Optimization of Backward Fuzzy Reasoning Based on Rule Knowledge

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Abstract. In [14], we have presented a fuzzy forward reasoning methodology for rule-based systems using the functional representation of rules (fuzzy implications). In this paper, we extend methodology for selecting relevant fuzzy implications from [14] in backward reasoning. The proposed methodology takes full advantage of the functional representation of fuzzy implications and the algebraic properties of the family of all fuzzy implications. It allows to compare two fuzzy implications. If the truth value of the conclusion and the truth value of the implication are given, we can easily optimize the truth value of the implication premise. This methodology can be useful for the design of an inference engine based on the rule knowledge for a given rule-based system.

Key words: fuzzy implication, knowledge representation, backward reasoning, rule-based system

1 Introduction

Recently we can observe further growth of an interest in the design and exploitation of rule-based systems built on the basis of uncertain knowledge. Various methods of knowledge representation and reasoning have already been proposed. One of the most popular approaches to knowledge representation are the fuzzy production rules. However, reasoning is mainly classified into two types: forward reasoning and backward reasoning. The inference mechanism of forward reasoning is based on a data-derived way, and has a powerful prediction ability. It is capable of warning against latent hazards, forthcoming accidents, and faults. By contrast, backward reasoning is based on a goal-derived manner, it has explicit objectives, which are generally used to search for the most possible causes related to an existing fact. Backward reasoning plays an essential role in fault diagnosis, accident analysis, and defect detection.

In this paper, we mainly focus on backward reasoning based on the fuzzy rules. They can be presented in the form of IF-THEN and interpreted as fuzzy implications [1]. There exist uncountably many implication functions in the field of fuzzy logic, and the nature of the fuzzy inference changes variously depending on the implication function to be used. The variety of implication functions existing in the fuzzy set framework has always been seen as a rich potential for modeling different shades of expert attitude in the inference process (e.g. [7]), although no precise, practical interpretation was provided for the different implication functions [10]. Moreover, it is very difficult to select a suitable implication function for actual applications.

From over eight decades a number of different fuzzy implications have been proposed [2],[4]-[6],[8]-[9],[11]-[12],[17]-[18]. In the family of basic fuzzy implications the partial order induced from [0,1] interval exists. Pairs of incomparable fuzzy implications can generate new fuzzy implications by using min(inf) and max(sup) operations. As a result the structure of lattice is created ([1], page 186). This leads to the following question: how to choose the relevant functions among basic fuzzy implications and other generated as described above.

In [14], we have presented a fuzzy forward reasoning methodology for rule-based systems using the functional representation of rules (fuzzy implications). In this paper, we extend a methodology for selecting relevant fuzzy implications from [14] in backward reasoning. The proposed methodology takes full advantage of the functional representation of fuzzy implications and the algebraic properties of the family of all fuzzy implications. It allows to compare two fuzzy implications. If the truth value of the conclusion and the truth value of the implication are given, we can easily optimize the truth value of the implication premise. This general methodology is considered in details in [13]. It can be useful for the design of an inference engine based on the rule knowledge for a given rule-based system. Using the proposed approach, we can reduce the efforts related to a selection of a suitable implication function.

The rest of this paper is organized as follows. In Sect. 2, we briefly recall some definitions related to partially ordered sets, the fuzzy production rules, fuzzy implications and basic algebraic properties of fuzzy implications. The research problem considered in the paper is formulated in Sect. 3. Sect. 4 presents the main theorem together with its proof concerning a selection of suitable implication function. Sect. 5 presents two algorithms solving the given research problem. The first algorithm allows to select the suitable implication function based on information concerning a given set of fuzzy implications, their truth-values, and the truth value of conclusion. The second algorithm allows to select the "optimal" fuzzy implication using the same information as for the first one. In Sect. 6, we present an example illustrating these algorithms in the use. Sect. 7 includes the summary of our research and some remarks.

2 Basic Notions and Definitions

2.1 Partially Ordered Sets

Let R be a binary relation on a set A. A relation R on A is said to be a *partial ordering on A* if it is reflexive, transitive and antisymmetric. A partial ordering R on A is said to be a *linear ordering on A* if at least one of the following conditions: $(x, y) \in R$, $(y, x) \in R$ or $x = y$ holds for any $x, y \in A$. If R is a partial ordering on A, then the pair $U = (A, R)$ is said to be a *partially ordered set* (abbreviated poset). If R is a linear ordering on A, then the pair $U = (A, R)$ is said to be a *linearly ordered set*.

