}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n-1)^2}\right)\frac{\Delta^{n-1}U_r^{2n-2}}{2^{2n-2}Q^{(n-1)r}} \mp \cdots\Bigg] , \end{aligned}$$

and at the limit of convergence

$$(120) \qquad \log^3 (1+\sqrt{2}) = 1 - \frac{3\cdot 3}{4\cdot 5}\left(1 + \frac{1}{3^2}\right) + \frac{3\cdot 5\cdot 3}{4\cdot 6\cdot 7}\left(1 + \frac{1}{3^2} + \frac{1}{5^2}\right) - \cdots$$

If we expand, by Lagrange's formula, one of the roots a^r or b^r of the equation

$$z^2 - zV_r + Q^r = 0 ,$$

we find

$$(121) \qquad \left\{ \begin{aligned} b^r &= \frac{Q^r}{V_r} + \frac{Q^{2r}}{V_r^3} + \frac{4}{2}\frac{Q^{3r}}{V_r^5} + \frac{5\cdot 6}{2\cdot 3}\frac{Q^{4r}}{V_r^7} + \cdots , \\ \log b^r &= \log \frac{Q^r}{V_r} + \frac{Q^r}{V_r^2} + \frac{3}{2}\frac{Q^{2r}}{V_r^4} + \frac{5\cdot 4}{2\cdot 3}\frac{Q^{3r}}{V_r^6} + \cdots , \\ \frac{1}{3} b^{2r} &= \frac{Q^{2r}}{2V_r^2} + \frac{Q^{3r}}{V_r^4} + \frac{5}{2}\frac{Q^{4r}}{V_r^6} + \frac{7\cdot 6}{2\cdot 3}\frac{Q^{5r}}{V_r^8} + \cdots . \end{aligned} \right.$$

Further if we expand y^{-n} in powers of z using Lagrange's formula, and denote by y one of the roots of the equation

$$y = 2 + \frac{z}{y} ,$$

we obtain

$$\left(\frac{2}{1+\sqrt{1+z}}\right)^n = 1 - \frac{n}{1}\frac{z}{4} + \frac{n(n+3)}{1\cdot 2}\left(\frac{z}{4}\right)^2 - \frac{n(n+4)(n+5)}{1\cdot 2\cdot 3}\left(\frac{z}{4}\right)^3 + \frac{n(n+5)(n+6)(n+7)}{1\cdot 2\cdot 3\cdot 4}\left(\frac{z}{4}\right)^4 - \cdots .$$

Letting

$$\frac{z}{4} = -\frac{Q^r}{V_r^2} ,$$

we have

$$(122)\qquad \frac{V_{nr}\alpha^{nr}}{Q^{nr}} = 1 + \frac{n}{1}\frac{Q^r}{V_r^2} + \frac{n(n+3)}{1\cdot 2}\frac{Q^{2r}}{V_r^4} + \frac{n(n+4)(n+5)}{1\cdot 2\cdot 3}\frac{Q^{3r}}{V_r^6} + \cdots$$

This series is convergent for

$$\frac{Q^r}{V_r^2} < 1 .$$

It is the generalization of formula (84).

We also have

$$b^r = \frac{V_r - \sqrt{V_r^2 - 4Q_r}}{2}$$

Expanding the radical by the binomial formula

$$(123)\qquad b^r = \frac{1}{2}\frac{2Q^r}{V_r} + \frac{1}{2\cdot 4}\frac{2^3Q^{2r}}{V_r^3} + \frac{1\cdot 3}{2\cdot 4\cdot 6}\frac{2^5Q^{3r}}{V_r^5} + \cdots ,$$

then, at the limit of convergence,

$$(124)\qquad \sqrt{2} - 1 = \frac{1}{2} - \frac{1}{2\cdot 4} + \frac{1\cdot 3}{2\cdot 4\cdot 6} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8} + \cdots$$

By applying Burmann's formula to the expansion of z in powers of

$$\frac{2z}{1 + z^2}$$

we would obtain, for all moduli of z less than unity, and $a = b^r/a^r$, the formula (121) given above.

XIX. ON THE RAPID CALCULATION OF CONTINUED PERIODIC FRACTIONS

We can improve the caluulation of the convergents of the continued periodic functions in a significant way by means of the following formulas. M. Catalan gave these relations:

$$\frac{z}{1-z^2} + \frac{z^2}{1-z^4} = \frac{z+z^2+z^3}{1-z^4} \quad ,$$

$$\frac{z}{1-z^2} + \frac{z^2}{1-z^4} + \frac{z^4}{1-z^8} = \frac{z+z^2+z^3+z^4+z^5+z^6+z^7}{1-z^8} \quad ,$$

$$\frac{z}{1-z^2} + \frac{z^2}{1-z^4} + \frac{z^4}{1-z^8} + \frac{z^5}{1-z^{10}} = \frac{z+z^2+\cdots+z^{14}+z^{15}}{1-z^{16}} \quad ,$$

More generally we have

$$\frac{z}{1-z^2} + \frac{z^2}{1-z^4} + \cdots + \frac{z^{2^{n-1}}}{1-z^{2^n}} = \frac{1}{1-z} \cdot \frac{z-z^{2^n}}{1-z^{2^n}} \quad .$$

Consequently if we let $z = b^r/a^r$ we obtain the formula

$$\text{(125)} \qquad \frac{Q^r}{U_{2r}} + \frac{Q^{2r}}{U_{4r}} + \cdots + \frac{Q^{2^{n-1}\cdot r}}{U_{2^n\cdot r}} = \frac{Q^r U_{(2^n-1)r}}{U_r U_{2^n\cdot r}} \quad .$$

When n increases indefinitely we have, for the series of the first and second kind

$$\text{(126)} \qquad \frac{b^r}{U_r} = \frac{Q^r}{U_{2r}} + \frac{Q^{2r}}{U_{4r}} + \frac{Q^{4r}}{U_{8r}} + \cdots \quad .$$

For example, in the Fibonacci series, for $r = 1$,

$$(127) \qquad \frac{1-\sqrt{5}}{2} = -\frac{1}{1} + \frac{1}{3} + \frac{1}{3\cdot 7} + \frac{1}{3\cdot 7\cdot 47} + \frac{1}{3\cdot 7\cdot 47\cdot 2207} + \cdots .$$

Each new factor in the denominator is equal to the square of the preceding factor diminished by two. Likewise in the Pell series

$$(128) \qquad 1-\sqrt{2} = -\frac{1}{2} + \frac{1}{2^2\cdot 3} + \frac{1}{2^3\cdot 3\cdot 17} + \frac{1}{2^4\cdot 3\cdot 17\cdot 577} + \cdots .$$

Each new factor in the denominator is equal, by the duplication formulas, to twise the square of the preceding factor diminished by one.

