Morphisms on Marked Graphs

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Abstract. Many kinds of morphisms on Petri nets have been defined and studied. They can be used as formal techniques supporting refinement/abstraction of models. In this paper we introduce a new notion of morphism on marked graphs, a class of Petri nets used for the representation of systems having deterministic behavior. Such morphisms can indeed be used to represent a form of abstraction on marked graphs, consisting in folding cycles and identifying chains. We will then prove that systems joined by these morphisms show behavioral similarities.

Keywords: Petri nets, marked graphs, morphisms, model abstraction, preservation of behavioral properties

1 Introduction

When working on concurrent and distributed systems, the dimensions and complexity of a model may lead to difficulties in the analysis of its features and properties. For this reason it is useful to have formal techniques allowing the decomposition of the entire model into separate modules which can be studied separately, then being recomposed maintaining their properties. Another way to reduce the dimension and complexity of a model is to use a multilevel approach to its analysis: we start working on a very abstract version of the model, then proceed through different levels of refinement by adding details to the model.

In order to obtain such functionalities we can use morphisms on Petri nets. In the literature (see, for example, [1], [2], [3], [4] and [5]) several kinds of morphism on different classes of Petri nets have been introduced. In this paper we propose a new definition of morphism on marked graphs, a class of Petri nets often used for representing systems having deterministic behavior. These so called F-morphisms and the subclass of $\hat{F}$-morphisms constitute a formal instrument which can be used to obtain a kind of abstraction of marked graphs.

Some kinds of morphisms defined in the literature, such as α-morphisms ([5]), allow to collapse part of the initial model on a single place or a single transition in order to obtain the abstract system. Differently, $\hat{F}$-morphisms map places on single places and transitions on single transitions, preserving the environment of each mapped element. Instead of collapsing portions of the detailed model into a single element, the abstraction is here obtained by "folding" cycles and identifying chains and cycles. Both these elements still remain in the reduced model.

Such kind of abstraction preserves the behavior of the mapped part of the original system. This means that, whenever we apply a $\hat{F}$-morphism on a system, all the sequences of actions executable in the reduced version can be found in the original model.

In the last part of this paper, an analysis of preserved and reflected behavioral properties and invariants of marked graphs joined by $\hat{F}$-morphisms is performed.

In the next section, basic definitions related to Petri nets and their unfoldings are recalled. In Section 3 F- and $\hat{F}$-morphisms are introduced together with their main features. Then the relationship between the unfoldings of two marked graphs joined by a $\hat{F}$-morphism is explicated. Section 4 shows the results of the analysis of behavioral and structural properties preserved and reflected by $\hat{F}$-morphisms. The paper is closed by a short concluding section.

2 Preliminary definitions

In this section we recall basic definitions about marked graph theory and unfoldings. These notions will be used in the next chapters to study important aspects of F-morphisms.

2.1 Petri nets

We first start introducing the notion of net as seen in [6], with some adjustements.

Definition 1. *A net is a triple $N = (S, T, F)$, where*

- *S is a set of places,*
- *T is a set of transitions such that $S \cap T = \emptyset$,*
- *F is a set of directed arcs (flow relation), $F \subseteq (S \times T) \cup (T \times S)$.*

All places and transitions are said to be *elements* of N. A net is *finite* if the set of elements is finite.

For an element x of $S \cup T$, its *pre-set* is defined by

$$^\bullet x = \{y \in S \cup T \mid (y, x) \in F\}$$

while its *post-set* is defined by

$$x^\bullet = \{y \in S \cup T \mid (x, y) \in F\}.$$

A *directed path* (*path* for short) in a net N is a nonempty sequence $x_0 \dots x_k$ satisfying $x_i \in x_{i-1}^\bullet$ for each i $(1 \leq i \leq k)$. We say that this path *leads* from x_0 to x_k. The net is *strongly connected* if for each two elements x and y there exists a directed path leading from x to y.

An *undirected path* is a nonempty sequence $x_0 \dots x_k$ of elements satisfying $x_i \in {^\bullet x_{i-1}} \cup x_{i-1}^\bullet$ for each i $(1 \leq i \leq k)$. Such undirected path *leads* from x_0 to

x_k. The net is *weakly connected* if, for each two elements x and y, there exists an undirected path leading from x to y. In this paper, we will call *connected* a weakly connected net.

A *directed circuit* is a directed path $x_0 \dots x_k x_0$ such that, for each $i, j \in \mathbb{N}$, $i, j \leq k$, $i \neq j$, $x_i \neq x_j$ holds.

The states of a Petri net are defined by its *markings*. State changes are caused by the occurrences of transitions. A *marking* of a net $N = (S, T, F)$ is a mapping $M : S \to \mathbb{N}$. A place $s \in S$ is *marked* by a marking M if $M(s) > 0$.

A transition t is *enabled* at a marking M if M marks every place in ${}^\bullet t$. Then t can *occur*. Its occurrence transforms M into the marking M', defined for each place s as

$$M'(s) = \begin{cases} M(s) - 1 & \text{if } s \in {}^\bullet t \setminus t^\bullet, \\ M(s) + 1 & \text{if } s \in t^\bullet \setminus {}^\bullet t, \\ M(s) & \text{otherwise.} \end{cases}$$

In this case we write $M \xrightarrow{t} M'$. Notice that a place in ${}^\bullet t \cap t^\bullet$ is marked whenever t is enabled but does not change its token count by the occurrence of t. A marking is called *dead* if it enables no transition of N. A net N together with an *initial marking* M_0 constitutes a *Petri Net System* (also called *place/transition system*), denoted (N, M_0).

