# Analysis of Large Graphs: Overlapping Communities

Mining of Massive Datasets
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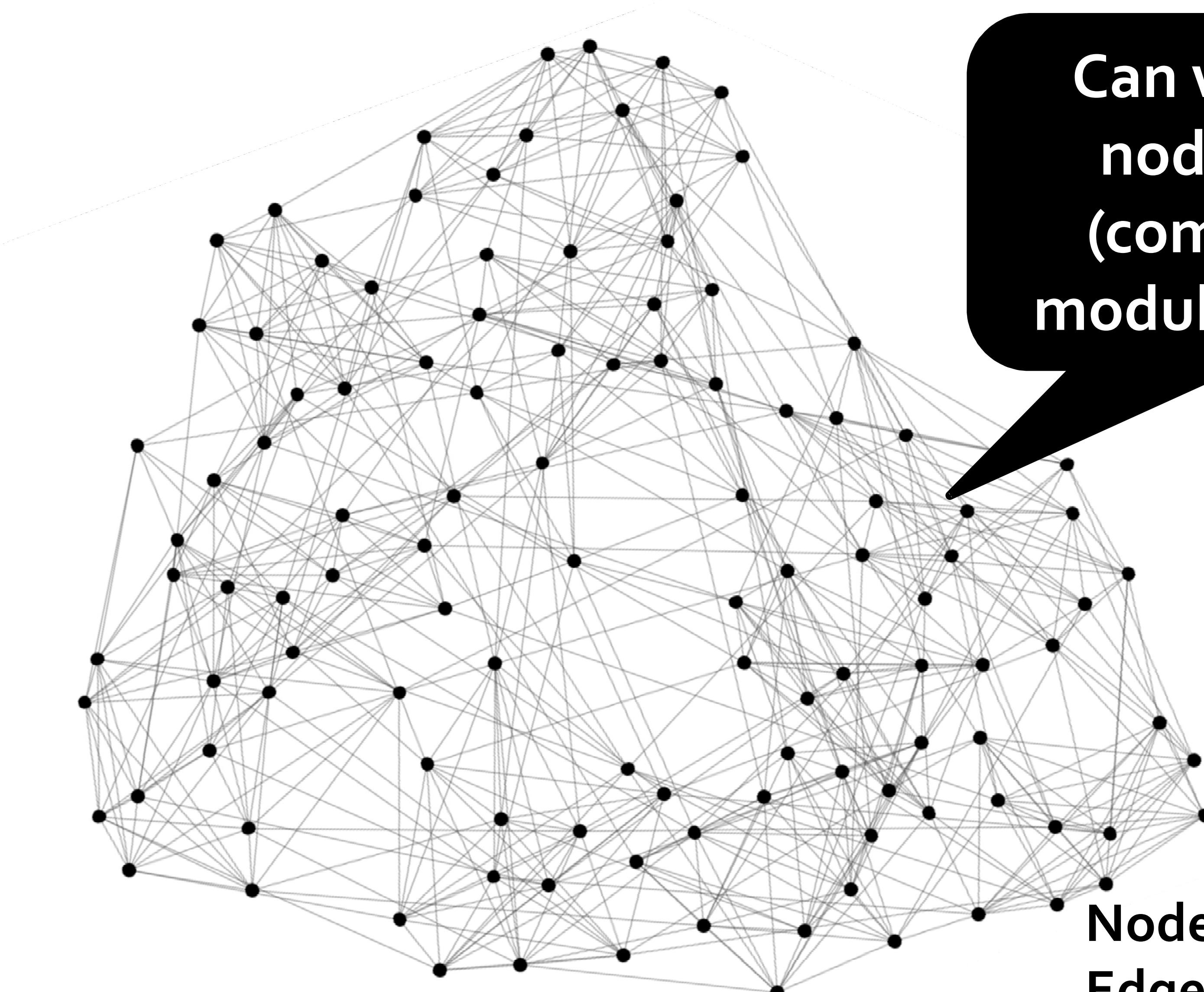
http://www.mmds.org

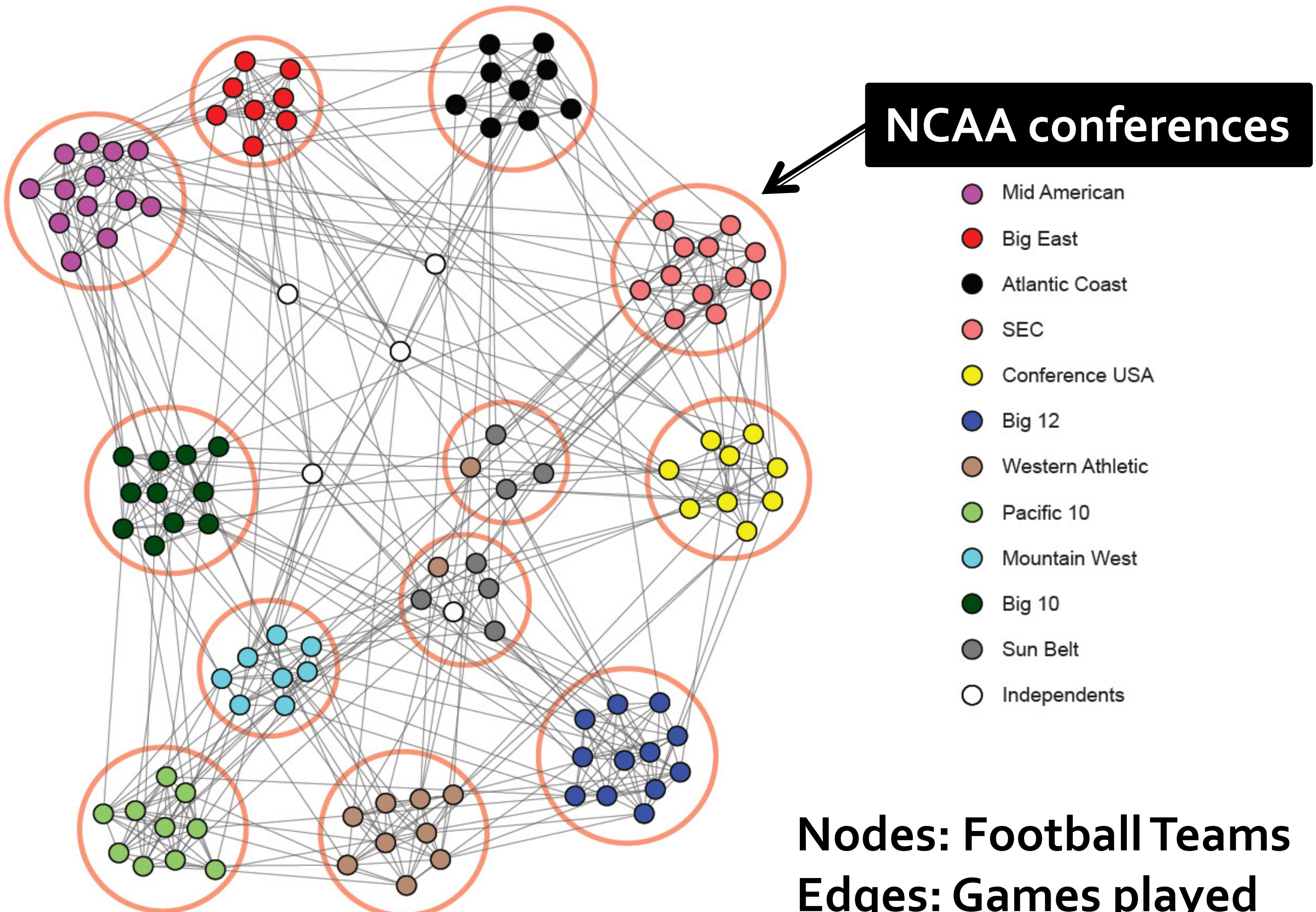
# Identifying Communities



Can we identify node groups? (communities, modules, clusters)

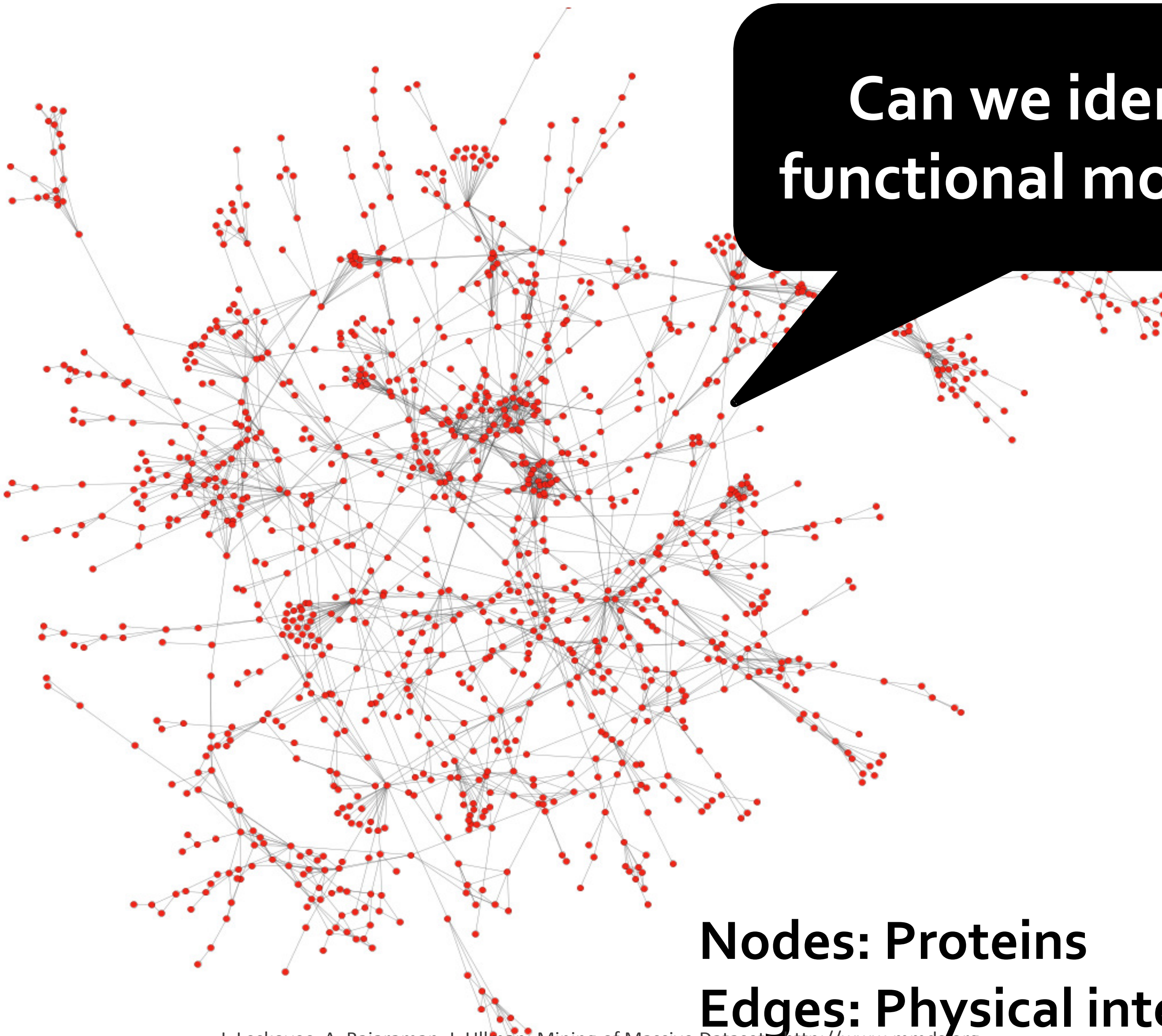
**Nodes: Football Teams**
**Edges: Games played**