These expansions converge very rapidly. It is, in a manner of speaking, a combination of a logarithmic calculation and a calculation by continued fractions. Thus the denominator of the thirty-second fraction of formula (127) is very nearly equal to

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{2^{32}},$$

and contains approximately two hundred million digits. Two hundred million centuries would be needed to write the sixty-fourth fraction of formula (128).

We have moreover shown (Section XI) that the different factors of the denominators are relatively prime in pairs and consequently they all contain different prime factors. It follows from this that in the sum of the first n terms of this series there will be no occasion to reduce this sum to a simpler expression. We will later show that all these factors, prime and different, appear in known linear and quadratic forms.

More generally we have the identity

$$(129) \qquad \frac{z-z^q}{(1-z)(1-z^q)} + \frac{z^q - z^{pq}}{(1-z^q)(1-z^{pq})} = \frac{z-z^{pq}}{(1-z)(1-z^{pq})} .$$

If we replace q by p^n we then have

$$(130)\qquad \frac{z - z^{p^n}}{(1 - z)(1 - z^{p^n})} + \frac{z^{p^n} - z^{p^{n+1}}}{(1 - z^{p^n})(1 - z^{p^{n+1}})} = \frac{z - z^{p^{n+1}}}{(1 - z)(1 - z^{p^{n+1}})} .$$

If we let n successively equal 1, 2, 3, $\cdots$, n and if we add the equalities obtained we have

$$(131)\qquad \frac{z - z^{p}}{(1 - z)(1 - z^{p})} + \frac{z^{p} - z^{p^2}}{(1 - z^{p})(1 - z^{p^2})} + \frac{z^{p^2} - z^{p^3}}{(1 - z^{p^2})(1 - z^{p^3})} + \cdots + \frac{z^{p^n} - z^{p^{n+1}}}{(1 - z^{p^n})(1 - z^{p^{n+1}})} = \frac{z - z^{p^{n+1}}}{(1 - z)(1 - z^{p^{n+1}})} .$$

Now let $z = b^r/a^r$. We obtain the formula

$$(132\star)\qquad \frac{Q^rU_{(p-1)r}}{U_rU_{pr}} + \frac{Q^{pr}U_{(p-1)pr}}{U_{pr}U_{p^2r}} + \frac{Q^{p^2r}U_{(p-1)p^2r}}{U_{p^2r}U_{p^3r}} + \cdots + \frac{Q^{p^nr}U_{(p-1)p^nr}}{U_{p^nr}U_{p^{n+1}r}} = \frac{Q^rU_{(p^{n+1}-1)r}}{U_rU_{p^{n+1}r}} .$$

We will moreover calculate the numerators and the denominators of these fractions by means of the formulas for the multiplication of the numerical functions which we have given. If p denotes an odd number we obtain an analogous formula by changing U to V. We may also apply those formulas to the circular functions.

Later we will give analogous formulas which we derive from the theory of elliptic functions and in particular the sums of the inverses of the terms U_n and of their like powers.

XX. RELATION OF THE FUNCTIONS U_n AND V_n WITH THE THEORY OF THE BINOMIAL EQUATION

We know, from the theory of the binomial equation, stated in the last section of the Disquisitiones Arithmeticae, that if p denotes an odd prime number the quotient

$$4\,\frac{z^p - 1}{z - 1} = 4(z^{p-1} + z^{p-2} + z^{p-3} + \cdots + z^2 + z + 1)$$

may be written in the form

$$4\,\frac{z^p - 1}{z - 1} = Y^2 \pm pZ^2 ,$$

in which Y and Z are polynomials in z with integer coefficients. We take the + sign when p denotes a prime number of the form $4q + 3$ and the − sign when p denotes a prime number of the form $4q + 1$. If we let $z = \sqrt{a^r/b^r}$ in this formula we derive from it the following results for p successively equal to 3, 5, 7, 11, 13, 17, 19, 23, 29, ⋯ (*: The correct formula, given here, is not an example of the process stated above — Ed. note)

$$\left\{\begin{aligned}
\frac{U_{3r}}{U_r} &= \Delta U_r^2 + 3Q^r ,\\
4\,\frac{U_{5r}}{U_r} &= [2V_{2r} + Q^r]^2 - 5Q^{2r} ,\\
4\,\frac{U_{7r}}{U_r} &= \Delta[2U_{3r} + Q^rU_r]^2 + 7Q^{2r}V_r^2 ,\\
4\,\frac{U_{11r}}{U_r} &= \Delta[2U_{5r} + Q^rU_{3r} - 2Q^{2r}U_r]^2 + 11Q^{2r}V_{3r}^2 ,\\
4\,\frac{U_{13r}}{U_r} &= [2V_{6r} + Q^rV_{4r} + 4Q^{2r}V_{2r} - Q^{3r}]^2 - 13Q^{2r}[V_{4r} + Q^{2r}]^2\\
4\,\frac{U_{17r}}{U_r} &= [2V_{8r} + Q^rV_{6r} + 5Q^{2r}V_{4r} + 7Q^{3r}V_{2r} + 4Q^{4r}]^2\\
&\quad - 17Q^{2r}[V_{6r} + Q^rV_{4r} + Q^{2r}V_{2r} + 2Q^{3r}]^2,
\end{aligned}\right.$$

$$
(133)\begin{cases}
4\,\dfrac{U_{19r}}{U_r} = \Delta\left[\,2U_{9r} + Q^rU_{7r} - 4Q^{2r}U_{5r} + 3Q^{3r}U_{3r} + 5Q^{4r}U_r\right]^2 \\
\qquad\qquad + 19Q^{2r}\left[V_{7r} - Q^{2r}V_{3r} + Q^{3r}V_r\right]^2, \\
4\,\dfrac{U_{23r}}{U_r} = \Delta\left[\,2U_{11r} + Q^rU_{9r} - 5Q^{2r}U_{7r} - 8Q^{3r}U_{5r} - 7Q^{4r}U_{3r} - 4Q^{5r}U_r\right]^2 \\
\qquad\qquad + 23Q^{2r}\left[V_{9r} + Q^rV_{7r} - Q^{3r}V_{3r} - 2Q^{4r}V_r\right]^2, \\
4\,\dfrac{U_{29r}}{U_r} = \left[2V_{14r} + Q^rV_{12r} + 8Q^{2r}V_{10r} - 3Q^{3r}V_{8r} + Q^{4r}V_{6r} - 2Q^{5r}V_{4r} + 3Q^{6r}V_{2r}\right. \\
\qquad \left. + 9Q^{7r}\right]^2 - 29Q^{2r}\left[V_{12r} + Q^{2r}V_{8r} = Q^{3r}V_{6r} + Q^{5r}_{2r}V + Q^{6r}\right]^2, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{cases}
$$