Let M be a marking of a net. A finite sequence $t_1 \dots t_k$ of transitions is called a *finite occurrence sequence, enabled at* M, if there are markings $M_1, \dots, M_k$ such that

$$M \xrightarrow{t_1} M_1 \xrightarrow{t_2} \dots \xrightarrow{t_k} M_k.$$

In this case we write $M \xrightarrow{\omega} M_k$, where $\omega = t_1 \dots t_k$. The empty sequence $\mathcal{E}$ is enabled at any marking M and satisfies $M \xrightarrow{\mathcal{E}} M$. A marking M' is said to be *reachable* from a marking M if there exists a finite occurrence sequence ω such that $M \xrightarrow{\omega} M'$.

In this paper we will mainly work on a particular kind of Petri nets, the *marked graphs*.

Definition 2. *A Petri net $N = (S, T, F, M_0)$ is a marked graph if, for every $s \in S$, $|{}^\bullet s| \leq 1$ and $|s^\bullet| \leq 1$.*

2.2 Behavioral properties

The presence of an initial marking M_0 allows to identify the *behavior* of the Petri net system (N, M_0), defined as the set of all markings reachable from M_0 together with the set of occurences of each transition which make the global state of the system change.

Properties of a net depending on the initial marking are known as *behavioral properties* of the net. We now introduce some behavioral properties ([7]) which will be used in the next sections.

Definition 3. *A Petri net* (N, M_0) *is said to be k-bounded or simply bounded if the number of tokens in each place does not exceed a finite number k for any marking reachable from* M_0*, i.e.,* $M(s) \leq k$ *for every place s and every reachable marking M.* (N, M_0) *is said to be safe if it is 1-bounded.*

While boundedness implies the presence of a finite number of global states for a finite net, *liveness* ensures that every event can potentially occur in the future.

Definition 4. *A Petri net* (N, M_0) *is said to be live (or equivalently* M_0 *is said to be a live marking for N) if, no matter which marking has been reached from* M_0*, it is possible to ultimately fire any transition of the net by progressing through some further firing sequence.*

2.3 Incidence matrix and structural invariants

Definitions recalled in this section are taken from [7], with some adaptations.

Definition 5. *Let* (N, M_0) *be a Petri net with n transitions and m places. Its incidence matrix* $A = [a_{ij}]$ *is an* $m \times n$ *matrix of integers and its typical entry is given by*

$$a_{ij} = a_{ij}^{+} - a_{ij}^{-}$$

where $a_{ij}^{+} = 1$ *if there is an arc of N going from transition j to its post-condition i, otherwise* $a_{ij}^{+} = 0$*, while* $a_{ij}^{-} = 1$ *if there is an arc to transition j from its pre-condition i, otherwise* $a_{ij}^{-} = 0$*.*

Some properties of a Petri net can be studied through the incidence matrix and its invariants. A S-invariant associates weights to places in a way such that the weighted sum of tokens is the same in all reachable markings.

Definition 6. *Let N be a net and let A be its incidence matrix. A vector* $\underline{I} : S \to \mathbb{Z}$ *is a S-invariant for N iff it is a solution of:* $\underline{I}A = \underline{0}$*.*

T-invariants allow to identify possible cyclic behaviors in a Petri net.

Definition 7. *Let N be a net and let A be its incidence matrix. A vector* $\underline{J} : T \to \mathbb{Z}$ *is a T-invariant for N iff it is a solution of:* $A\underline{J}^T = \underline{0}$*.*

2.4 Branching processes and unfoldings

The *behavior* of a Petri net N can be represented in different ways. One of these is to use the so called *unfolding* of N. In order to understand what the unfolding of a net is, we first need to introduce some formal definitions. The theoretical notions we will relate in this subsection are all taken from [7]. From now on, we will only consider Petri nets such that, for every transition t, ${}^{\bullet}t$ and $t^{\bullet}$ are finite sets and, moreover, we assume them to be nonempty. Furthermore, we do not allow more than one token on a place in the initial marking. Such constraints do not result too restrictive with respect to the behavior of the studied systems.

Definition 8. *Let $N = (S, T, F, M_0)$ be a Petri net. For $x, y \in S \cup T$ we say that x precedes y if there is a (possibly empty) directed path from x to y in N. N is finitary if for every $y \in S \cup T$ the set $\{x \in S \cup T \mid x \text{ precedes } y\}$ is finite.*

The relation *precedes* defines a partial order on $S \cup T$, and $Min(N)$ is the set of minimal elements of that partial order. We now introduce the notion of *conflict*.