# NCAA Football Network




**Nodes: Football Teams**
**Edges: Games played**

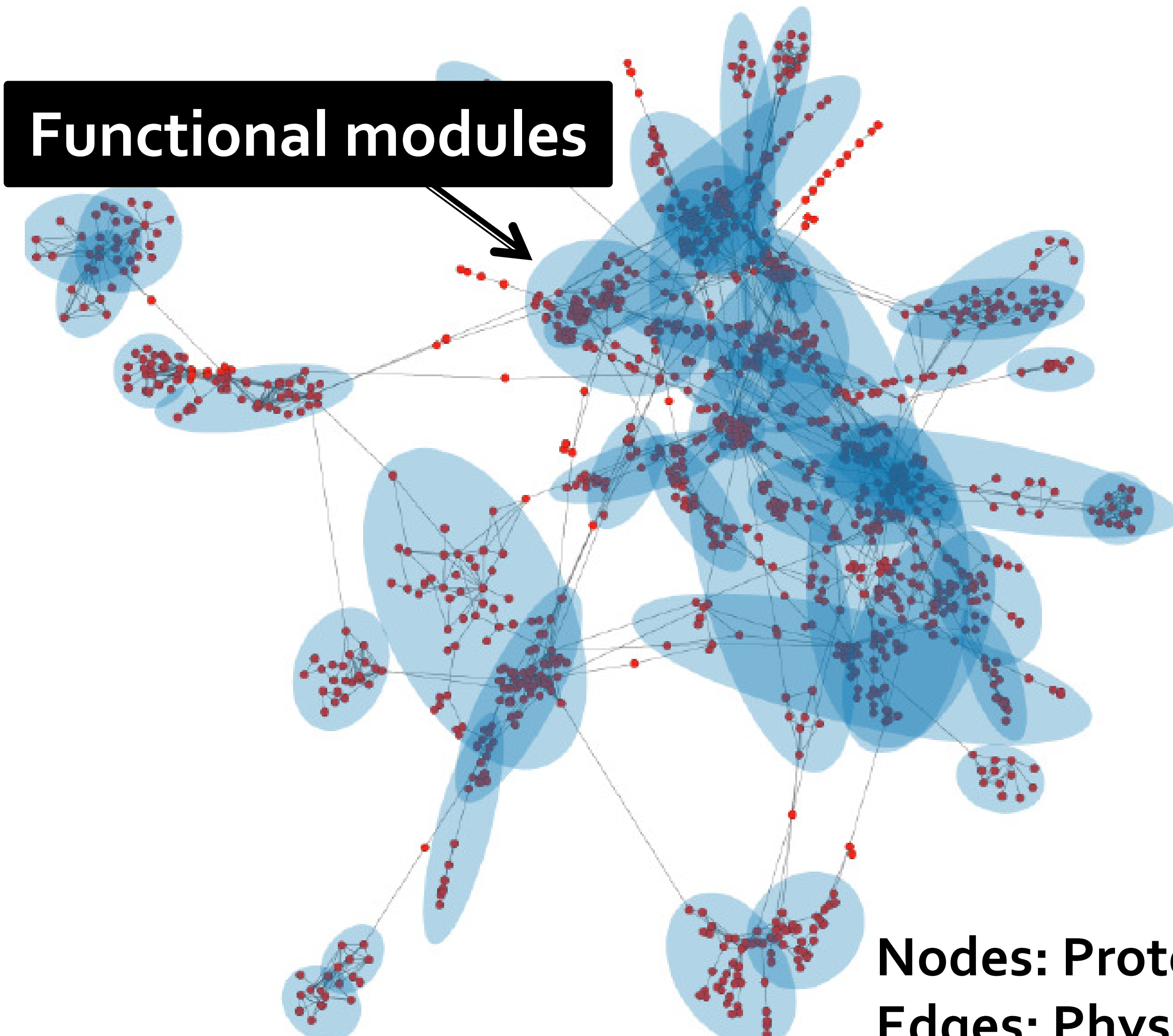
# Protein-Protein Interactions



Can we identify functional modules?

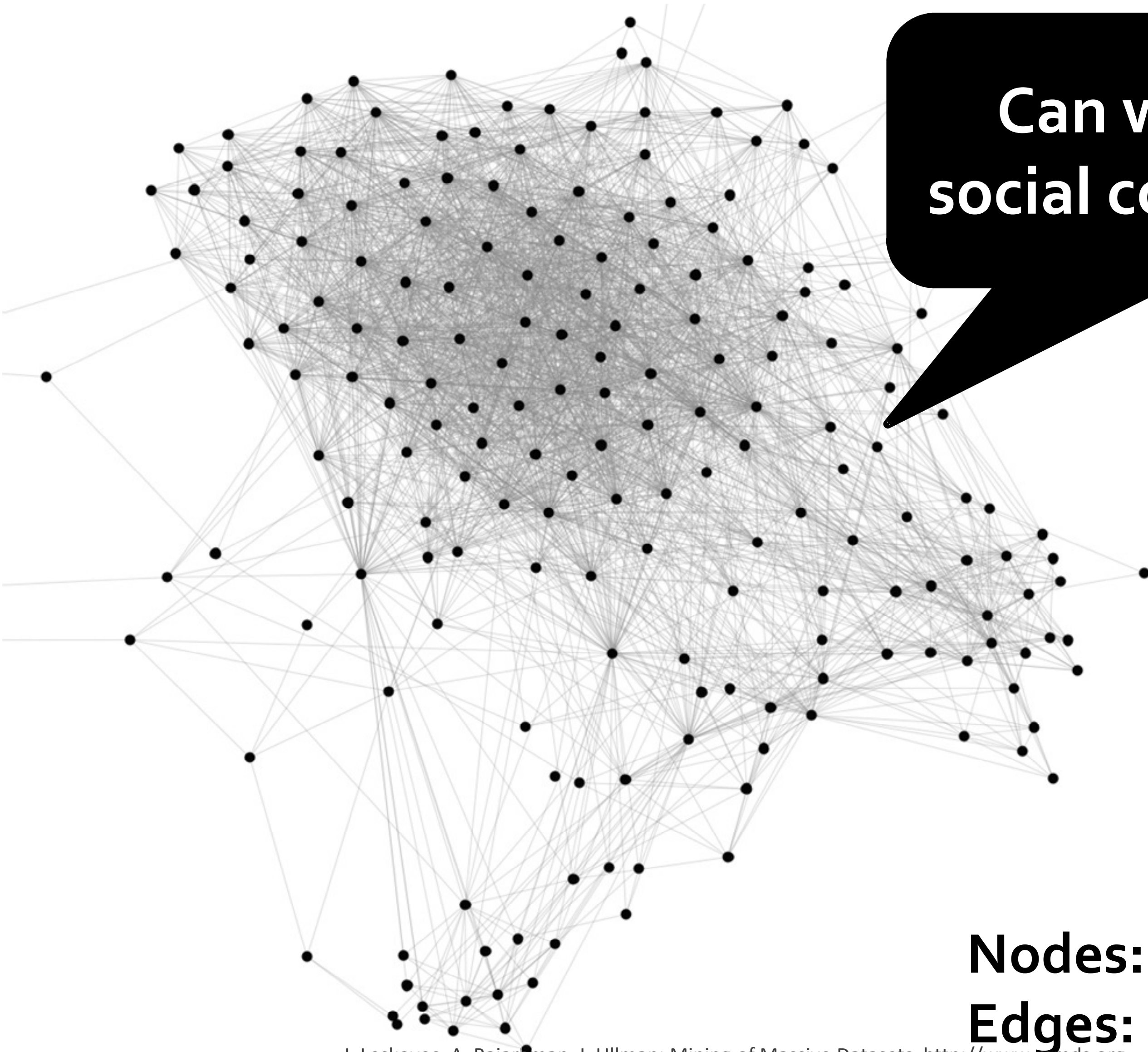
**Nodes: Proteins**
**Edges: Physical interactions**

# Protein-Protein Interactions




**Nodes: Proteins**
**Edges: Physical interactions**

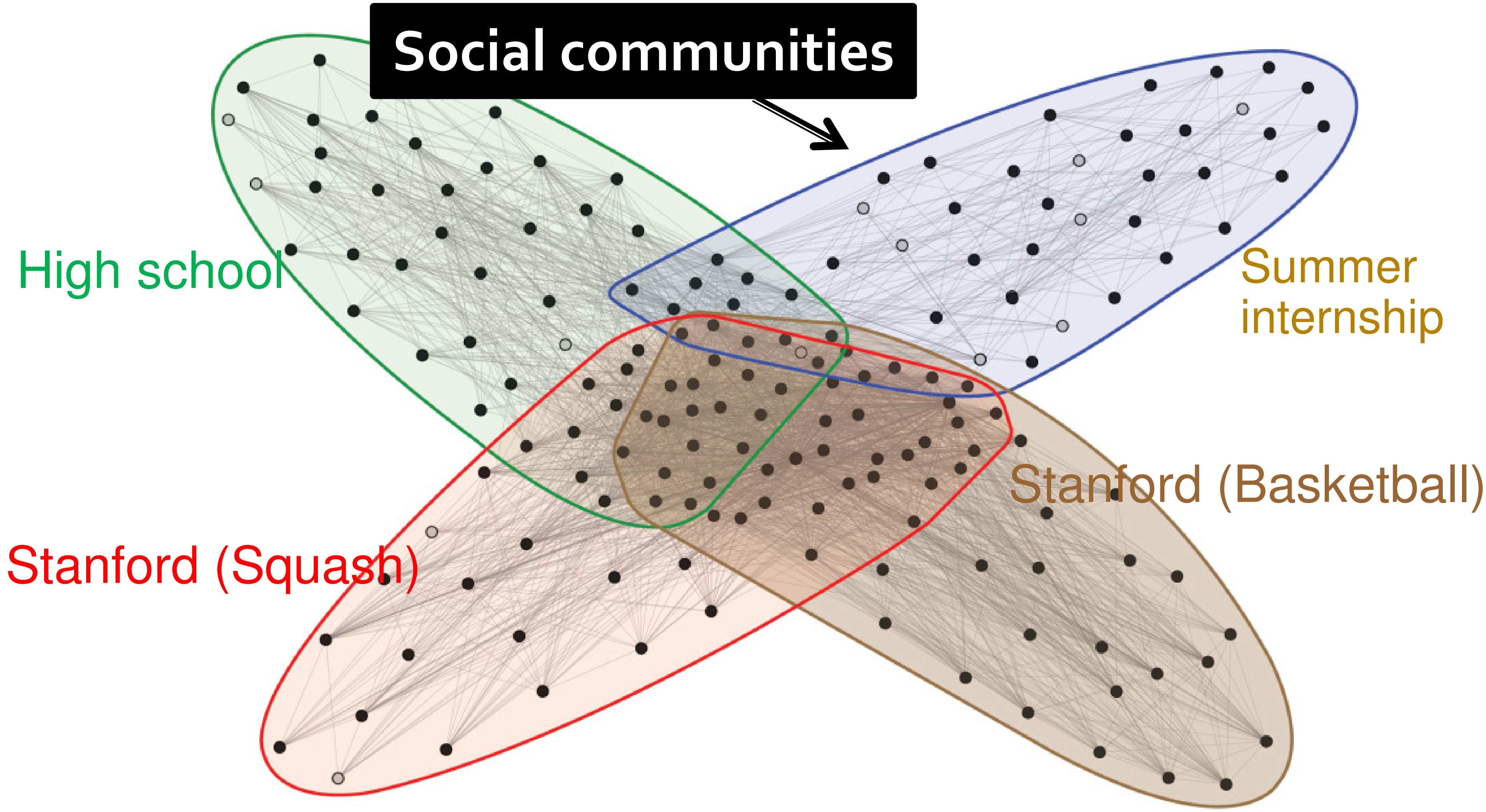
# Facebook Network



Can we identify social communities?