We consequently have the following proposition

<u>Theorem</u>: If p denotes a prime number of the form $4q + 1$ the quotient $4U_{pr}/U_r$ may be put in the form $U^2 - pZ^2$, and if p denotes a prime number of the form $4q + 3$ the quotient $4U_{pr}/U_r$ may be put in the form $\Delta Y^2 + pZ^2$.

Moreover, by changing z to $-z$ we will obtain a similar result for the quotient $4V_{pr}/V_r$. Thus we generalize a theorem given by Legendre whose proof by this means is simplified. Another important consequence results from the formulas (133). In fact, up to now, we have let Δ be arbitrary, but if we consider the functions of the third kind we may assume $-\Delta$ equal to the product of a square and a prime number of the form $4q + 3$*. We then see that the quotients $4U_{pr}/pU_r$ and $4V_{pr}/pV_r$ are equal to the difference of squares and consequently decomposable into the product of two factors. We then have this proposition:

<u>Theorem</u>: If $-\Delta$ is equal to the product of a prime number p of the form $4q + 3$ and a square, the quotients $4U_{pr}/U_r$ and $4V_{pr}/V_r$ are, whatever the integer value of r, decomposable into the product of two integer factors.

If we consider the fundamental equation

$$x^2 = x - 2$$

---

* In fact, it suffices to determine Q from the relation $4Q - P^2 = pK^2$.

in which $\Delta = -7$, we obtain for example

$$U_{11} = +23, \quad U_{77} = -26\,4721893121\,;$$

and as a result

$$U_{77} = -7 \times 23 \times 11087 \times 148303\,.$$

We will later show that the prime divisors of $4U_{77}/7U_{11}$ have the linear forms $77q \pm 1$. Consequently the number 11087 is prime without the need to test these divisors since the first of the numbers of the indicated linear form is greater than the square root of 11087. For the factor 148303 only the divisor 307 need be tested. We also have, in the same series

$$U_{13} = -1, \quad U_{91} = -384\,17168\,38057\,,$$

and as a result

$$U_{91} = -7 \times 712711 \times 770041\,.$$

The last two factors are prime. There were but two divisors to test. We thus understand how it is possible to apply the preceding theorem in the direct search for large prime numbers from the consideration of the series of the third kind.

XXI. ON THE CONGRUENCES OF PASCAL'S ARITHMETIC TRIANGLE AND ON A GENERALIZATION OF FERMAT'S THEOREM

Denoting by $\binom{m}{n}$ the number of combinations of m things taken n at a time we have the two findamental formulas

$$\binom{m}{n} = \frac{m(m-1)\cdots(m-n+1)}{1\cdot 2\cdot \cdots (n)}\,,$$

$$\binom{m}{n} = \binom{m-1}{m} + \binom{m-1}{n-1}.$$

Consequently when p is prime we have for integer n between 0 and p the congruence

$$(134)\qquad \binom{p}{n} \equiv 0 \pmod{p} .$$

For n between 0 and p - 1

$$(135)\qquad \binom{p+1}{n} \equiv (-1)^n \pmod{p}$$

For n between 1 and p

$$(136)\qquad \binom{p+1}{n} \equiv 0 \pmod{p} .$$

In other words in Pascal's arithmetic triangle all the numbers on the p^{th} line are, for p prime, divisible by p with the exception of the coefficients at the extremities which are equal to unity. The coefficients of the $(p-1)^{th}$ line alternately give residues of +1 and -1; those of the $(p+1)^{th}$ line are divisible by p except for the four extreme coefficients which are equal to unity.

If we continue the formation of the arithmetic triangle saving only the residues modulo p, we rewrite the arithmetic triangle of the first (p - 1) lines twice; then after the $(2p)^{th}$ line we rewrite it three times but the residues of the intermediate triangle are multiplied by 2. After the third line the triangle of residues is reproduced four times but the numbers in these triangles are respectively multiplied by 1, 3, 3, 1 of the third power of the binomial, etc.

We then have in general

$$\binom{m}{n} \equiv \binom{m_1}{n_1}\binom{\mu}{\nu} \pmod{p}$$

m_1 and n_1 denoting the integral parts of m/p and n/p and μ and ν the residues of m and of n. Likewise we have

$$\binom{m_1}{n_1} \equiv \binom{m_2}{n_2}\binom{\mu_1}{\nu_1} \pmod{p} ,$$

and consequently

$$\binom{m}{n} \equiv \binom{\mu_1}{\nu_1}\binom{\mu_2}{\nu_2}\binom{\mu_3}{\nu_3} \cdots \pmod{p} , \tag{137}$$

$\mu_1, \mu_2, \mu_3, \cdots$ denoting the integral parts of m/p, m/p^2, $m/p^3, \cdots$, and likewise for $\nu_1, \nu_2, \nu_3, \cdots$.

Consequently if we wish to find the remainder of the division of $\binom{n}{m}$ by a prime number it suffices to apply the preceding formula until we have reached the two indices of C for numbers less than p.