Definition 9. *Let $N = (S, T, F, M_0)$ be a Petri net. For $x_1, x_2 \in S \cup T$, x_1 and x_2 are in conflict, denoted $x_1 \# x_2$, if there exist distinct transitions $t_1, t_2 \in T$ such that ${}^\bullet t_1 \cap {}^\bullet t_2 \neq \emptyset$ and t_i precedes x_i, for $i = 1, 2$. For $x \in S \cup T$, x is in self-conflict if $x \# x$.*

The concept of conflict is used to define *occurrence net*.

Definition 10. *An occurrence net is a finitary acyclic net $N = (S, T, F, M_0)$ such that*

- *for every $s \in S$, $|{}^\bullet s| \leq 1$,*
- *no transition $t \in T$ is in self-conflict, and*
- *$M_0 = Min(N)$.*

We now define a particular kind of morphism called “folding” in [8]. Intuitively, a *homomorphism* from net N_1 to net N_2 formalizes the fact that N_1 can be folded onto a part of N_2, or, in other words, that N_1 can be obtained by partially unfolding a part of N_2.

Definition 11. *Let $N_i = (S_i, T_i, F_i, M_0^i)$ be nets, $i = 1, 2$. A homomorphism from N_1 to N_2 is a mapping $h : S_1 \cup T_1 \to S_2 \cup T_2$ such that*

- *$h(S_1) \subseteq S_2$ and $h(T_1) \subseteq T_2$,*
- *for every $t \in T_1$, the restriction of h to ${}^\bullet t$ is a bijection between ${}^\bullet t$ and ${}^\bullet h(t)$, and similarly for $t^\bullet$ and $h(t)^\bullet$, and*
- *the restriction of h to M_0^1 is a bijection between M_0^1 and M_0^2.*

The notions of homomorphism and occurrence net are necessary to formally define *branching processes*.

Definition 12. *Let $N = (S, T, F, M_0)$ be a net. A branching process of N is a pair (N', π), where $N' = (S', T', F', M_0')$ is an occurrence net and π is a homomorphism from N' to N, such that, for every $t_1, t_2 \in T$, if ${}^\bullet t_1 = {}^\bullet t_2$ and $\pi(t_1) = \pi(t_2)$, then $t_1 = t_2$.*

In [9], a notion of homomorphism between branching processes of the same net N is also defined. Injective homomorphisms define a partial order for the branching processes of N, called *approximation*. The set of the isomorphism classes of the branching processes of N, together with approximation, form a complete lattice. The least upper bound of such lattice is the *unfolding* of N.

3 A new class of morphisms on marked graphs

In this section we introduce a new kind of morphism on marked graphs, the *F-morphisms*. We will then focus on a subclass of such morphisms, the *$\hat{F}$-morphisms*, analysing some interesting features of theirs. Finally, we will study the relationship between the unfoldings of two marked graphs joined by a $\hat{F}$-morphism. In this paper we only consider a particular kind of marked graphs.

Remark *From now on, we only consider connected marked graphs without self-loops.*

It is now possible to introduce the main notion of this work.

Definition 13. *Let $N_i = (S_i, T_i, F_i, M_0^i)$, $i = 1, 2$, be two marked graphs. A F-morphism from N_1 to N_2 is a pair (σ, τ), where $\sigma : S_1 \to S_2$ and $\tau : T_1 \to T_2$ are partial surjective functions, such that:*

- *if $\tau(t_1)$ is undefined, then $\sigma({}^\bullet t_1) = \emptyset = \sigma(t_1^\bullet)$,*
- *if $\tau(t_1) = t_2$, then the restriction of σ to ${}^\bullet t_1$ is an injective and surjective partial function from ${}^\bullet t_1$ to ${}^\bullet t_2$ and, similarly, the restriction of σ to $t_1^\bullet$ is an injective and surjective partial function from $t_1^\bullet$ to $t_2^\bullet$,*
- *for every $s' \in S_2$*
$$M_0^2(s') = \sum_{s \in \sigma^{-1}(s')} M_0^1(s).$$

We define the composition of two F-morphisms $(\sigma_1, \tau_1) : N_1 \to N_2$ and $(\sigma_2, \tau_2) : N_2 \to N_3$ by using the notion of composition of functions, i.e., $(\sigma_1, \tau_1) \circ (\sigma_2, \tau_2) = (\sigma_2 \circ \sigma_1, \tau_2 \circ \tau_1) : N_1 \to N_3$. F-morphisms are closed by composition.

Theorem 1. *Let $N_i = (S_i, T_i, F_i, M_0^i)$ be marked graphs for $i = 1, \ldots, 3$. Let (σ_i, τ_i), $i = 1, 2$, be F-morphisms from N_i to N_{i+1}. The function $(\sigma, \tau) : N_1 \to N_3$, where $\sigma = \sigma_2 \circ \sigma_1$ and $\tau = \tau_2 \circ \tau_1$ is a F-morphism.*

This theorem is proved in [10]. The identity function $1_N = (id_S, id_T)$ is a F-morphism, where $id_S : S \to S$ and $id_T : T \to T$ are the total identity functions. The composition is associative. Hence, the family of F-morphisms, together with marked graphs, form a category which takes the name of *Marked Graph System*, denoted $\mathcal{MGS}$.

With these morphisms we allow to map chains on cycles, as shown in Figure 1, representing an example of F-morphism from N_1 to N_2. The labels suggest the arrows of the morphism. Notice that the cardinality of the pre-images of the elements labelled by `1`, `b` and `2` of N_2 is one, while the place labelled by `ac` has two elements in its pre-image.