Let $U = (A, R)$ be a poset, and $X \subseteq A$. The element $a_0 \in A$ is said to be the *upper (lower) bound in U* of a subset $X \subseteq A$ if $(x, a_0) \in R$ $((a_0, x) \in R)$ for all $x \in X$.

The upper (lower) bound in U of A is the *greatest* (*least*) element in U. An element $a \in A$ is said to be *maximal* (*minimal*) in U if $(a, x) \in U$ (respectively $(x, a) \in R$) implies $x = a$. It is clear that the greatest (least) element is maximal (minimal), and if R is a linear ordering, then the element maximal (minimal) in U is also the greatest (least) in U. It is obvious that if the greatest (least) element in U exists, then all the maximal (minimal) elements are equal. If B is a set of upper bounds in $U = (A, R)$ of a set $A_1 \subseteq A$, then the least element in $(B, R \cap B^2)$ is said to be the *least upper bound* in U of the set A_1 and is denoting sup(A_1, U). Replacing in the preceding definition "upper" and "least" respectively by "lower" and "greatest" we obtain the definition of the *greatest lower bound* of A_1 in U which will be denoted by inf(A_1, U). It is clear that sup(A_1, U) and inf(A_1, U) are uniquely determined by A_1 and U if they exist. A poset U is said to be a *lattice* if for any $a, b \in A$ in U there are sup$(\{a, b\}, U)$ and inf$(\{a, b\}, U)$. If $R \cap X^2$ is a linear ordering on X, then X is said to be a *chain in U*.

For more detailed information about partially ordered sets the reader is referred to [3].

2.2 Fuzzy Production Rules and Fuzzy Implications

Let R be a set of fuzzy production rules, $R = \{r_1, r_2, ..., r_n\}$. The general formulation of the $i-$th fuzzy production rule is as follows:

$$r_i : \text{IF } d_j \text{ THEN } d_k \text{ (CF=}z_i\text{)}$$

where: (1) d_j and d_k are statements; the truth degree of each statement is a real value between zero and one. (2) z_i is the value of the certainty factor (CF), $z_i \in [0, 1]$. The larger the value of z_i, the more the rule is believed in.

We can use a fuzzy implication model [1] to represent the fuzzy production rules of a rule-based system.

Fuzzy implications are one of the main operations in fuzzy logic [1]. Now we recall a definition of a fuzzy implication and some of its properties that will be used in the paper.

A function $I : [0, 1]^2 \to [0, 1]$ is said to be a *fuzzy implication* if it satisfies, for all $x, x_1, x_2, y, y_1, y_2 \in [0, 1]$, the following conditions:

1. $I(., y)$ is decreasing (i.e., if $x_1 \leq x_2$, then $I(x_1, y) \geq I(x_2, y)$).
2. $I(x, .)$ is increasing (i.e., if $y_1 \leq y_2$, then $I(x, y_1) \leq I(x, y_2)$).
3. $I(0, 0) = 1$, $I(1, 1) = 1$, and $I(1, 0) = 0$.

The family of all fuzzy implications will be denoted by *FI*.

Remark 1. Let us observe that each fuzzy implication I is constant for $x = 0$ and for $y = 1$ (i.e., I fulfils the following conditions, respectively: (1) $I(0, y) = 1$ for $y \in [0, 1]$, (2) $I(x, 1) = 1$ for $x \in [0, 1]$).

If, for two fuzzy implications I_1 and I_2, the inequality $I_1(x, y) \leq I_2(x, y)$ holds for all $(x, y) \in [0, 1]^2$, then we say that I_1 is less than or equal to I_2 and we write $I_1 \leq I_2$. We shall write $I_1 < I_2$ whenever $I_1 \leq I_2$ and $I_1 \neq I_2$, i.e., if $I_1 \leq I_2$ and for some $(x_0, y_0) \in [0, 1]^2$ we have $I_1(x_0, y_0) < I_2(x_0, y_0)$. In this case we also say that I_1 is comparable with I_2. Moreover, if, for two fuzzy implications I_1 and I_2, the inequality $I_1(x, y) < I_2(x, y)$ holds for all $(x, y) \in D \subset [0, 1]^2$, then we say that I_1 is less than I_2 and we write $I_1 \prec I_2$.

Example 1. Since there exist uncountably many fuzzy implications, we list below only a few of basic fuzzy implications known from the subject literature. Figures 1 and 2 illustrate the plots of I_{LK}, I_{RC}, I_{KD} and I_{YG} implications, respectively.