We see that the coefficients of order p of the binomial are integers and divisible by p when p denotes a prime number except for the coefficients of the p^{th} powers. Denoting by $\alpha, \beta, \gamma, \cdots \lambda$, any n integers, we then have

$$[\alpha + \beta + \gamma + \cdots + \lambda]^p - [\alpha^p + \beta^p + \gamma^p + \cdots + \lambda^p] \pmod{p} ,$$

and for $\alpha = \beta = \gamma = \cdots = \lambda = 1$, we obtain

$$n^p - n \equiv 0 \pmod{p} .$$

It is from this congruence, which contains Fermat's Theorem, that we may generalize in the following manner, which is different than Euler's approach. If $\alpha, \beta, \gamma, \cdots \lambda$ denote the q^{th} powers of the roots of an equation with integer coefficients and S_q their sum, the first member of the preceding congruence represents the product of p and a symmetric integer function (with integer coefficients) of the roots and consequently of the coefficients of the proposed equation. We then have

$$S_{pq} \equiv S_q^p \pmod{p}$$

with the application of Fermat's Theorem

$$S_{pq} \equiv S_q \pmod{p}. \tag{138}$$

The study of the prime divisors of the numerical function S_n and of several other analogous ones is very important. We have, in particular, for $n = 1$ and $S_1 = 0$ as in the equation

$$x^3 = x + 1$$

The congruence

$$S_p \equiv 0 \pmod{p}.$$

We derive from it inversely that if, in the case $S_1 = 0$, we have S_n divisible by p for $n = p$ and not before, the number p is prime. In fact let p equal, for example, the product of the two prime numbers g and h. We have

$$S_{gh} \equiv -S_h \pmod{g},$$
$$S_{gh} \equiv S_g \pmod{h}.$$

Consequently if we have found

$$S_{gh} \equiv 0 \pmod{gh},$$

we will also have

$$S_g \equiv 0 \pmod{h},$$
$$S_h \equiv 0 \pmod{g},$$

and by the theorem shown

$$S_g \equiv S_h \equiv 0 \pmod{gh}.$$

Thus S_{gh} would not be the first of the numbers S_n divisible by gh.

We may obtain, in this way, a large number of theorems, like those of Wilson, serving to verify prime numbers. We will bypass, for the moment, the new and curious facts that we have thus found to consider only those which can be derived from the numerical functions with a simple period.

XXII. ON THE THEORY OF PRIME NUMBERS AND THEIR RELATIONSHIP TO ARITHMETIC PROGRESSIONS

The theory of prime numbers was outlined by Euclid and Eratosthenes. We owe to Euclid the theory of the divisors and of the common multiples of two or several given numbers, the representation of composite numbers by means of their factors, and the proof of the infinity of prime numbers, which we can easily extend to the proof of the infinity of prime numbers of the linear forms $4x + 3$ and $6x + 5$. In Section XXIV we will given an elementary proof concerning the infinity of prime numbers of the form $mx + 1$ whatever the value of m. We know, moreover, that by the use of infinite series Lejuene-Dirichlet was able to show the infinity of prime numbers of the linear form $a + bx$ where a and b are any two relatively prime integers.[1]

We owe to Eratosthenes an ingeneous method known under the name of the Sieve of Eratosthenes which leads to the formation of a table of prime numbers and of composite numbers. Due to the work of Chernac, of Burckhardt and of Dase we have a table of the first nine millions. Lebesgue has indicated a method which diminishes the volume of these tables.[2] On the other hand M. Glaisher counted the many prime numbers contained in these tables in order to compare the theoretical formulas given by Gauss, Legendre, Tchebychoff, and Heargrave for the expression of the number of prime numbers less than a given integer. M. Glaisher, counting 1 and 2 as primes found the following values.[3]

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[1]Abhandlungen der Berliner Akademie, Berlin, 1837.

[2]Chernac.—Cribrum Arithmeticum from 1 to 1020000. Deventer, 1811.
Burkhardt.—Tables of divisors up to 3036000. Paris, 1814-1817.
Dase.—Factoren Tafeln from 6000000 to 9000000. Vienna, 1862-1865.
Lebesgue.—Various tables for the decomposition of numbers into their prime factors, Paris, 1864.

[3]Preliminary accounts of the results of an enumeration of the primes in Dase's and Burckhardt's tables. Cambridge 1876-1877.

For the first million,	78499 prime numbers[1],				
" "	second	"	, 70433	" "	,
" "	third	"	, 67885	" "	,
" "	seventh	"	, 63799	" "	,
" "	eighth	"	, 63158	" "	,
" "	ninth	"	, 62760	" "	.

The principles of Euclid and of Eratosthenes thus lead to a first method for the verification of prime numbers which are not contained in the tables, and of the decomposition of large numbers into their prime factors, by the successive division of a fixed given number by all prime numbers less than its square root. But it is an indirect method which becomes impractical as soon as the given number has ten digits.

Following along these lines M. Dormoy, using ingeneous considerations, arrived at deductions from the theory of certain numbers which he called objectives (in which we again find the different terms of the Fibonacci series under the name of objectives of unity) leading to the establishment of a general formula for prime numbers. Unfortunately for slightly larger limits this formula contains large coefficients which render its application illusory.[2]

The prime numbers are distributed very irregularly among the integers. In part this is due to the fact that if μ denotes the smallest common multiple of the numbers $2, 3, \cdots, m$, then the numbers

$$\mu + 2, \quad \mu + 3, \quad \cdots, \quad \mu + m ,$$

are respectively divisible by

$$2, 3, \cdots, m .$$

As a consequence we can always find m consecutive composite numbers whatever the value of m. On the other hand, an examination of the tables permits

[1]This table contains several errors. — Ed. note.

[2]E. Dormoy. —General formula for prime numbers and the theory of objectives. Paris, 1867.

us to verify the existence of very large consecutive odd prime numbers. M. Glaisher gave a list of the groups contained in the tables which comprise at least fifty consecutive composite numbers. Thus, for example, the following:

111	consecutive	composite	numbers	between	370261 and 370373,
113	"	"	"	"	492113 and 492227,
131	"	"	"	"	1357201 and 1357333,
131	"	"	"	"	1561919 and 1562051,
147	"	"	"	"	2010733 and 2010881,

(London Mathematical Society, 10 May, 1877).

We also know how to show that a rational function of n

$$p = \phi(n)$$

cannot continually give only prime numbers since we have for any integer k

$$\phi(n + kp) \equiv \phi(n) \pmod{p} ,$$

i.e., that $\phi(n)$ is a numerical periodic function of period p. It is thus very difficult to arrive at a rule for the distribution of the prime numbers among the ordinary series of integers.