By adding a further constraint to the definition of F-morphisms, we get a subclass of morphisms which preserve cycles and chains.

Definition 14. *Let $N_i = (S_i, T_i, F_i, M_0^i)$ be marked graphs for $i = 1, 2$. A $\hat{F}$-morphism from N_1 to N_2 is a F-morphism (σ, τ) with the following restriction:*

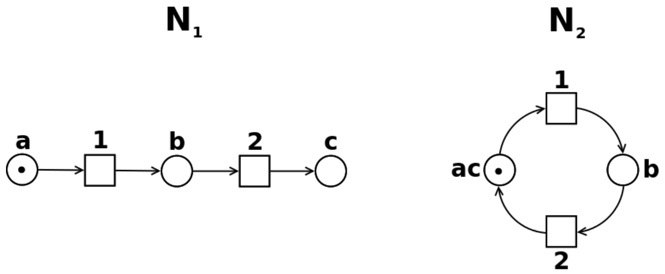


Fig. 1

– *for all $s_1 \in S_1$ such that $\sigma(s_1) = s_2$, the restriction of τ to ${}^\bullet s_1$ is a bijection from ${}^\bullet s_1$ to ${}^\bullet s_2$ and, similarly, the restriction of τ to $s_1^\bullet$ is a bijection from $s_1^\bullet$ to $s_2^\bullet$.*

It is easy to see that $\hat{F}$-morphisms are closed by composition. In fact, since we already know that a $\hat{F}$-morphism (σ, τ) is a F-morphism, it is sufficient to prove that the additional constraint that characterizes $\hat{F}$-morphisms is preserved by composition. We prove it simply by observing that the composition of two bijections is also a bijection.

The example in Figure 1 shows a F-morphism (σ, τ) which is not a $\hat{F}$-morphism: let s_1 be the place of N_1 labelled with c and let $\sigma(s_1) = s_2$ (therefore, s_2 is the place of N_2 labelled with ac). The restriction of τ to $s_1^\bullet$ is not a bijection from $s_1^\bullet$ to $s_2^\bullet$, in fact we have that $s_1^\bullet = \emptyset \neq s_2^\bullet$.

In Figure 2 three examples of $\hat{F}$-morphisms are shown: the first two of them, $((\sigma_1, \tau_1) : N_1 \to N_2$ and $(\sigma_2, \tau_2) : N_3 \to N_4$, respectively, Figure 2a and Figure 2b), allow us to observe that, using $\hat{F}$-morphisms, it is possible to compress cycles and to identify chains; in the last one, $((\sigma_3, \tau_3) : N_5 \to N_6$, Figure 2c), an identification of cycles is represented.

Let us now compare $\hat{F}$-morphisms with another kind of morphisms defined in [2], N-morphisms, corresponding to a kind of partial simulation. We want to do this since we will later show that we can always find a N-morphism between the unfoldings of two marked graphs joined by a $\hat{F}$-morphism. First of all, N-morphisms are defined on elementary net systems, while $\hat{F}$-morphisms are defined on marked graphs. N-morphisms define a relation between the places of the joined systems, such that its inverse is a partial function. Differently, $\hat{F}$-morphisms allow two places to have the same image. Furthermore, for $\hat{F}$-morphisms the mapping between events is surjective, while N-morphisms do not require such constraint. The last main difference is that, if two places s and s' of different elementary net systems are joined by a N-morphism, s belongs to the initial case of the first system if and only if s' is in the initial case of the second one, whereas whith $\hat{F}$-morphism a place of the starting system contain-