**Nodes: Facebook Users**
**Edges: Friendships**

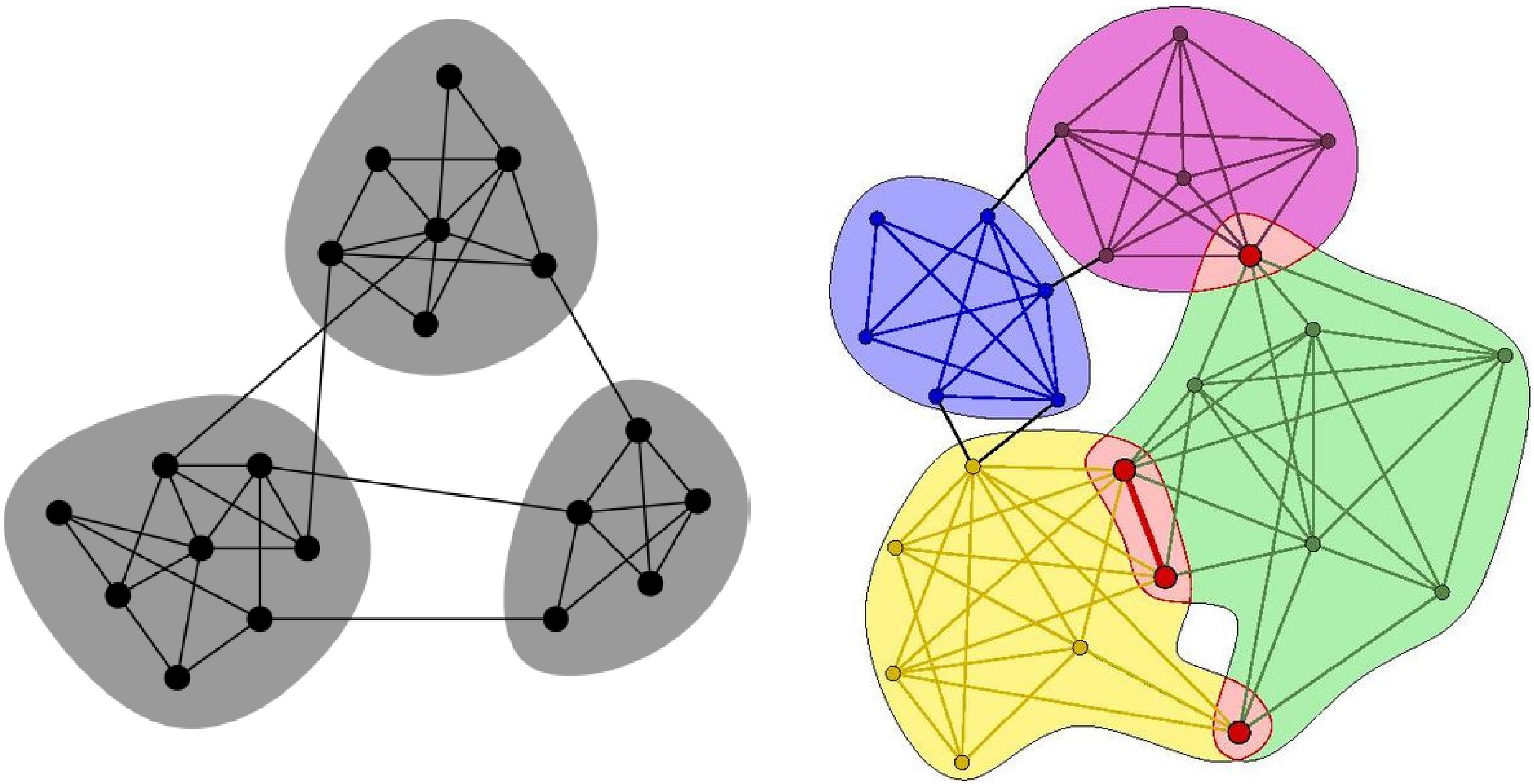
# Facebook Network



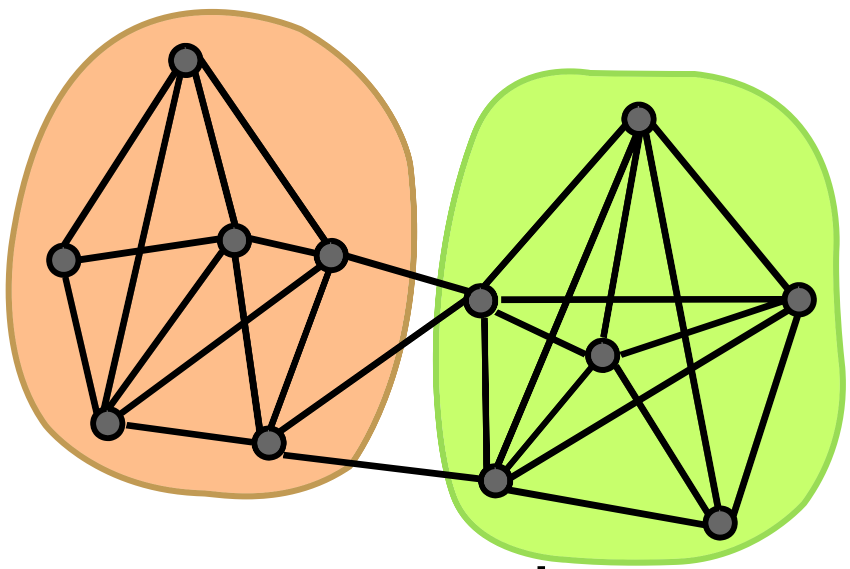

**Nodes: Facebook Users**
**Edges: Friendships**

# Overlapping Communities

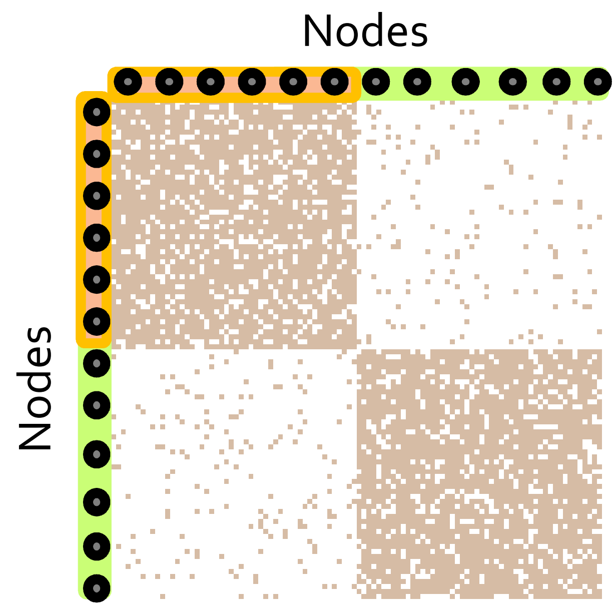
- **Non-overlapping vs. overlapping communities**



# Non-overlapping Communities



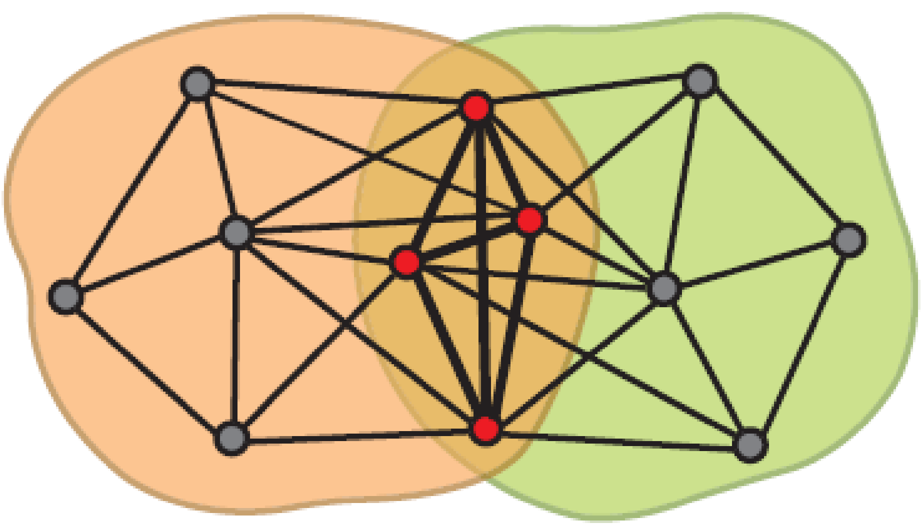
Network

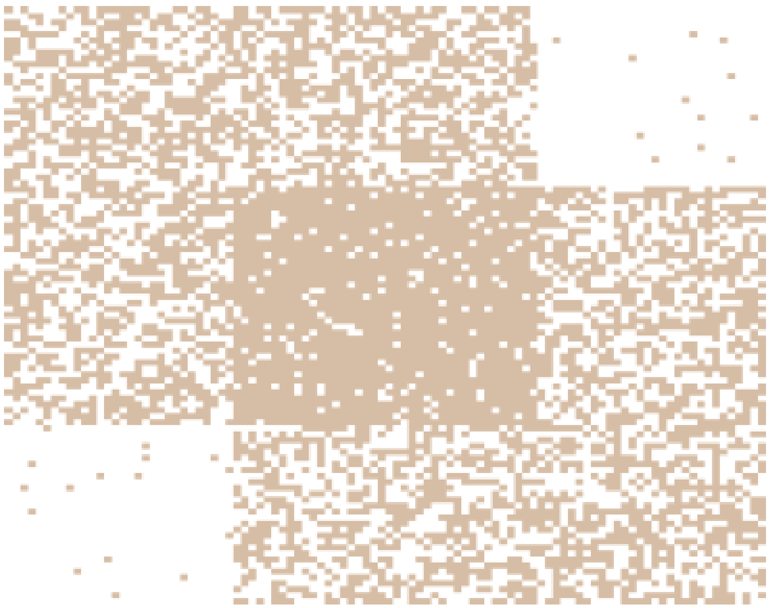



Adjacency matrix

# Communities as Tiles!

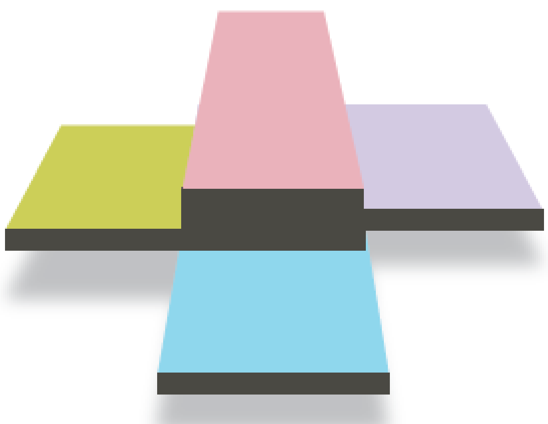
- **What is the structure of community overlaps: Edge density in the overlaps is higher!**