However it appears natural to study the prime numbers acoording to their rule of formation. An exhaustive study of Eratothenes' method led Prince A. de Polignac to the interesting properties of the Diatomic series. [1] At the time M. Tchebychoff had arrived from slightly different considerations to the proof of this remarkable theorem. For $a > 3$ there is at least one prime number between a and $2a - 2$. We immediately deduce from this that the product

$$1 \cdot 2 \cdot 3 \cdots n$$

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[1]New research in prime numbers by M. A. Polignac; Paris, 1851. It is curious to note that, under the name of median series, we again find in the diatomic series the different terms of the Fermat series.

[2]Journal of Liouville, vol. XVII.

cannot be a power nor a product of powers as shown by M. Liouville. (Journal de Liouville, 2nd series, vol. II). In summary, this research was based on the consideration of arithmetic progressions.

XXIII. ON THE THEORY OF PRIME NUMBERS AND THEIR RELATIONSHIP WITH THE GEOMETRIC PROGRESSIONS

We owe to Fermat the profound investigations on the theory of prime numbers based on the consideration of the geometric progressions. It is this idea, as distinct from the preceding, that gave rise to the theory of residues, and more particularly to those of quadratic residues. In this way we simplify the verification of large prime numbers, and divisors of the form $a^n - 1$, or more generally of the form $a^n - b^n$ for a and b integers, as well as the decomposition of numbers of this form into prime factors. Fermat noted the linear form $nx + 1$ of the divisors and himself gave the decomposition of several terms of the series $2^n - 1$ and thus of the number $2^{37} - 1$ which he found divisible by 223 (Letter from Fermat dated 12 October 1640).

Lately M. Genocchi shed some light on a curious passage from the works of P. Mersenne. But to better understand its importance we recall some definitions from the theory of perfect numbers. We say that a number is perfect when it is equal to the sum of its aliquot parts, that is to say of all its divisors except itself. If we restrict ourselves to the case of even perfect numbers and denote different prime numbers by b, c, $\cdots$, and express the supposed perfect number n by $a^{\alpha}b^{\beta}c^{\gamma}d^{\delta}$ we have

$$2^{\alpha+1}b^{\beta}c^{\gamma}\cdots = (1 + 2 + \cdots + 2^{\alpha})(1 + b + b^2 + \cdots + b^{\beta}) \cdot (1 + c + c^2 + \cdots + c^{\gamma})\cdots$$

or clearly

$$b^{\beta}c^{\gamma}\cdots + \frac{b^{\beta}c^{\gamma}\cdots}{2^{\alpha+1} - 1} = (1 + b + b^2 + \cdots + b^{\beta}) \cdot (1 + c + c^2 + c^{\gamma})\cdots$$

The second term of the first member is thus an integer and becomes, upon division, of the form $b^{\beta'}c^{\gamma'}\cdots$; on the other hand, the second member which contains the number of terms

$$\mu = (\beta + 1)(\gamma + 1)\cdots$$

reduces to the two terms of the first member. As a result $\mu = 2$, $\beta = 1$, $\gamma = \delta = \cdots = 0$. Hence $n = 2^\alpha b$, and b is prime. Thus the even perfect numbers are of the form $n = 2^\alpha b$ in which b is prime. We readily have, with this condition

$$b = 2^{\alpha+1} - 1 .$$

In short, even perfect numbers are only of the form

$$2^\alpha(2^{\alpha+1} - 1)$$

in which the second factor is a prime number. This rule was known to Euclid, but this geometrician did not know how to show that this form contains all the even perfect numbers without exception.

Here now is the passage from the Oeuvres de Mersenne:

XIX.

To what has been said concerning numbers at the end of Proposition 20 on Ballistics and at[1] Point 14 of the Prefact to Hydraulics, add the art which has been discovered[2] whereby however many numbers you wish may be found which not only are twice the sum of their aliquot parts[3] (such are 120 (the smallest of all), 672, 523776, 1476304896, and 459818240, which, when multiplied by[4] 3 produces the number 1379454720, the sum of whose aliquot parts

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[1]This "at" is one interpretation. The Latin is here carelessly written and might equally well mean "on Ballistics and Point 14 of the Prefact to Hydraulics," That is, Mersenne might have written about Ballistics and about Point 14 of the Preface to Hydraulics when he was writing proposition 20.

[2]What "inventam" ("found") means here is enigmatic. Discovered by whom?

[3]The literal translation is "which not only have a double ratio with their aliquot parts reduced to one sum."

[4]Literally, "led into."

is triple the number itself,[1] as is also the case in the following: 30240, 32760, 23569920, and limitless others, concerning which refer to our Harmony[2], in which are found[3] 14182439040 and other quadruples[4] of their own aliquot parts) but also are in a given ration with their aliquot parts.

There are also other numbers which they call amicable[5] because each is the sum of the other's aliquot parts.[6] Such are the smallest of all, 220 and 284; for the aliquot parts of the latter produce the former, and vice-versa the aliquot parts of the former render the latter perfectly. Such also are 18416 and 17296; you will further find 9437036, 4363584, and numberless others.

At this point[7] it will be worth while to note that the 28[8] numbers exhibited by Petrus Bungus as perfect in Chapter 28 of his book on numbers[9] are not all perfect. Indeed, 20 are imperfect, so that he has only 8 perfect ones, namely 6, 28, 496, 8128, 33550336[10], 8589869056, 137438691328, and 2305843008139952128. These are from Bungus' table,[11] lines 1, 2, 3, 4, 8, 10, 12, and 19;[12] and these alone are perfect, so that those who have Bungus may remedy the error.

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[1]This is an expansion of the literal "whose aliquot parts are triple."

[2]Literally, "let our Harmony be seen." By this, Mersenne seems to be referring to his book Harmonie universelle, published in 1636-7, seven years before the present Cogitata physico-mathematica. See the Encyclopaedia Britannica article on Mersenne.

[3]Not in the Latin, but this seems to be what he means.

[4]The Latin has "subquadruples" which means no more than "quadruples." Perhaps it is a misreading of the manuscript by the printer.

[5]A technical expression used by Dickson, Uspensky and others.

[6]This is a paraphrase of the Latin, which says, literally, "because they have aliquot parts from which they are mutually remade."

[7]Literally, "where."

[8]Actually, only 24. See Dickson, History of the Theory of Numbers, Vol. 1, p. 12, note.

[9]Dickson, p. 9, note 42.