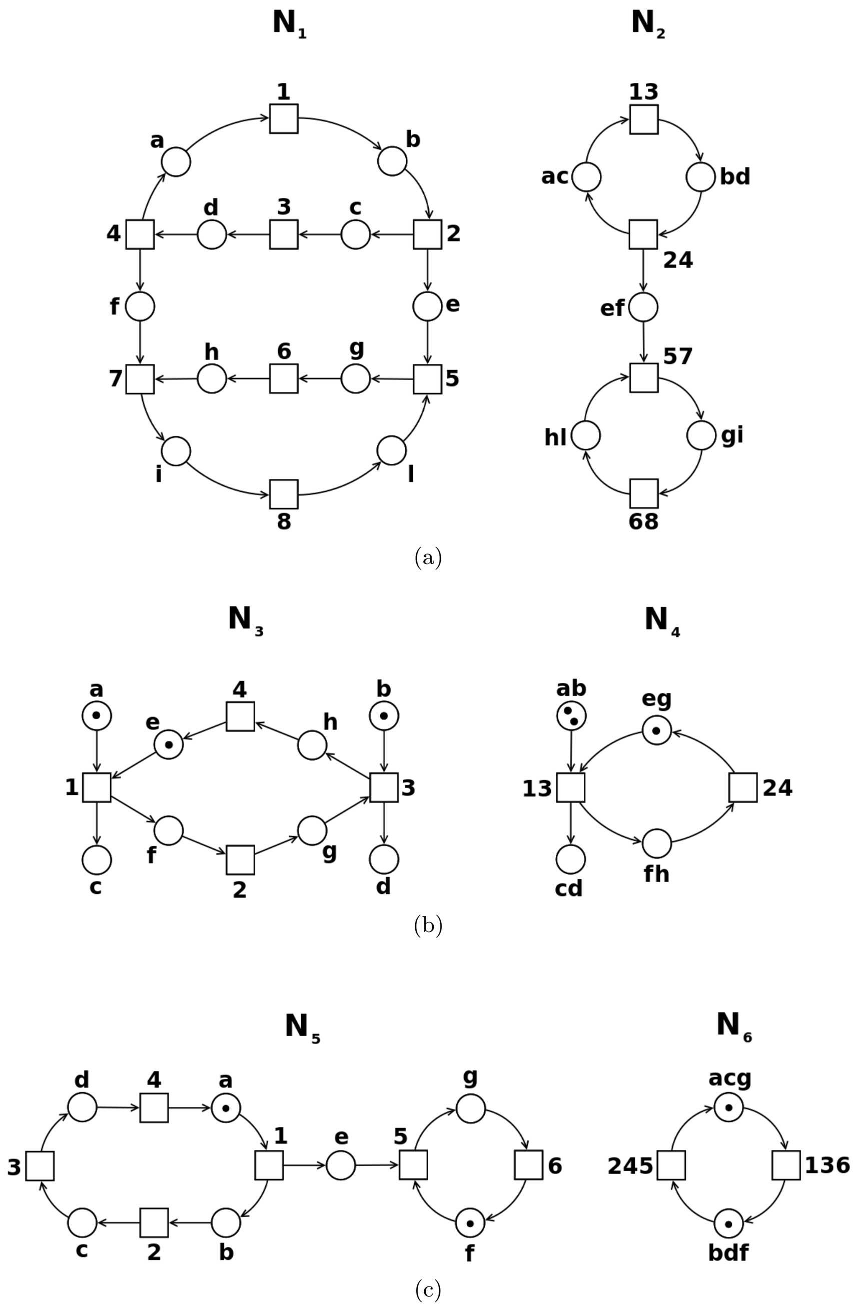

N1
1
a
b
4
d
3
c
2
f
e
7
h
6
g
5
i
8
l
N2
13
ac
bd
24
ef
57
hl
gi
68
(a)
N3
a
4
b
e
h
1
3
f
g
c
2
d
N4
ab
eg
13
24
cd
fh
(b)
N5
d
4
a
1
e
5
g
3
6
c
2
b
f
N6
acg
245
136
bdf
(c)

Fig. 2

ing no tokens in the initial marking can be mapped on a place containing tokens.

We now show some interesting features of $\hat{F}$-morphisms.

Theorem 2. *Let $N_i = (S_i, T_i, F_i, M_0^i)$ be marked graphs, for $i = 1, 2$, joined by a $\hat{F}$-morphism $(\sigma, \tau) : N_1 \to N_2$. Let A_1 and A_2 be the incidence matrices of, respectively, N_1 and N_2. Let $s' \in S_2$ be a place of N_2 such that $\sigma^{-1}(s') = \{s_1, s_2, \ldots, s_n\}$. For every transition $t \in T_1$ such that $\tau(t)$ is defined, the following equation holds:*

$$\sum_{i=1}^{n} A_1(s_i, t) = A_2(s', \tau(t)). \tag{1}$$

Proof. In order to prove the theorem, we need to compare the incidence matrices of N_1 and N_2. Let A_i, $i = 1, 2$, be the incidence matrices of, respectively, N_1 and N_2. Because of the structure of a marked graph, it is possible to say that every row of A_i contain one 1 or -1 value or both of them, while the remaining entries of that row contain 0 values. Let us now consider n distinct places $s_1, \ldots, s_n$ of N_1, such that $\sigma(s_i) = s'$, $1 \leq i \leq n$. For each $s_i \in \sigma^{-1}(s')$, if $|{}^\bullet s_i| = 1$ we denote t_{pre} the input transition of $|s_i|$ and, similarly, if $|s_i^\bullet| = 1$, we denote t_{post} the input transition of $|s_i|$. So, if such entries exist, $A_1(s_i, t_{pre}) = 1$ and $A_1(s_i, t_{post}) = -1$. For definition of $\hat{F}$-morphism, $A_2(s', \tau(t_{pre})) = 1$ and $A_2(s', \tau(t_{post})) = -1$. Furthermore, since we consider marked graphs without self-loops and σ defines an injective and surjective partial function between the pre-conditions of transitions joined by τ, for each $s_j \in \sigma^{-1}(s')$, $j \neq i$, we have $A_1(s_j, t_{pre}) = 0$ and $A_1(s_j, t_{post}) = 0$. This proof about one generic s' place of N_2 can be extended to all the places of N_2: so the theorem is proved.

The previous theorem allows us to introduce another interesting feature of $\hat{F}$-morphisms. Intuitively, if two marked graphs N_1 and N_2 are joined by a $\hat{F}$-morphism $(\sigma, \tau) : N_1 \to N_2$, the pre-images of any element of N_2 contain the same number n of elements.