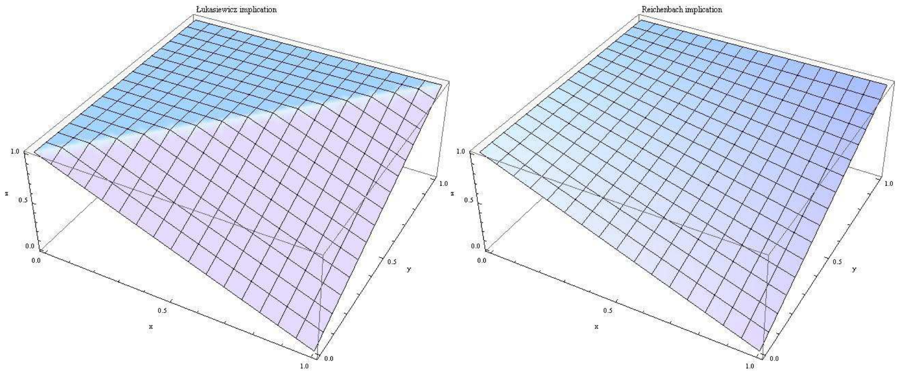

Fig. 1. Plots of I_{LK} and I_{RC} fuzzy implications

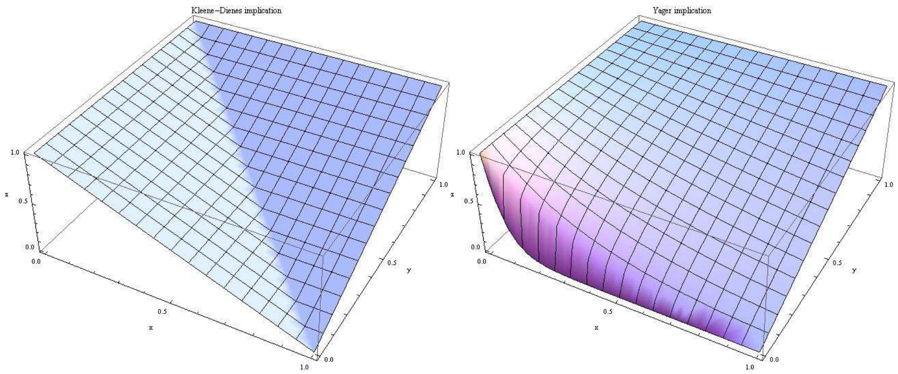

Fig. 2. Plots of I_{KD} and I_{YG} fuzzy implications

1. $I_{LK}(x, y) = min(1, 1 - x + y)$ (the Łukasiewicz implication) [9];
2. $I_{GD}(x, y) = 1$, if $x \leq y$, and $I_{GD}(x, y) = y$ otherwise (the Gődel implication) [5];
3. $I_{RC}(x, y) = 1 - x + xy$ (the Reichenbach implication) [11];
4. $I_{KD}(x, y) = max(1 - x, y)$ (the Kleene-Dienes implication) [2],[8];
5. $I_{GG}(x, y) = 1$, if $x \leq y$, and $I_{GG}(x, y) = \frac{y}{x}$ otherwise (the Goguen implication) [6];

6. $I_{RS}(x, y) = 1$, if $x \leq y$, and $I_{RS}(x, y) = 0$ otherwise (the Rescher implication) [12];
7. $I_{WB}(x, y) = 1$, if $x < 1$, and $I_{WB}(x, y) = y$, if $x = 1$ (the Weber implication) [17];
8. $I_{FD}(x, y) = 1$, if $x \leq y$, and $I_{FD}(x, y) = max(1 - x, y)$ otherwise (the Fodor implication) [4];
9. $I_{YG}(x, y) = 1$, if $x = 0$ and $y = 0$, and $I_{YG}(x, y) = y^x$, if $x > 0$ or $y > 0$ (the Yager implication) [18].

Example 2. Let A be the basic fuzzy implications from Example 1, and R be the relation $<$. It is easy to check that the pair $U = (A, R)$ is a poset. A graphical representation of five chains: $C_1 = \{I_{KD}, I_{RC}, I_{LK}, I_{WB}\}$, $C_2 = \{I_{RS}, I_{GD}, I_{GG}, I_{LK}, I_{WB}\}$, $C_3 = \{I_{YG}, I_{RC}, I_{LK}, I_{WB}\}$, $C_4 = \{I_{KD}, I_{FD}, I_{LK}, I_{WB}\}$, $C_5 = \{I_{RS}, I_{GD}, I_{FD}, I_{LK}, I_{WB}\}$ in U is shown in Figure 3.