[10]This seems to be Mersenne's correction for Bungus' error 23···. See Dickson, p. 13.

[11]Literally, "from the region of Bungus' table." These "regions" are lines marked 1, 2, etc., where each line number indicates the number of digits. Dickson, ibid.

[12]Mersenne's or the printer's error since there are only 19 digits in the last named perfect number.

Further, perfect numbers are so rare that up to now only eleven have been able to be found, that is, three others differing from those of Bungus;[1] for there is no other perfect number outside of those eight, unless you go beyond the exponent 62[2], in $1 + 2 + 2^2 + \cdots$[3]. The ninth perfect number is the power of the exponent 68 minus 1;[4] the tenth, the power of the exponent 128 minus 1[5]; the eleventh, finally, the power 258 minus 1, that is, the power 257, decreased by unity, multiplied by the power 256.[6]

The person who finds eleven others will know that he has surpassed every analysis previously made and will remember meanwhile that there is no perfect number from the power 17000 to 32000, and that no interval of powers can be assigned so great but that it may be given without perfect numbers. E.G., if there is an exponent 1050000, all the way to 2090000 there will be no number of double progression such as to serve perfect numbers,

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[1]"Bougianis" in the text is an obvious mis-print for "Bungianus."

[2]Evidently this is an error, since according to Uspensky and Heaslet, Elementary Number Theory, p. 82, $2^{61} - 1$ is a prime number, and thus $2^{60}(2^{61} - 1)$ is perfect. W. W. R. Ball speculated that the printer had made an error and had printed a 7 for a 1. R. C. Archibald dismissed this as "ridiculous." (Scripta Mathematica, Vol. 3, p. 112). However, if Dickson is correct in saying that Mersenne was reporting information that he got from correspondence with Frenicle and Fermat, it is possible that Mersenne mis-read his correspondent and reported that 67 instead of 61 produces a prime, $2^{67} - 1$. This would explain both errors, and would also account for the peculiar oversight on his part since he was certainly acute enough to realize that Bungus included several numbers in his list which are not perfect.

[3]This is Dickson's rendering of what Mersenne calls "of double progression beginning from 1."

[4]This is evidently in error. See Dickson, p. 13; Uspensky, p. 82.

[5]Actually, $2^{126}(2^{127} - 1)$ is the 12^{th} perfect number since Mersenne has omitted $2^{88}(2^{89} - 1)$ and $2^{106}(2^{107} - 1)$ which are perfect. See Uspensky, p. 82.

[6]Another error. See "Mersenne and Fermat Numbers," by R. M. Robinson Proceedings of the A. M. S. Vol. 5, p. 842-846 for a list of perfect numbers obtained through the use of a modern high-speed computer. The 13^{th} perfect number is $2^{520}(2^{521} - 1)$.

that is, such as to be a prime minus a unity.[1]

From this it is clear how rare perfect numbers are, and how deservedly they are compared to perfect men; and (it is clear) that one of the greatest difficulties in all mathematics is to show a prescribed multitude[2] of perfect numbers, as also to recognize whether given numbers consisting of 15 or 20 digits[3] are prime or not, since not even an entire century is sufficient for this investigation, in any way known up to now.

According to this passage the list of even perfect numbers would be the following:

First perfect number	$2(2^2 - 1)$,	Second perfect number	$2^2(2^3 - 1)$,
Third perfect number	$2^4(2^5 - 1)$,	Fourth perfect number	$2^6(2^7 - 1)$,
Fifth perfect number	$2^{12}(2^{13} - 1)$,	Sixth perfect number	$2^{16}(2^{17} - 1)$,
Seventh perfect number	$2^{18}(2^{19} - 1)$,	Eighth perfect number	$2^{30}(2^{31} - 1)$,
Ninth perfect number	$2^{66}(2^{67} - 1)$,	Tenth perfect number	$2^{126}(2^{127} - 1)$,
Eleventh perfect number	$2^{256}(2^{257} - 1), \cdots$		

This passage is moreover reported in a paper by C. N. Winsheim inserted in the Novi Commentarii Academiae Petropolitanae, ad annum MDCCXLIX (vol. II, page 78), preceded by the following thoughts.

"For there appears to be distrust as to whether the ninth number[4] can keep the place of a perfect number, since it is excluded by the very intelligent Mersenne, who substituted in its place the power of the binary $(2^{67} - 1)^{66}$ or the ninteenth[5] perfect number of Hansch 147573952589676412927; certainly the words of a very perspicacious man seem to me to be worthy to be set forth here verbatim."

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[1]It is puzzling that Mersenne puts in all those zeroes since he was certainly aware that the exponent must be a prime. Is it possible that he wrote 17···? It also seems that it should be 33 not 32 since $(2 \times 17) - 1 = 33$. Unfortunately 33 is not prime. Similar objections apply to 105 and 209. Eventually somebody conjectured that if $2^n - 1$ is prime, then this number used as an exponent will produce another prime of the same form. Robinson reports in his article that D. J. Wheeler disproved this conjecture in 1953 using a high-speed computer.

[2]The text here is oorrupt. Multitudinem, not multitudinum must be read.

[3]Omitting the comma after 15 — an obvious error.

[4]The reference is to a table by Euler which listed $(2^{41} - 1)2^{40}$ as the ninth perfect number. See Dickson, History of the Theory of Numbers, Vol. 1, p. 18.

[5]Hansch had stated (erroneously) that $2^n - 1$ is a prime if n is any of the twenty-two primes ≤ 79. See Dickson, p. 17.

In this way Mersenne showed that, for n between 31 and 257, there do not exist any prime numbers of the form $2^n - 1$ except for n equal to 31, 67, 127, or 257. The proof of the non-decomposition of the first of these numbers, $2^{31} - 1$, was not given until much later, by Euler. Going much further, M. F. Landry, by means of an unpublished method, probably very simple, arrived at the decomposition of certain very large numbers and their prime factors. In fact he gave the decomposition of the numbers

$$2^{41} - 1, \quad 2^{43} - 1, \quad 2^{47} - 1, \quad 2^{53} - 1, \quad 2^{59} - 1,$$

into their prime factors. In addition we find that $2^{73} - 1$, $2^{79} - 1$, and $2^{113} - 1$ are respectively divisible by 439, 2687, and 3391. Finally we have the following theorem.