Theorem 3. *For $i = 1, 2$, let $N_i = (S_i, T_i, F_i, M_0^i)$ be marked graphs and let $(\sigma, \tau) : N_1 \to N_2$ be a $\hat{F}$-morphism. Every $x \in P_2 \cup T_2$ has pre-image containing the same number n of elements.*

Proof. Let A_i, $i = 1, 2$, be the incidence matrices of, respectively, N_1 and N_2. For every place $s' \in S_2$, if $|\sigma^{-1}(s')| = n$, then it is possible to find n distinct columns $t_1, \ldots, t_n$ of A_1 such that $A_1(s_i, t_i) = 1$ or $A_1(s_i, t_i) = -1$, with $s_i \in \sigma^{-1}(s')$. Let t' be the input or output transition of p'; it is easy to verify that $\tau^{-1}(t') = \{t_1, \ldots, t_n\}$. This means that, if the pre-image of a place of N_2 contains n elements, the pre-images of its input and output transitions also contain n elements. We can extend this proof to every place of N_2, thus proving the theorem.

We call n the *reduction factor* of (σ, τ). The $\hat{F}$-morphism shown in Figure 2b has reduction factor 2, while the one in Figure 2c has reduction factor 3.

3.1 $\hat{F}$-morphisms and behavioral relationships

We now want to show the relationship between the behaviors of two marked graphs joined by a $\hat{F}$-morphism. In this paper we assume that the behavior of a system can be entirely described by means of its *unfolding*, according to the definition given in [9]. For this reason, from now on, we will only consider marked graphs with one technical restriction: in the initial marking there should not be more than one token on each place.

Marked graphs are used to model deterministic systems. The absence of choices in the behavior of deterministic systems can be used to observe that the unfolding of a marked graph does not contain conflicts. In [9] the unfolding of a net N is formally defined as a pair (N', π), where N' is an occurrence net and π is a homomorphism from N' to N. An occurrence net containing no conflicts is called causal net, which is an acyclic marked graph.

Let us now consider N-morphisms defined in [2] for elementary net systems, and compared to $\hat{F}$-morphisms in the previous subsection. Causal nets, used to represent the unfoldings of marked graphs, form a subclass of elementary net systems. This allows us to explicit the relationship between the behaviors of two marked graphs joined by a total $\hat{F}$-morphism.

Theorem 4. *For $i = 1, 2$, let $N_i = (S_i, T_i, F_i, M_0^i)$ be marked graphs joined by a $\hat{F}$-morphism $(\sigma, \tau) : N_1 \to N_2$ and let (N_1', π_1) and (N_2', π_2) be, respectively, the unfoldings of N_1 and N_2. Then, there exists a N-morphism $(\beta, \eta) : N_1' \to N_2'$ which makes the following diagram commute.*

$$\begin{tikzcd} N_1 \arrow[r, "\sigma,\tau"] & N_2 \\ N_1' \arrow[u, "\pi_1"'] \arrow[r, "\beta,\eta"] & N_2' \arrow[u, "\pi_2"'] \end{tikzcd}$$

In particular, β^{-1} is an injective partial function and, if (σ, τ) is total, (β, η) is an isomorphism.

The proof of this theorem can be found in [10], together with the necessary theoretical notions. Such proof uses an improved version of McMillan's unfolding algorithm (see [11]) with some modifications.

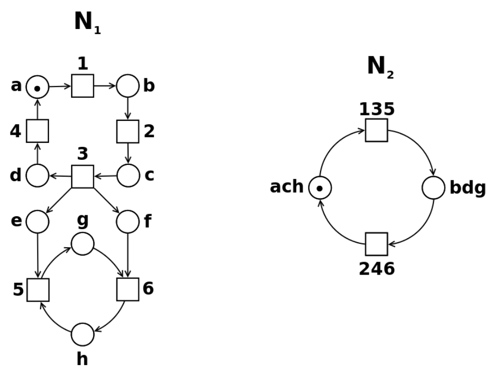
4 $\hat{F}$-morphisms and their properties

In this section we want to analyze some properties about liveness, boundedness, safeness, S and T-invariants of two marked graphs N_1 and N_2, joined by a $\hat{F}$-morphism $(\sigma, \tau) : N_1 \to N_2$. We will first analyze behavioral properties and then structural invariants.

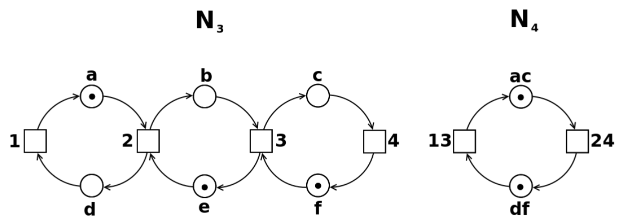
4.1 Analysis of behavioral properties

First of all, it is useful to observe that directed circuits are preserved by $\hat{F}$-morphisms. Intuitively, this means that, given two marked graphs N_1 and N_2 and a $\hat{F}$-morphism $(\sigma, \tau) : N_1 \to N_2$, if $\gamma = x_1 x_2 \dots x_k x_1$ is a directed circuit of N_1, $x_i \in S_1 \cup T_1$, (σ, τ) maps γ on a directed circuit of N_2.