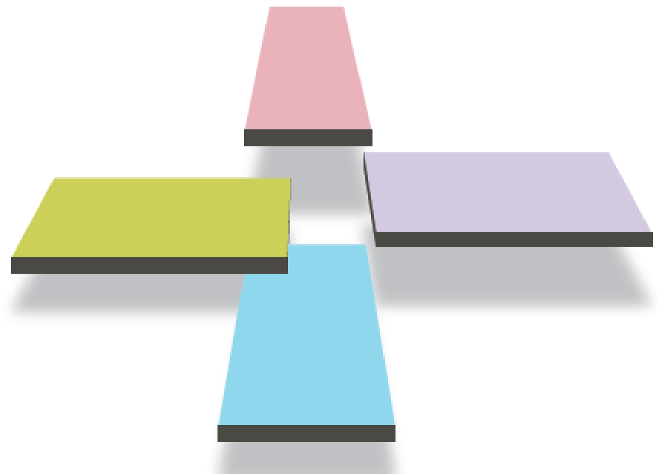


**Communities as "tiles"**

# Recap so far…



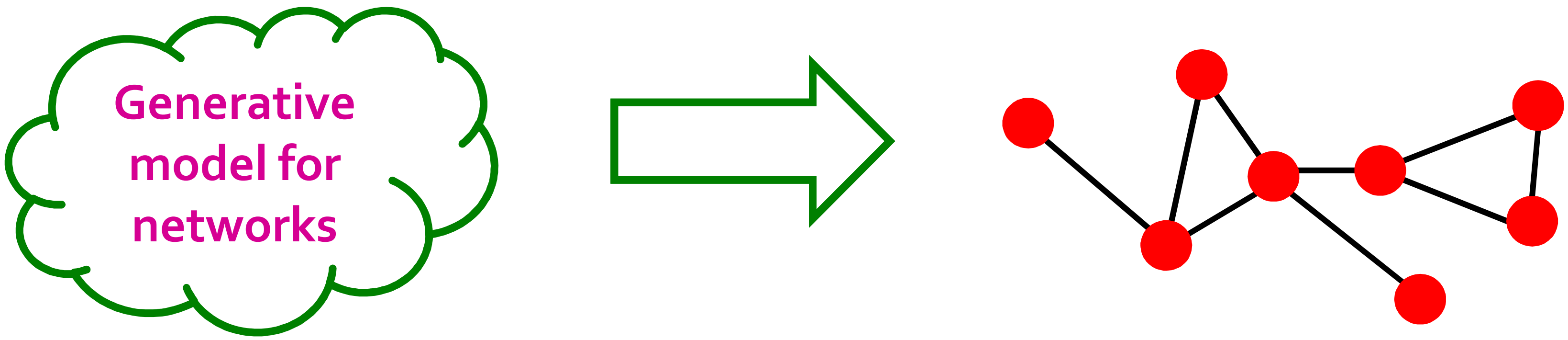
Communities in a network



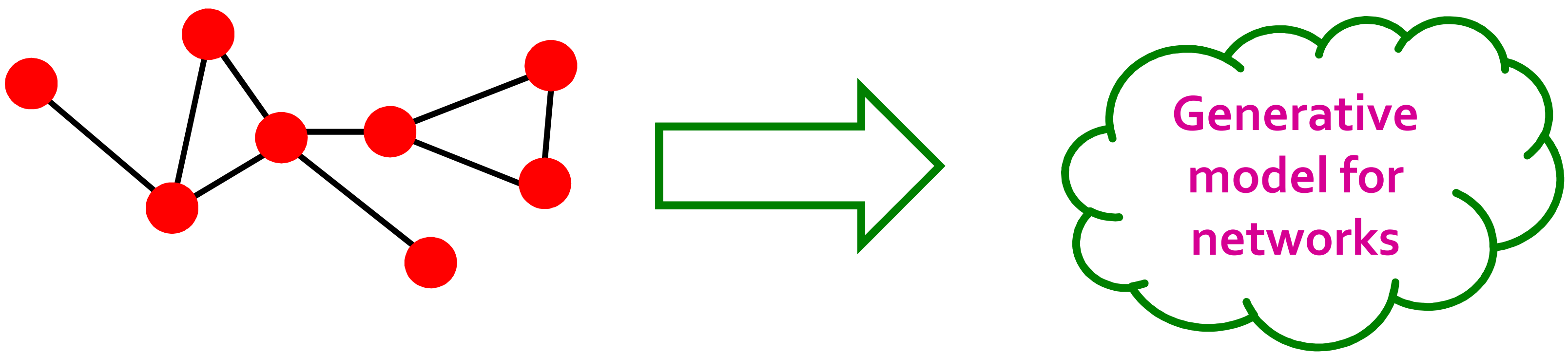
This is what we want!

# Plan of attack

- **1) Given a model, we generate the network:**

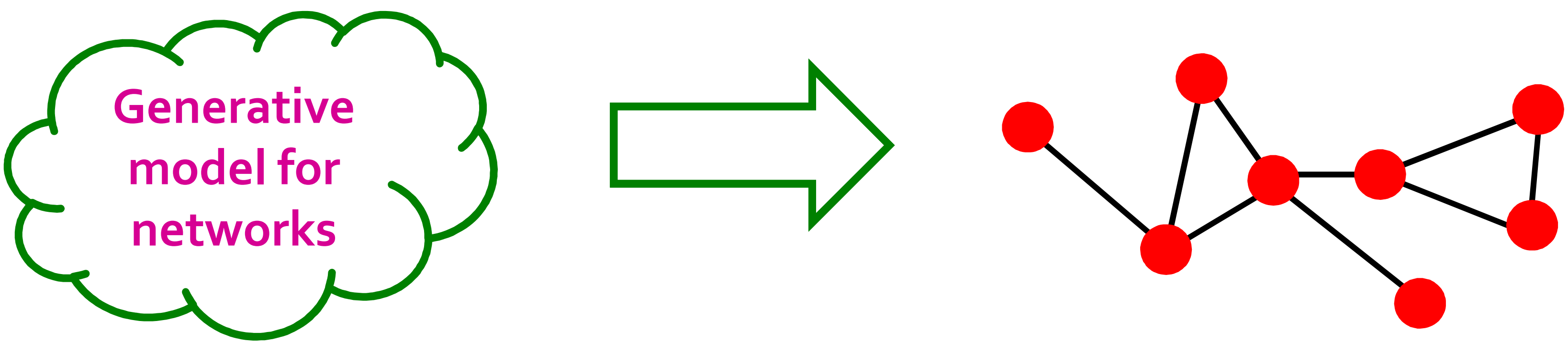



- **2) Given a network, find the "best" model**



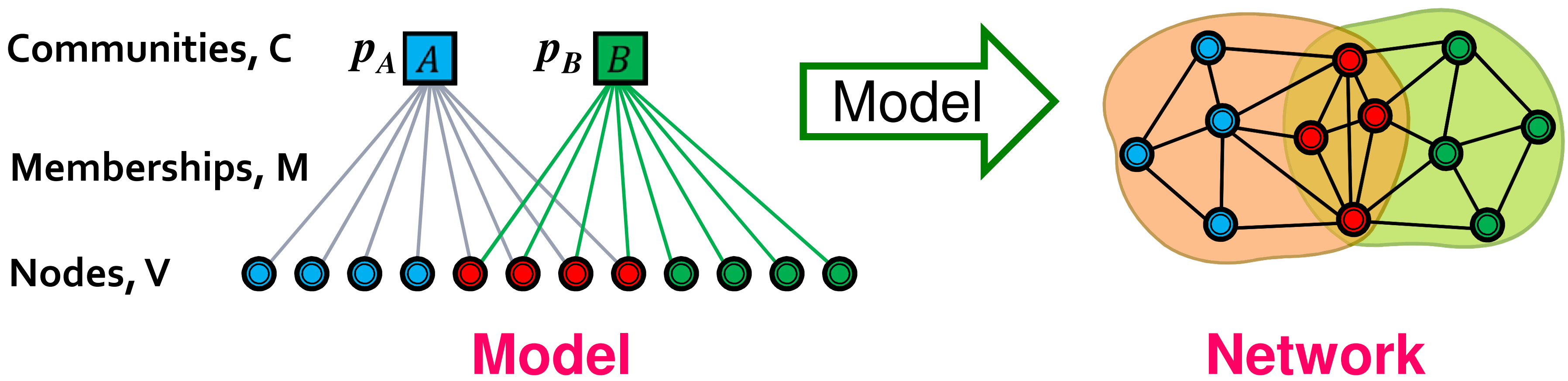
# Model of networks

- **Goal:** **Define a model that can generate networks**
  - The model will have a set of "parameters" that we will later want to estimate (and detect communities)



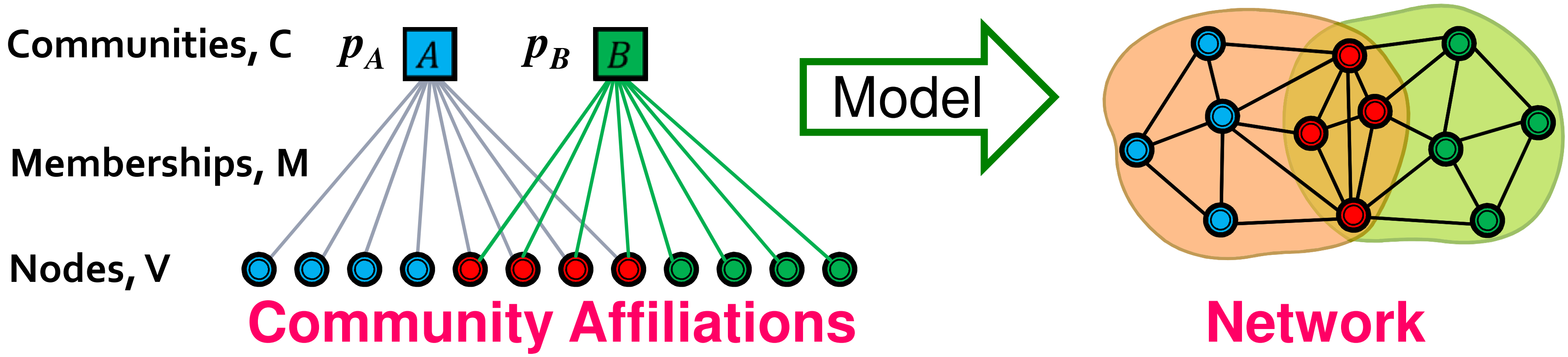

- **Q: Given a set of nodes, how do communities "generate" edges of the network?**