Theorem: If $4q + 3$ and $8q + 7$ are prime numbers then the number

$$2^{4q+3} - 1$$

is divisible by $8q + 7$.

In fact, according to Fermat's Theorem we have

$$2^{8q+6} - 1 \equiv 0 \pmod{8q + 7},$$

and as a result one of the two factors $2^{4q+3} + 1$ or $2^{4q+3} - 1$ of the first member of the congruence is divisible by the modulus. On the other hand we know that 2 is a quadratic residue of all prime numbers of the form $8n + 1$ and $8n + 7$. Consequently we have

$$2^{4q+3} - 1 \equiv 0 \pmod{8q + 7} .$$

Consulting a table of prime numbers we conclude that for n successively equal to

11, 23, 83, 131, 179, 191, 239, 251, 359, 419, 431, 443, 491,

the numbers $2^n - 1$ are respectively divisible by the factors

23, 47, 167, 263, 359, 383, 479, 503, 719, 839, 863, 887, 983.

The result of these various considerations is that Mersenne was in possession of an arithmetic method which we have not been able to attain. However it appears natural to believe that this method is not too far from Fermat's principles and consequently does not differ essentially from those which we will later derive from the inverse of Fermat's theorem. We show, in fact, how it is possible to rapidly arrive at the nature of the factorization of the large numbers which we mentioned above.

In the following table we give the decomposition of the numbers U_n and V_n of the Fermat series for all values of n up to 64. Among the large prime numbers in this table we will note

1) Five numbers of ten digits

42782 55361 a factor of $2^{40} + 1$,
88314 18697 a factor of $2^{41} + 1$,
29315 42417 a factor of $2^{44} + 1$,
18247 26041 a factor of $2^{59} + 1$,
45622 84561 a factor of $2^{60} + 1$;

2) Two numbers of eleven digits

5 44109 72897 a factor of $2^{56} + 1$,
7 71586 73929 a factor of $2^{63} + 1$;

3) A number of twelve digits

16 57685 37521 a factor of $2^{47} + 1$;

4) Four numbers of thirteen digits

293 20310 07403 a factor of $2^{43} + 1$,
443 26767 98593 a factor of $2^{49} - 1$,
436 39531 27297 a factor of $2^{49} + 1$,
320 34317 80337 a factor of $2^{59} - 1$;

5) A number of fourteen digits

2805 98107 62433 a factor of $2^{53} + 1$.

It remains to determine the nature of the numbers $2^{61} - 1$, $(2^{61} + 1)/3$, and $2^{64} + 1$. Landry feels that these numbers are prime but on the other hand, according to Mersenne, the first of these numbers would be composite. In addition, from the consideration of the calculations which I have made, and whose theory is given later, the last of these numbers would also be composite. There is nothing further that can be said at the moment.

Outside of the decompositions contained in the table, M. Landry has also obtained the proper divisors of a certain number of other terms of this series, to wit

For $2^{65}+1$ 4 09891 and 76 23851,

$2^{69}+1$ 16 87499 65921, (prime)

$2^{75}-1$ 1 00801 and 105 67201,

$2^{75}+1$ 113 38367 30401, (prime)

$2^{105}+1$ 6 64441 and 15 64921 .

In his way, M. le Lasseiur arrived at the same results but he did this by means of the identity

$$2^{4n+2}+1 = (2^{2n+1}+2^{n+1}+1)(2^{2n+1}-2^{n+1}+1),$$

which permits the calculations to be abbreviated. This very important identity will be later generalized.

TABLE OF THE PRIME FACTORS OF FERMAT'S RECURRENT SERIES ACCORDING TO M. F. LANDRY

U_n	Divisors of U_n		Values of 2^n	V_n	Divisors of V_n
2^1-1	1	2^1	2	2^1+1	3
		2^2	4	2^2+1	5
2^3-1	7	2^3	8	2^3+1	3^2
		2^4	16	2^4+1	17
2^5-1	31	2^5	32	2^5+1	$3\cdot 11$
		2^6	64	2^6+1	$5\cdot 13$
2^7-1	127	2^7	128	2^7+1	$3\cdot 43$
		2^8	256	2^8+1	257
2^9-1	$7\cdot 73$	2^9	512	2^9+1	$3^2\cdot 19$
		2^{10}	1024	$2^{10}+1$	$5^2\cdot 41$
$2^{11}-1$	$23\cdot 89$	2^{11}	2048	$2^{11}+1$	$3\cdot 683$
		2^{12}	4096	$2^{12}+1$	$17\cdot 241$
$2^{13}-1$	8191	2^{13}	8192	$2^{13}+1$	$3\cdot 2731$
		2^{14}	16384	$2^{14}+1$	$5\cdot 29\cdot 113$
$2^{15}-1$	$7\cdot 31\cdot 151$	2^{15}	32768	$2^{15}+1$	$3^2\cdot 11\cdot 331$
		2^{16}	65536	$2^{16}+1$	65537

TABLE OF THE PRIME FACTORS OF FERMAT'S RECURRENT SERIES ACCORDING TO M. F. LANDRY

(Cont.)