In [7] marked graphs are defined as Petri nets $N = (S, T, F, M_0)$ in which, for each $s \in S$, it holds $|{}^\bullet s| = |s^\bullet| = 1$. Then, they prove that a marked graph N is live *iff* the initial marking places at least one token on each directed circuit in N. In this paper we consider a more general notion of marked graph: for each place s we have $|{}^\bullet s| \leq 1$ and $|s^\bullet| \leq 1$. It is well known (for example, see [7]) that, given a marked graph N such that $|{}^\bullet s| = 1$ for each place s, N is live if and only if the initial marking places at least one token on each directed circuit in N.



(a)



(b)

Fig. 3

The previous remarks allow to prove that $\hat{F}$-morphisms preserve liveness.

Theorem 5. *For $i = 1, 2$, let $N_i = (S_i, T_i, F_i, M_0^i)$ be two marked graphs joined by a $\hat{F}$-morphism $(\sigma, \tau) : N_1 \to N_2$. If N_1 is live, then N_2 is also live.*

Generally, liveness is not reflected by $\hat{F}$-morphisms. In Figure 3a an example of $\hat{F}$-morphism from N_1 to N_2 is shown. N_2 is a live net, while N_1 is not live: transitions labelled with `5` and `6` are never enabled.

Since we proved that there is a N-morphism between the unfoldings of two marked graphs joined by a $\hat{F}$-morphism, it is easy to observe that $\hat{F}$-morphisms also preserve occurrence sequences.

Theorem 6. *Let $N_i = (S_i, T_i, F_i, M_0^i)$, $i = 1, 2$, be two marked graphs joined by a $\hat{F}$-morphism $(\sigma, \tau) : N_1 \to N_2$. Let $\omega = t_1 \dots t_k$, be an occurrence sequence of N_1 enabled at the initial marking M_0^1. Therefore $\omega' = \tau(t_1) \dots \tau(t_2)$ is an occurrence sequence of N_2 enabled at M_0^2.*

From the definition of $\hat{F}$-morphism, it follows immediately that, if two marked graphs N_1 and N_2 are joined by a $\hat{F}$-morphism $(\sigma, \tau) : N_1 \to N_2$, for each place s of N_2, the sum of the number of tokens placed by the initial marking of N_1 in the elements of the pre-image of s is equal to the number of tokens placed by the initial marking of N_2 in s. It is possible to extend this condition to every reachable marking of the two systems.

Theorem 7. *For $i = 1, 2$, let $N_i = (S_i, T_i, F_i, M_0^i)$ be two marked graphs joined by a $\hat{F}$-morphism $(\sigma, \tau) : N_1 \to N_2$. Let $\omega = t_1 \dots t_k$ be an occurrence sequence of N_1 enabled at M_0^1 such that $M_0^1 \xrightarrow{\omega} M$. Then, $\omega' = \tau(t_1) \dots \tau(t_k)$ is an occurrence sequence of N_2 enabled at M_0^2 such that $M_0^2 \xrightarrow{\omega'} M'$ and, for each $s' \in S_2$, the following equation holds*

$$M'(s') = \sum_{s \in \sigma^{-1}(s')} M(s).$$

Using Theorem 7 it is easy to prove that boundedness is preserved by $\hat{F}$-morphisms.

Theorem 8. *For $i = 1, 2$, let $N_i = (S_i, T_i, F_i, M_0^i)$ be two marked graphs joined by a $\hat{F}$-morphism $(\sigma, \tau) : N_1 \to N_2$. If N_1 is bounded, then N_2 is also bounded.*

So $\hat{F}$-morphisms preserve boundedness but, generally, they do not reflect it. The $\hat{F}$-morphism from N_1 to N_2 represented in Figure 3a does not preserve boundedness: N_2 is a 1-bounded net, while in N_1 the places labelled with `e` and `f` can be filled with an infinite number of tokens.

Note that the reflection of boundedness is obtained if (σ, τ) is total.

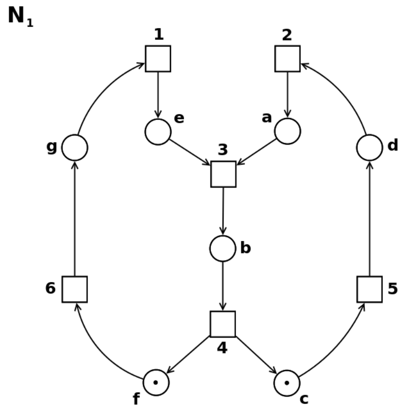
Theorem 9. *For $i = 1, 2$, let $N_i = (S_i, T_i, F_i, M_0^i)$ be two marked graphs joined by a $\hat{F}$-morphism $(\sigma, \tau) : N_1 \to N_2$ such that σ is total. If N_2 is bounded, then N_1 is also bounded.*

Proof. Each place of N_1 is mapped on a place of N_2. If N_2 is bounded, by theorem 7 it is easy to see that N_1 is also bounded.

Notice that, in general, safeness (1-boundedness) is not preserved. Let us consider the example shown in Figure 3b: there is a $\hat{F}$-morphism from N_3 to N_4 and, while N_1 is a safe net, N_2 is 2-bounded.