# Community-Affiliation Graph




- **Generative model B($V$, $C$, $M$, $\{p_c\}$) for graphs:**
  - Nodes **V**, Communities **C**, Memberships **M**
  - Each community **c** has a single probability $\mathbf{p_c}$
  - Later we fit the model to networks to detect communities

# AGM: Generative Process




- **AGM generates the links: For each**
  - For each pair of nodes in community $\boldsymbol{A}$, we connect them with prob. $\boldsymbol{p_A}$
  - **The overall edge probability is:**

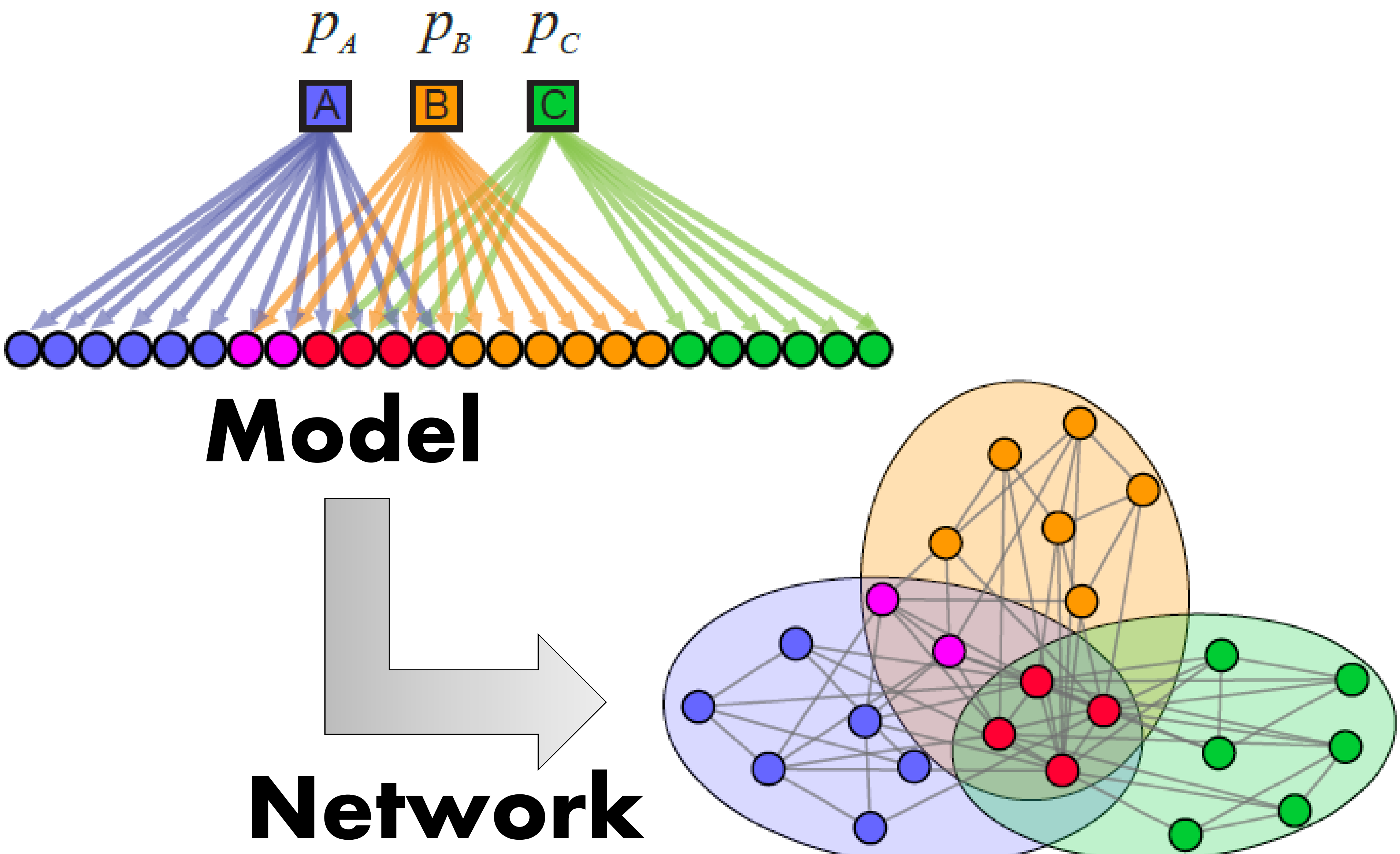
$$P(u,v) = 1 - \prod_{c \in M_u \cap M_v} (1 - p_c)$$

If $\boldsymbol{u}, \boldsymbol{v}$ share no communities: $\boldsymbol{P}(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{\varepsilon}$

$\boldsymbol{M_u}$ … set of communities node $\boldsymbol{u}$ belongs to

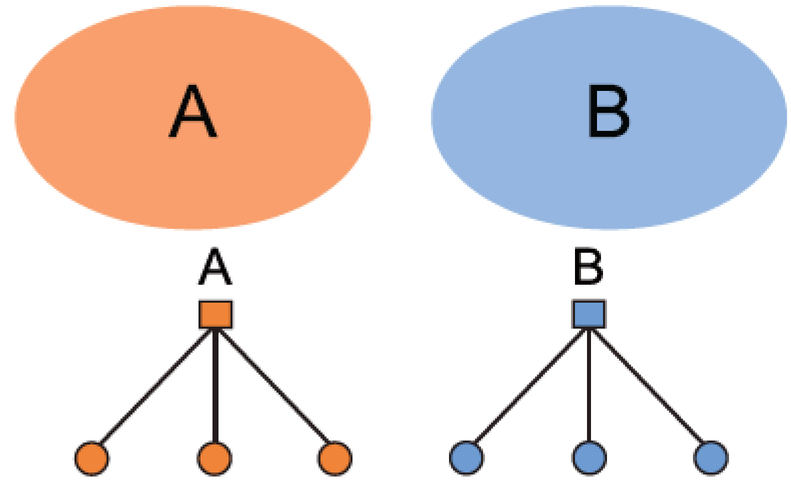
Think of this as an "OR" function: If at least 1 community says "YES" we create an edge

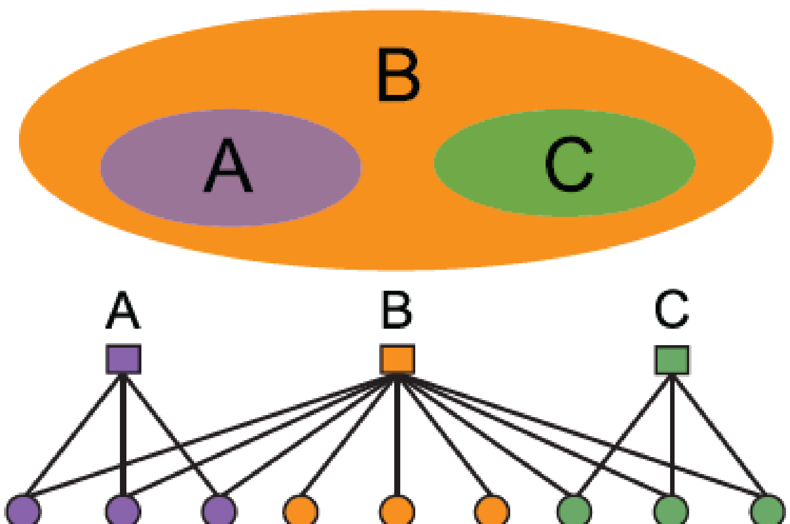
# Recap: AGM networks

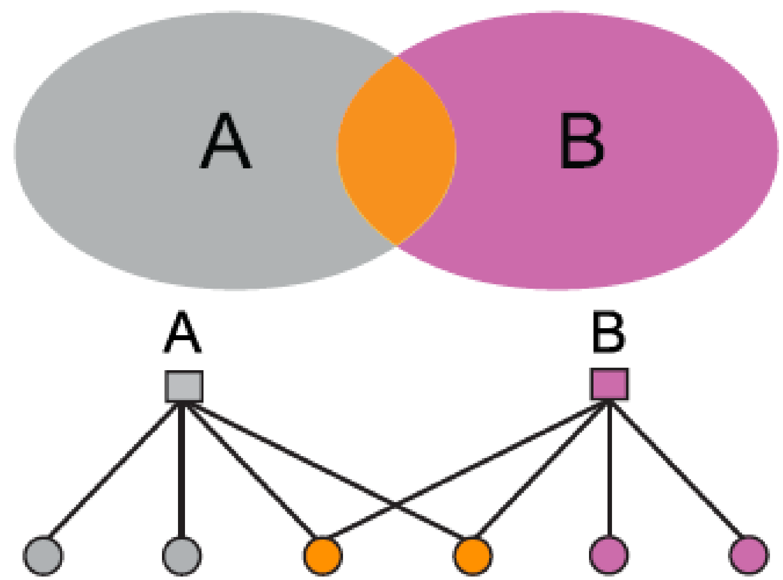


# AGM: Flexibility

- **AGM can express a variety of community structures:** Non-overlapping, Overlapping, Nested



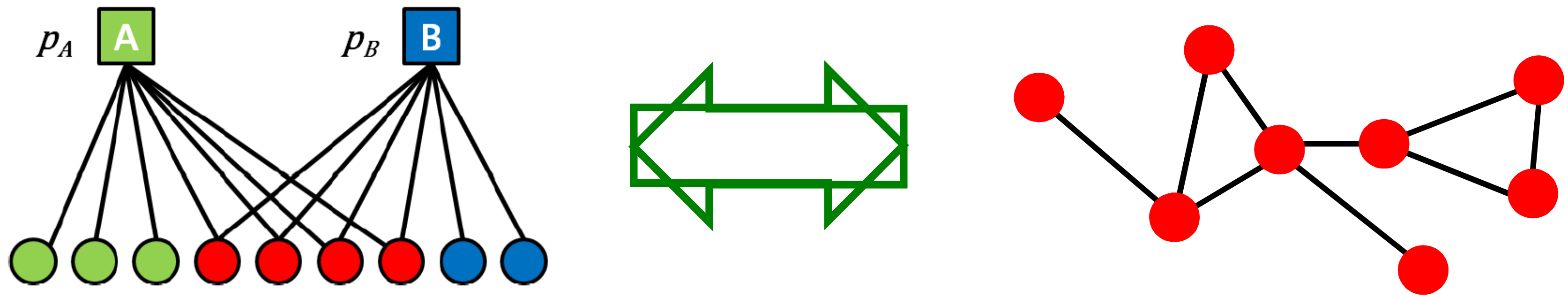



# How do we detect communities with AGM?

# Detecting Communities

- **Detecting communities with AGM:**




**Given a Graph $G(V, E)$, find the Model**

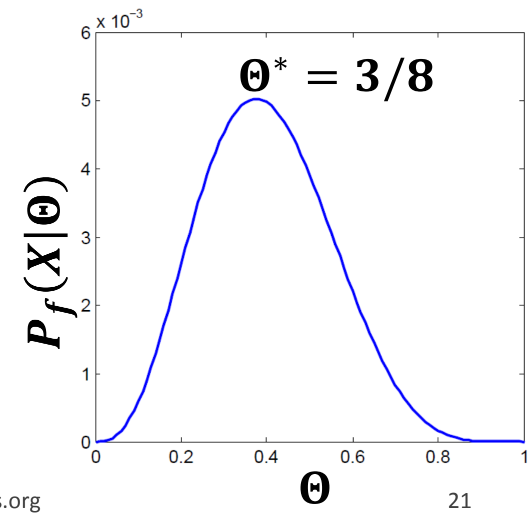
1) Affiliation graph ***M***
2) Number of communities **C**
3) Parameters $\boldsymbol{p_c}$