U_n	Divisors of U_n		Values of 2^n	V_n	Divisors of V_n
$2^{17}-1$	131071	2^{17}	131072	$2^{17}+1$	$3 \cdot 43691$
		2^{18}	262144	$2^{18}+1$	$5 \cdot 13 \cdot 37 \cdot 109$
$2^{19}-1$	524287	2^{19}	524288	$2^{19}+1$	$3 \cdot 174763$
		2^{20}	1048576	$2^{20}+1$	$17 \cdot 61681$
$2^{21}-1$	$7^2 \cdot 127 \cdot 337$	2^{21}	2097152	$2^{21}+1$	$3^2 \cdot 43 . 5419$
		2^{22}	4194304	$2^{22}+1$	$5 \cdot 397 \cdot 2113$
$2^{23}-1$	$47 \cdot 178481$	2^{23}	8388608	$2^{23}+1$	$3 \cdot 2796203$
		2^{24}	16777216	$2^{24}+1$	$97 \cdot 257 \cdot 673$
$2^{25}-1$	$31 \cdot 601 \cdot 1801$	2^{25}	33554432	$2^{25}+1$	$3 \cdot 11 \cdot 251 \cdot 4051$
		2^{26}	67108864	$2^{26}+1$	$5 \cdot 53 \cdot 157 \cdot 1613$
$2^{27}-1$	$7 \cdot 73 \cdot 262657$	2^{27}	134217728	$2^{27}+1$	$3^4 \cdot 19 \cdot 87211$
		2^{28}	268435456	$2^{28}+1$	$17 \cdot 15790321$
$2^{29}-1$	$233 \cdot 1103 \cdot 2089$	2^{29}	536870912	$2^{29}+1$	$8 \cdot 59 \cdot 3033169$
		2^{30}	1073741824	$2^{30}+1$	$5^2 \cdot 13 \cdot 41 \cdot 61 \cdot 1321$
$2^{31}-1$	2147483647	2^{31}	2147483648	$2^{31}+1$	$3 . 715827883$
		2^{32}	4294967296	$2^{32}+1$	$641 \cdot 6700417$
$2^{33}-1$	$7 \cdot 23 \cdot 89 \cdot 599479$	2^{33}	8589934592	$2^{33}+1$	$3^2 \cdot 67 \cdot 683 \cdot 20857$
		2^{34}	17179869184	$2^{34}+1$	$5 \cdot 137 \cdot 953 \cdot 26317$
$2^{35}-1$	$31 \cdot 71 \cdot 127 \cdot 122921$	2^{35}	34359738368	$2^{35}+1$	$3 \cdot 11 \cdot 43 \cdot 281 \cdot 86171$
		2^{36}	68719476736	$2^{36}+1$	$17 \cdot 241 \cdot 433 \cdot 38737$

TABLE OF THE PRIME FACTORS OF FERMAT'S RECURRENT SERIES ACCORDING TO M. F. LANDRY

(Cont.)

U_n	Divisors of U_n		Values of 2^n	V_n	Divisors of V_n
$2^{37}-1$	$223 \cdot 616318177$	2^{37}	137438953472	$2^{37}+1$	$3 \cdot 1777 \cdot 25781083$
		2^{38}	274877906944	$2^{38}+1$	$5 \cdot 229 \cdot 457 \cdot 525313$
$2^{39}-1$	$7 \cdot 79 \cdot 8191 \cdot 121369$	2^{39}	549755813888	$2^{39}+1$	$3^2 \cdot 2731 \cdot 22366891$
		2^{40}	1099511627776	$2^{40}+1$	$257 \cdot 4278255361$
$2^{41}-1$	$13367 \cdot 164511353$	2^{41}	2199023255552	$2^{41}+1$	$3 \cdot 83 \cdot 8831418697$
		2^{42}	4898046511104	$2^{42}+1$	$5 \cdot 13 \cdot 29 \cdot 113 \cdot 1429 \cdot 14449$
$2^{43}-1$	$431 \cdot 9719 \cdot 2099863$	2^{43}	8796093022208	$2^{43}+1$	$3 \cdot 2932031007403$
		2^{44}	17592186044416	$2^{44}+1$	$17 \cdot 353 \cdot 2931542417$
$2^{45}-1$	$7 \cdot 31 \cdot 73 \cdot 151 \cdot 631 \cdot 23311$	2^{45}	35184372088832	$2^{45}+1$	$3^3 \cdot 11 \cdot 19 \cdot 331 \cdot 18837001$
		2^{46}	70368744177664	$2^{46}+1$	$5 \cdot 277 \cdot 1013 \cdot 1657 \cdot 30269$
$2^{47}-1$	$2351 \cdot 4513 \cdot 13264529$	2^{47}	140737488355328	$2^{47}+1$	$3 \cdot 283 \cdot 165768537521$
		2^{48}	281474976710656	$2^{48}+1$	$193 \cdot 65537 \cdot 22253377$
$2^{49}-1$	$127 \cdot 4432676798593$	2^{49}	562949953421312	$2^{49}+1$	$3 \cdot 43 \cdot 4363953127297$
		2^{50}	1125899906842624	$2^{50}+1$	$5^3 \cdot 41 \cdot 101 \cdot 8101 \cdot 268501$
$2^{51}-1$	$7 \cdot 103 \cdot 2143 \cdot 11119 \cdot 131071$	2^{51}	2251799813685248	$2^{51}+1$	$3^2 \cdot 307 \cdot 2857 \cdot 6529 \cdot 43691$
		2^{52}	4503599627370496	$2^{52}+1$	$17 \cdot 858001 \cdot 308761441$
$2^{53}-1$	$6361 \cdot 69431 \cdot 20394401$	2^{53}	9007199254740992	$2^{53}+1$	$3 \cdot 107 \cdot 28059810762433$

		2^{54}	18014398509481984	$2^{54}+1$	$5 \cdot 13 \cdot 37 \cdot 109 \cdot 246241 \cdot 279073$
$2^{55}-1$	$23 \cdot 31 \cdot 89 \cdot 881 \cdot 3191 \cdot 201961$	2^{55}	36028797018963968	$2^{55}+1$	$3 \cdot 11^2 \cdot 683 \cdot 2971 \cdot 48912491$
		2^{56}	72057594037927936	$2^{56}+1$	$257 \cdot 5153 \cdot 54410972807$
$2^{57}-1$	$7 \cdot 32377 \cdot 524287 \cdot 1212847$	2^{57}	144115188075855872	$2^{57}+1$	$3^2 \cdot 571 \cdot 174763 \cdot 160465489$
		2^{58}	288230376151711744	$2^{58}+1$	$5 \cdot 107367629 \cdot 536903681$
$2^{59}-1$	$179951 \cdot 3203431780337$	2^{59}	576460752303423488	$2^{59}+1$	$3 \cdot 2833 \cdot 37171 \cdot 1824726041$
		2^{60}	1152921504606846976	$2^{60}+1$	$17 \cdot 241 \cdot 61681 \cdot 4562284561$
$2^{61}-1$	Prime	2^{61}	2305843009213693952	$2^{61}+1$	. . . .
		2^{62}	4611686018427387904	$2^{62}+1$	$5 \cdot 5581 \cdot 8681 \cdot 49477 \cdot 384773$
$2^{63}-1$	$7^2 \cdot 73 \cdot 127 \cdot 337 \cdot 92737 \cdot 649657$	2^{63}	9223372036854775808	$2^{63}+1$	$3^3 \cdot 19 \cdot 43 \cdot 5419 \cdot 77158673929$
		2^{64}	18446744073709551616	$2^{64}+1$	. . . .