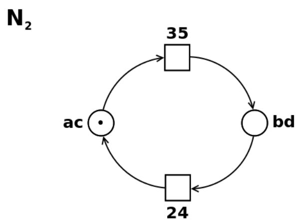
4.2 On structural invariants

We now focus on some properties about S and T-invariants of two marked graphs N_1 and N_2 joined by a $\hat{F}$-morphism $(\sigma, \tau) : N_1 \to N_2$. It is possible to prove that $\hat{F}$-morphisms reflect S-invariants. In order to obtain such result, we need to order the rows of the incidence matrix A_1 of N_1 in the following way. Let A_2 be the incidence matrix of N_2 and let n be the reduction factor of (σ). Given the first row of A_2, representing the place s of N_2, let us consider the n rows of A_1 corresponding to places of N_1 mapped by σ on s. We will put such rows in the first n positions of the matrix. The same procedure can be used to order the remaining rows of A_1. The rows corresponding to places not mapped by σ will occupy the last positions of A_1.



A_1

	1	2	3	4	5	6
a	0	1	-1	0	0	0
c	0	0	0	1	-1	0
b	0	0	1	-1	0	0
d	0	-1	0	0	1	0
e	1	0	-1	0	0	0
f	0	0	0	1	0	-1
g	-1	0	0	0	0	1



A_2

	35	24
ac	-1	1
bd	1	-1

Fig. 4

Theorem 10. *For $i = 1, 2$, let $N_i = (S_i, T_i, F_i, M_0^i)$ be two marked graphs joined by a $\hat{F}$-morphism $(\sigma, \tau) : N_1 \to N_2$. Let A_1, A_2 and n be, respectively, the incidence matrices of N_1 and N_2, ordered as seen before, and the reduction factor of (σ, τ). If $\underline{I}_2 = (\alpha_1 \alpha_2 \dots \alpha_P)$, with $\alpha_j \in \mathbb{N}$ and $P = |S_2|$, is a S-invariant for*

N_2, then

$$\underline{I}_1 = (\overbrace{\alpha_1\alpha_1 \dots \alpha_1}^{n\ times}\overbrace{\alpha_2\alpha_2 \dots \alpha_2}^{n\ times}\ \dots\ \overbrace{\alpha_P\alpha_P \dots \alpha_P}^{n\ times} 0 \dots 0)$$

is a S-invariant for N_1.

The previous theorem is proved in [10]. Let us now consider the $\hat{F}$-morphism $(\sigma, \tau) : N_1 \to N_2$ shown in Figure 4, having reduction factor $n = 2$. The incidence matrix of N_1 is ordered as explained. $\underline{I}_2 = (11)$ is a S-invariant for N_2. The corresponding S-invariant for N_1 is built by taking n times each single value of $\underline{I}_2$ as the first components and adding 0s in the remaining positions. Thus, we obtain $\underline{I}_1 = (1111000)$.

$\hat{F}$-morphisms reflect S-invariants but do not preserve them. The S-invariant $\underline{I}_A = (0100111)$ for N_1 in Figure 4 can not be used to build a corresponding S-invariant for N_2. It is impossible to assign to each place of N_2 the weight of the elements of its pre-image. For example, let s be the place of N_2 labelled with `bd`: $\underline{I}_A$ assigns a different weights to the elements of $\sigma^{-1}(s)$. $\underline{I}_B$, built by assigning to each place of N_2 the sum of the weights of the elements of its pre-image, is not a S-invariant of N_2.

Regarding T-invariants, we observe that, in marked graphs, an occurrence sequence leads back to the initial marking if and only if it fires every transition an equal number of times. Then, since $\hat{F}$-morphisms are surjective, by Theorem 7 they preserve T-invariants.

In general, T-invariants are not reflected by $\hat{F}$-morphisms. For instance, let us consider the example in Figure 4. $\underline{J}_2^T = (11)$ is a T-invariant for N_2. For each transition t of N_2, we assign to the elements of its pre-image the weight given by $\underline{J}_2^T$ to t, and we use 0s for the other transitions of N_1. So, we obtain $\underline{J}_1^T = (011110)$, which is not a T-invariant for N_1.

5 Remarks and conclusions

We have introduced F- and $\hat{F}$-morphisms, new kinds of morphisms on marked graphs, a basic class of Petri nets. These morphisms can be used as a formal technique to deal with a kind of abstraction on marked graphs, consisting in the folding of cycles and the identification of chains. We have also proved that the unfoldings of two systems joined by a $\hat{F}$-morphism are joined by a N-morphism (see [2]). We have finally shown that liveness, boundedness and T-invariants are preserved by such morphisms, while S-invariants are reflected.

We now plan to define a new operation for the composition of marked graphs driven by $\hat{F}$-morphisms mapping the components on a net which works as an interface, similarly to what described in [12], [13] for $\hat{N}$-morphisms. We also intend to extend the theory related to F-morphisms to other classes of Petri nets, such as persistent, free choice and Place/Transition Petri nets, thus applying such functions to systems having conflicts. Finally, we want to apply $\hat{F}$-morphisms to models representing real systems having deterministic behavior (such as, for example, manufacturing systems or cyclic processes) to formally analyze them by using a step-by-step approach based on different levels of refinement of the modelled system.

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