# Maximum Likelihood Estimation

- **Maximum Likelihood Principle (MLE):**
  - **Given:** Data $\boldsymbol{X}$
  - **Assumption:** Data is generated by some model $\boldsymbol{f}(\boldsymbol{\Theta})$
    - $\boldsymbol{f}$ **… model**
    - $\boldsymbol{\Theta}$ **… model parameters**
  - Want to estimate $\boldsymbol{P_f(X|\Theta)}$:
    - The probability that our model $\boldsymbol{f}$ (with parameters $\boldsymbol{\Theta}$) generated the data
  - **Now let's find the most likely model that could have generated the data:** $\arg\max_{\Theta} \boldsymbol{P_f(X|\Theta)}$

# Example: MLE

- **Imagine we are given a set of coin flips**
- **Task:** Figure out the bias of a coin!
  - **Data:** Sequence of coin flips: $\boldsymbol{X = [1, 0, 0, 0, 1, 0, 0, 1]}$
  - **Model:** $\boldsymbol{f(\Theta)}$ = return 1 with prob. $\Theta$, else return 0
  - **What is $\boldsymbol{P_f(X|\Theta)}$?** Assuming coin flips are independent
    - **So,** $\boldsymbol{P_f(X|\Theta) = P_f(1|\Theta) * P_f(0|\Theta) * P_f(0|\Theta) \dots * P_f(1|\Theta)}$
      - What is $\boldsymbol{P_f(1|\Theta)}$? **Simple,** $\boldsymbol{P_f(1|\Theta) = \Theta}$
    - **Then,** $\boldsymbol{P_f(X|\Theta) = \Theta^3(1-\Theta)^5}$
    - **For example:**
      - $\boldsymbol{P_f(X|\Theta = 0.5) = 0.003906}$
      - $\boldsymbol{P_f\left(X\middle|\Theta = \frac{3}{8}\right) = 0.005029}$
  - **What did we learn?** Our data was most likely generated by coin with bias $\boldsymbol{\Theta = 3/8}$



# MLE for Graphs

- **How do we do MLE for graphs?**
  - **Model generates a probabilistic adjacency matrix**
  - We then flip all the entries of the probabilistic matrix to obtain the **binary adjacency matrix $A$**

For every pair of nodes $u, v$ AGM gives the prob. $p_{uv}$ of them being linked

| | | | |
|---|---|---|---|
| 0 | 0.10 | 0.10 | 0.04 |
| 0.10 | 0 | 0.02 | 0.06 |
| 0.10 | 0.02 | 0 | 0.06 |
| 0.04 | 0.06 | 0.06 | 0 |

Flip biased coins

$A$

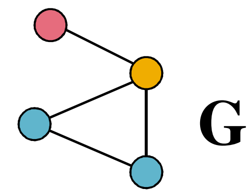
| | | | |
|---|---|---|---|
| 0 | 1 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 |

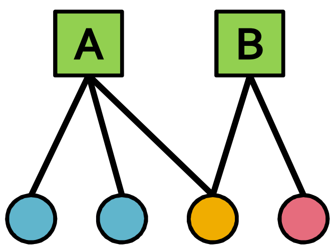
- **The likelihood of AGM generating graph G:**

$$P(G \mid \Theta) = \prod_{(u,v)\in E} P(u,v) \prod_{(u,v)\notin E} (1 - P(u,v))$$

# Graphs: Likelihood $P(G|\Theta)$

- Given graph $G(V,E)$ and $\Theta$, we calculate likelihood that $\Theta$ generated $G$: $P(G|\Theta)$




$\Theta$=B($V$, $C$, $M$, $\{p_c\}$)

| | | | |
|---|---|---|---|
| 0 | 0.9 | 0.9 | 0 |
| 0.9 | 0 | 0.9 | 0 |
| 0.9 | 0.9 | 0 | 0.9 |
| 0 | 0 | 0.9 | 0 |

| | | | |
|---|---|---|---|
| 0 | 1 | 1 | 0 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 1 |
| 0 | 0 | 1 | 0 |

$G$

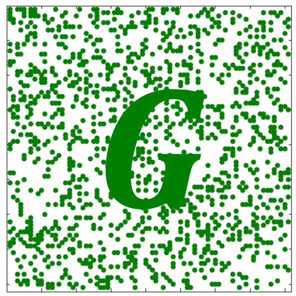
$\boxed{P(G|\Theta)}$

$$P(G \mid \Theta) = \prod_{(u,v)\in E} P(u,v) \prod_{(u,v)\notin E} (1 - P(u,v))$$

# MLE for Graphs

- **Our goal:** Find $\boldsymbol{\Theta} = \boldsymbol{B}(\boldsymbol{V}, \boldsymbol{C}, \boldsymbol{M}, \{\boldsymbol{p_C}\})$ **such that:**

$\arg\max_{\Theta} P($

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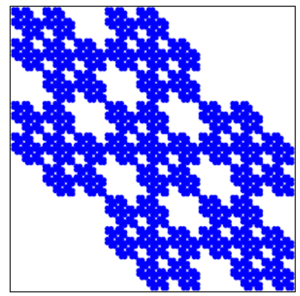
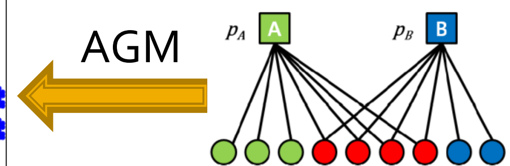





$)$

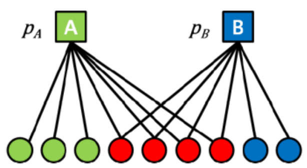
- **How do we find $\boldsymbol{B}(\boldsymbol{V}, \boldsymbol{C}, \boldsymbol{M}, \{\boldsymbol{p_C}\})$ that maximizes the likelihood?**

# MLE for AGM

- **Our goal is to find $B(V, C, M, \{p_C\})$ such that:**

$$\underset{B(V,C,M,\{p_C\})}{\arg\max} \prod_{u,v \in E} P(u,v) \prod_{uv \notin E} (1 - P(u,v))$$

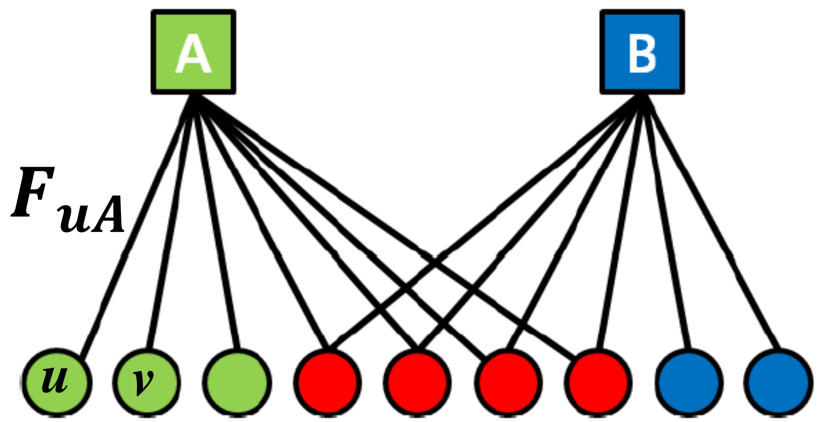
- **Problem:** Finding **B** means finding the bipartite affiliation network.




  - **There is no nice way to do this.**
  - **Fitting $B(V, C, M, \{p_C\})$ is too hard, let's change the model (so it is easier to fit)!**

# From AGM to BigCLAM

- **Relaxation: Memberships have strengths**



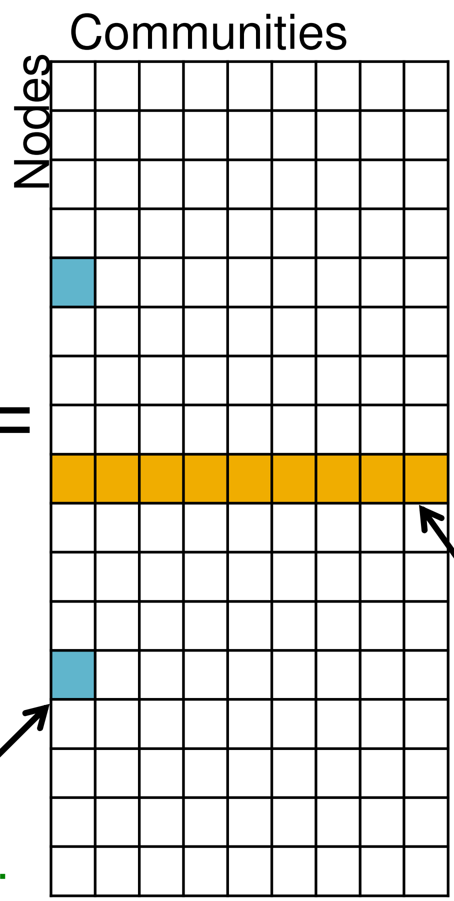

- $F_{uA}$: The membership strength of node $u$ to community $A$ ($F_{uA} = 0$: no membership)
- **Each community $A$ links nodes independently:**

$$P_A(u,v) = 1 - \exp(-F_{uA} \cdot F_{vA})$$

# Factor Matrix $F$

- **Community membership strength matrix $F$**

$\boldsymbol{F} =$




$F_{vA}$ … strength of $u$'s membership to $A$

$F_u$ … vector of community membership strengths of $u$

- $P_A(u, v) = 1 - \exp(-F_{uA} \cdot F_{vA})$
  - **Probability of connection is proportional to the product of strengths**
    - **Notice:** If one node doesn't belong to the community ($F_{uC} = 0$) then $P(u, v) = 0$
- Prob. that **at least one** common community $C$ links the nodes:
  - $P(u, v) = 1 - \prod_C (1 - P_C(u, v))$