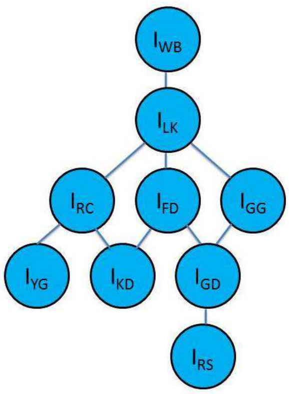

Fig. 3. A graphical representation of the chains from Example 2

Remark 3. It is also worth to point out that incomparable pairs of fuzzy implications generate new fuzzy implications by using the standard min and max operations. In particular, incomparable pairs of basic implications from Example 1 generate new implications in the lattice of fuzzy implications. Elements obtained in such way can be combined with other implications, which leads to the new fuzzy implications forming the lattice of fuzzy implications. This issue will not be dealt with here, and we will refer the reader to ([1], page 186).

We can use fuzzy implications to represent the fuzzy production rules of a rule-based system. For example, the following fuzzy production rule r_i : IF d_j THEN d_k (CF=z_i) can be interpreted as a fuzzy implication $z = I(x, y)$, where values for z, x, y correspond to CF, the truth degree of a statement d_j (premise), and the truth degree of a statement d_k (conclusion), respectively. The value of z_i is given by a domain expert.

However, the value for x (or y) is given by the user of a rule-based system dependently on a selected reasoning method (forward or backward, respectively).

3 Problem Statement

Let us consider a lattice $(FI, <)$ ([1], page 183), where FI is the family of all fuzzy implications and $<$ is the inequality relation between fuzzy implications from FI induced in the standard way from the unit interval [0,1] (see Sect. 2). Let U be a finite subset of FI.

Our goal is to elaborate on two algorithms which using information on a value of an argument y of a given fuzzy implication J from U and a truth-value of the implication J find in the set U form:

1. a "worse" fuzzy implication I than J (if there exists) such that: $I(x_1, y) = J(x_2, y)$ for the given argument y and some arguments $x_1, x_2 \in [0, 1]$, and $x_1 < x_2$, i.e., a fuzzy implication I with the strictly less value of the argument x_1 than it is possible to compute using the implication J;
2. an "optimal" (minimal) fuzzy implication I_{opt} (if there exists), i.e., a fuzzy implication that fulfils the following two requirements:
   - $I_{opt}(x_1, y) = J(x, y)$,
   - x_1 is the least value among all values x' possible to obtain using any fuzzy implication K comparable with J belonging to the set U and satisfying the condition: $K(x', y) = J(x, y)$.

4 Theorem

Now we are ready to formulate and prove a theorem which suggests how to select from a given finite set of fuzzy implications U the suitable implication function for a given fuzzy implication J in order to obtain a less truth-value of its premise x in reasoning taking into account information on the truth-value $J(x, y)$ of this implication and the truth-value of its conclusion y.

Theorem. Let I and J be fuzzy implications such that $I \prec J$ on a set $D \subset [0, 1]^2$, and $x_1, x_2, y \in [0, 1]$ such that $I(x_1, y) = J(x_2, y)$. Then $x_1 < x_2$.

Proof: Proof by contradiction. Suppose $x_1 \geq x_2$. Then from the definition of fuzzy implication (see item 1) it follows that $I(x_1, y) \leq I(x_2, y)$. From that and from the equality $I(x_1, y) = J(x_2, y)$ it follows that $I(x_2, y) \geq J(x_2, y)$. Since $I \prec J$, $I(x_2, y) < J(x_2, y)$. Thus, we have reached a contradiction. Therefore, we conclude that the theorem is correct.

Remark 4. The analogous theorem, but for forward fuzzy reasoning has been presented in [14]. Moreover, the detailed considerations related to a set D (the domain) for particular basic fuzzy implications used in forward/backward fuzzy reasoning are presented in [15] and [13], respectively.

As a simple consequence of the above theorem is the following fact.

Conclusion. The above theorem is false for $y = 1$.

Proof: From the property of a fuzzy implication presented in Remark 1 (item 1) we have $I(x, 1) = 1$ for any $x \in [0, 1]$. It means that for any two fuzzy implications I and J the following double dependency $I(x_1, 1) = J(x_2, 1) = 1$ is true for any $x_1, x_2 \in [0, 1]$. Hence, we get that this equality is true not only for $x_1 < x_2$.