# From AGM to BigCLAM

- Community $\boldsymbol{A}$ links nodes $\boldsymbol{u}, \boldsymbol{v}$ independently:

$$P_A(u,v) = 1 - \exp(-F_{uA} \cdot F_{vA})$$

- Then prob. at least one common $\boldsymbol{C}$ links them:

$$P(u,v) = 1 - \prod_C \big(1 - P_C(u,v)\big)$$
$$= 1 - \exp(-\textstyle\sum_C F_{uC} \cdot F_{vC})$$
$$= 1 - \exp(-F_u \cdot F_v^T)$$

- **Example $F$ matrix:**

| | | | | |
|---|---|---|---|---|
| $F_u$ : | 0 | 1.2 | 0 | 0.2 |
| $F_v$ : | 0.5 | 0 | 0 | 0.8 |
| $F_w$ : | 0 | 1.8 | 1 | 0 |

Node community membership strengths

**Then:** $F_u \cdot F_v^T = 0.16$

**And:** $P(u,v) = 1 - exp(-0.16) = 0.14$

**But:** $P(u,w) = 0.88$

$P(v,w) = 0$

# BigCLAM: How to find F

- **Task:** **Given a network $G(V, E)$, estimate $F$**
  - **Find $F$ that maximizes the likelihood:**

$$\arg\max_F \prod_{(u,v)\in E} P(u,v) \prod_{(u,v)\notin E} (1 - P(u,v))$$

    - **where:** $P(u,v) = 1 - \exp(-F_u \cdot F_v^T)$
    - Many times we take the logarithm of the likelihood, and call it log-likelihood: $l(F) = \log P(G|F)$
- **Goal:** Find $F$ that maximizes $l(F)$:

$$l(F) = \sum_{(u,v)\in E} \log(1 - \exp(-F_u F_v^T)) - \sum_{(u,v)\notin E} F_u F_v^T$$

# BigCLAM: V1.0

$$l(F_u) = \sum_{v \in \mathcal{N}(u)} \log(1 - \exp(-F_u F_v^T)) - \sum_{v \notin \mathcal{N}(u)} F_u F_v^T$$

- **Compute gradient of a single row $\boldsymbol{F_u}$ of $\boldsymbol{F}$:**

$$\nabla l(F_u) = \sum_{v \in \mathcal{N}(u)} F_v \frac{\exp(-F_u F_v^T)}{1 - \exp(-F_u F_v^T)} - \sum_{v \notin \mathcal{N}(u)} F_v$$

$\boldsymbol{\mathcal{N}(u)}$.. Set out outgoing neighbors

- **Coordinate gradient ascent:**
  - **Iterate over the rows of $\boldsymbol{F}$:**
    - Compute gradient $\boldsymbol{\nabla l(F_u)}$ of row $\boldsymbol{u}$ (while keeping others fixed)
    - Update the row $\boldsymbol{F_u}$: $\boldsymbol{F_u} \leftarrow \boldsymbol{F_u} + \boldsymbol{\eta}\, \boldsymbol{\nabla l(F_u)}$
    - Project $\boldsymbol{F_u}$ back to a non-negative vector: If $\boldsymbol{F_{uC}} < \boldsymbol{0}$: $\boldsymbol{F_{uC}} = \boldsymbol{0}$
- **This is slow!** Computing $\boldsymbol{\nabla l(F_u)}$ takes linear time!

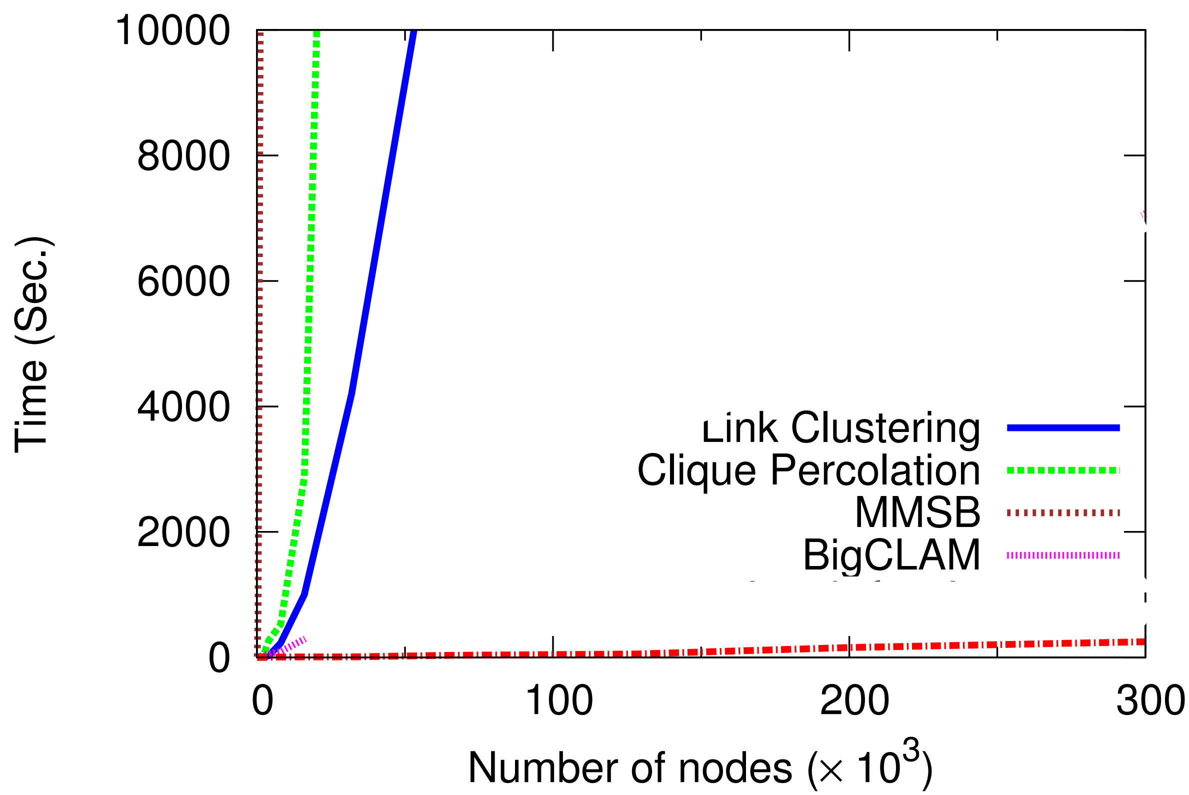
# BigCLAM: V2.0

- **However, we notice:**

$$\sum_{v \notin \mathcal{N}(u)} F_v = \left(\sum_v F_v - F_u - \sum_{v \in \mathcal{N}(u)} F_v\right)$$

  - We cache $\sum_v \boldsymbol{F_v}$
  - So, computing $\sum_{v \notin \mathcal{N}(u)} \boldsymbol{F_v}$ now takes **linear time** in the degree $|\mathcal{N}(\boldsymbol{u})|$ of $\boldsymbol{u}$
    - In networks degree of a node is much smaller to the total number of nodes in the network, so this is a significant speedup!

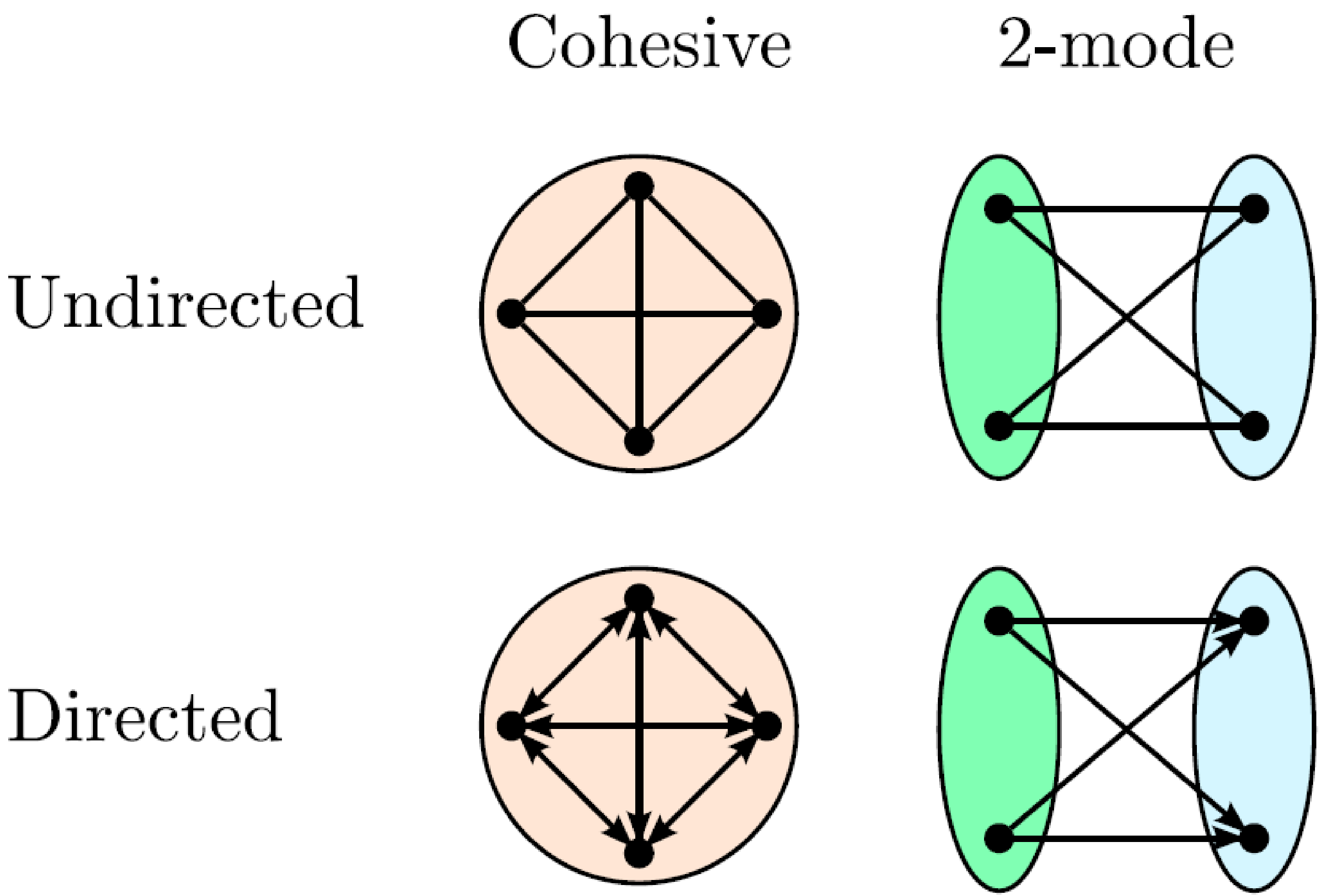
# BigClam: Scalability




- **BigCLAM takes 5 minutes for 300k node nets**
  - **Other methods take 10 days**
- **Can process networks with 100M edges!**

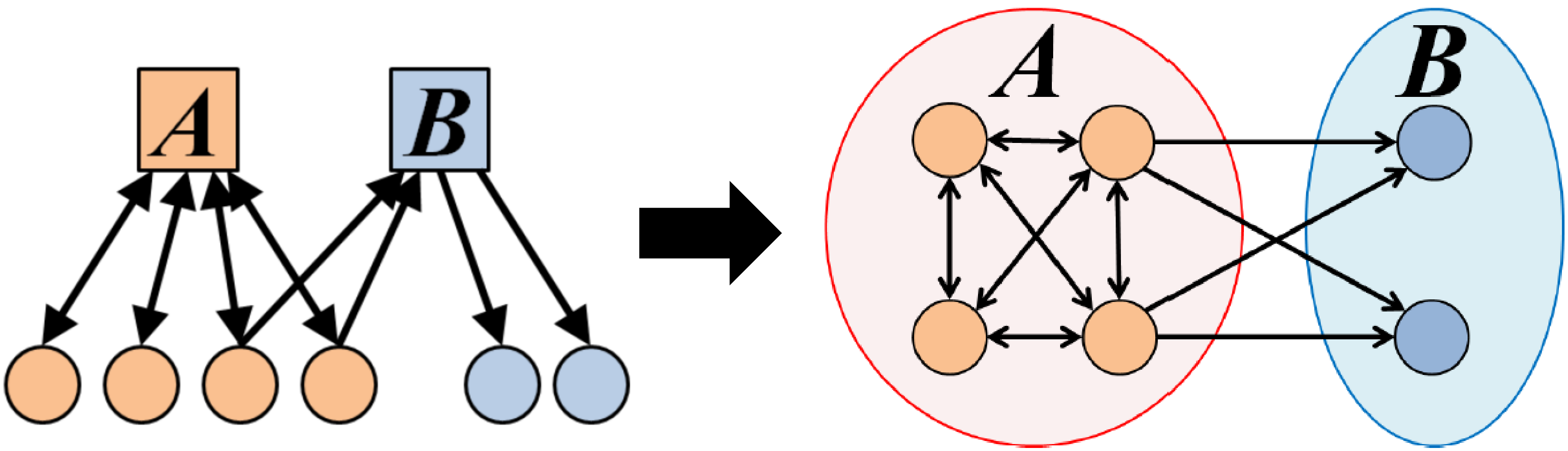
# Extension:
# Directed memberships

# Extension: Beyond Clusters

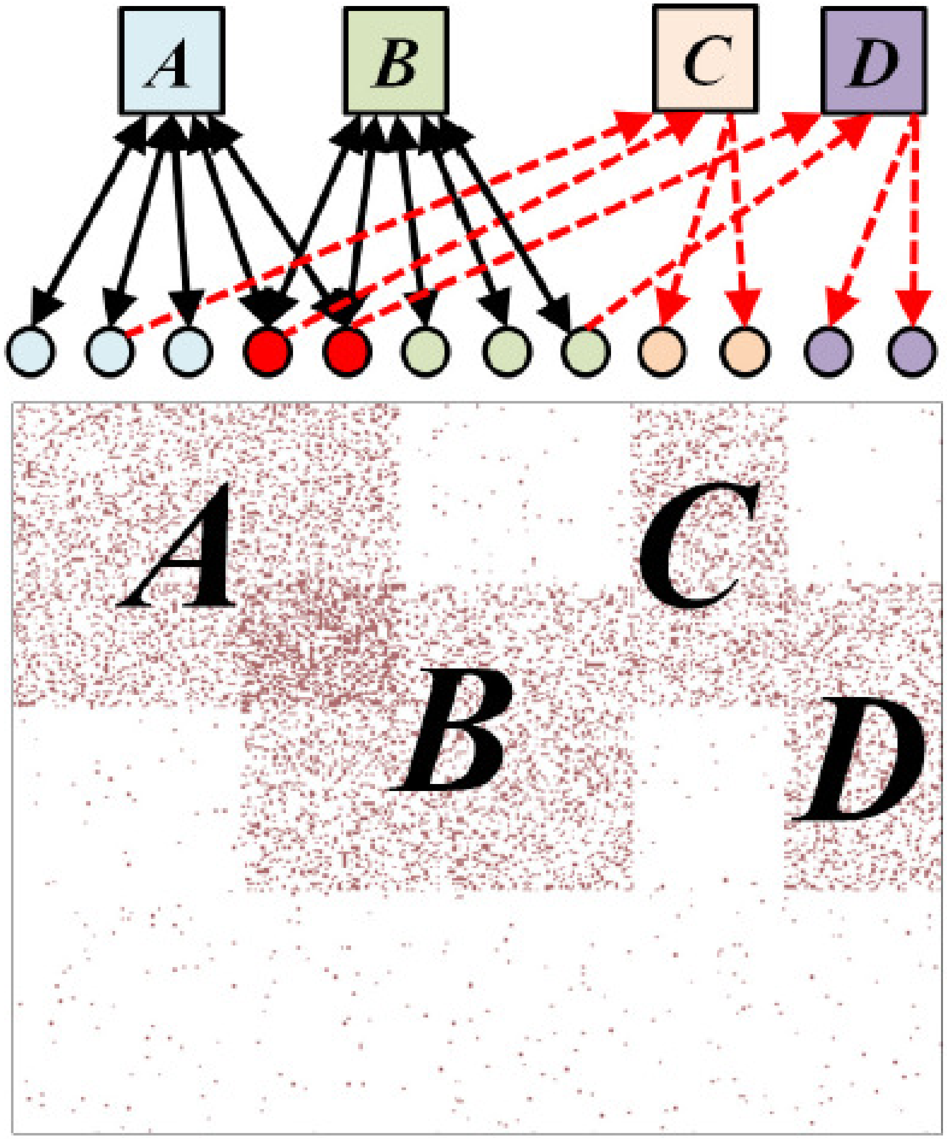


# Extension: Directed AGM

- **Extension:**
  **Make community membership edges directed!**
  - **Outgoing membership:** Nodes "**sends**" edges
  - **Incoming membership:** Node "**receives**" edges

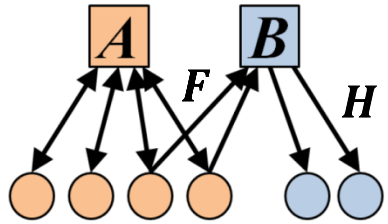


# Example: Model and Network



# Directed AGM

- **Everything is almost the same except now we have 2 matrices: $\boldsymbol{F}$ and $\boldsymbol{H}$**
  - $\boldsymbol{F}$**…** out-going community memberships
  - $\boldsymbol{H}$**…** in-coming community memberships

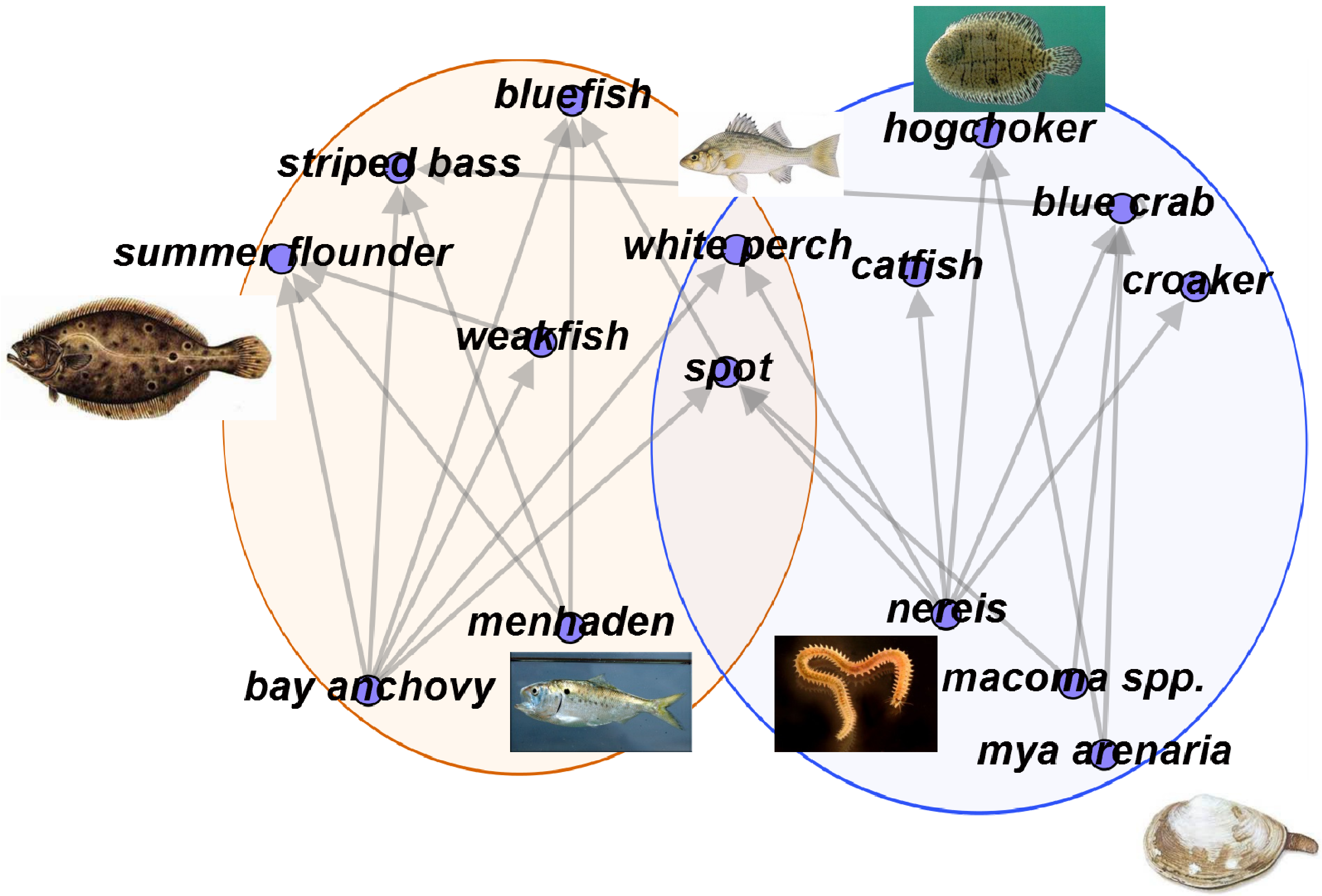



- **Edge prob.:** $\boldsymbol{P(u,v) = 1 - exp(-F_u H_v^T)}$
- **Network log-likelihood:**

$$l(F,H) = \sum_{(u,v)\in E} \log(1 - \exp(-F_u H_v^T)) - \sum_{(u,v)\notin E} F_u H_v^T$$

which we optimize the same way as before

# Predator-prey Communities



# More details at…

- Overlapping Community Detection at Scale: A Nonnegative Matrix Factorization Approach by J. Yang, J. Leskovec. *ACM International Conference on Web Search and Data Mining (WSDM)*, 2013.
- Detecting Cohesive and 2-mode Communities in Directed and Undirected Networks by J. Yang, J. McAuley, J. Leskovec. *ACM International Conference on Web Search and Data Mining (WSDM)*, 2014.
- Community Detection in Networks with Node Attributes by J. Yang, J. McAuley, J. Leskovec. *IEEE International Conference On Data Mining (ICDM)*, 2013.