5 Algorithms

In this section, we present two algorithms formulated on the basis of the theorem from Sect. 4. The first algorithm allows to select the suitable (worse) implication function (see the condition 1, Sect. 3) based on information concerning a given set of fuzzy implications, their truth-values, and the truth value of conclusion. The second one allows to select the optimal fuzzy implication (see the condition 2, Sect. 3) using the same information as for the first algorithm.

Let $(FI, <)$ be a lattice of all fuzzy implications, a finite set $U \subset FI$, and $J \in U$.

Algorithm 1 finding a worse implication $I \in U$ (in the sense of the condition 1, Sect. 3).
Input: U - a given finite set of fuzzy implications, $J \in U$, $y \in [0, 1)$, and $k \in [0, 1]$ - a truth-value of J.
Output: A worse implication $I \in U$ than J.

1. Order the set U with respect to the relation $<$.
2. Identify the implication $J \in U$.
3. **if** there exists an implication $I \in U$ such that $I \prec J$
   then
   Compute a value x_1 from the dependency $I(x_1, y) = k$.
   Return x_1.
   else Stop.

Remark 5. The correctness of the Algorithm 1 follows immediately from the theorem presented in Sect. 4.

Example 3. Consider a set of fuzzy implications $U = \{I_{LK}, I_{RC}, I_{KD}, I_{WB}\}$, the Łukasiewicz implication I_{LK}, a given argument $y = a$ $(a < 1)$, and the truth-value of $I_{LK} = b$ $(b > a)$. After executing the first step of the Algorithm 1 we obtain only one maximal chain c: $I_{KD} < I_{RC} < I_{LK} < I_{WB}$ (see Example 2, item 1). Let us observe that the Łukasiewicz implication I_{LK} belongs to the chain c. Moreover, it is easy to verify that there are two other implications less than I_{LK} with respect to the relation $<$ in this chain, i.e., the Reichenbach implication I_{RC} and the Kleene-Dienes implication I_{KD}. If, for example, we select the implication I_{RC}, then from the dependency $I_{RC}(x_1, a) = b$ we can compute a value $x_1 = \frac{b-1}{a-1}$. Whereas a value x computed for the dependency $I_{LK}(x, a) = b$ equals $a - b + 1$. It is easy to see that $x_1 < x$.

Algorithm 2 finding an optimal implication in U (in the sense of the condition 2, Sect. 3).

Input: U - a given finite set of fuzzy implications, $J \in U$, $y \in [0, 1)$, and $k \in [0, 1]$ - a truth-value of J.
Output: An optimal implication $I_{opt} \in U$ and a value x_{opt}.

1. Order the set U with respect to the relation $<$.
2. Compute a set C of all maximal chains in U such that J belongs to each of them.
3. **for** each chain $c \in C$ **do** find (if there exists) the least implication $I_c \prec J$.
   for each implication I_c **do** compute a value x_c (if there exists) from the dependency $I_c(x_c, y) = k$.
4. Compute a value x_{opt} = min$\{x_c : c \in C\}$.
5. Return (I_{opt}, x_{opt}).

Remark 6. The correctness of the Algorithm 2 follows from the theorem (see Sect. 4) and the finiteness of set U.

Example 4. Now consider a set of fuzzy implications $U' = \{I_{LK}, I_{RC}, I_{KD}, I_{WB}, I_{YG}\}$, the Łukasiewicz implication I_{LK}, a given argument $y = a$ $(a < 1)$, and the truth-value of $I_{LK} = b$ $(b > a)$. After executing the steps 1 and 2 of the Algorithm 2 we obtain two maximal chains as follows: $c_1 = I_{KD} < I_{RC} < I_{LK} < I_{WB}$ and $c_2 = I_{YG} < I_{RC} < I_{LK} < I_{WB}$ (see Example 2, items 1 and 3). We can identify the Łukasiewicz implication in these two chains. Moreover, it is easy to check that I_{KD} is the least implication in the chain c_1 with respect to the relation $\prec$, while I_{YG} is the least implication in the chain c_2. Next, solving the equations $I_{KD}(x_{c_1}, a) = b$ and $I_{YG}(x_{c_2}, a) = b$, we obtain $x_{c_1} = 1 - b$ for $b > a$, and $x_{c_2} = log_a b$ for $0 < a < b < 1$. Hence, we have $I_{opt} = I_{YD}$ and $x_{opt} = x_{c_2}$.

6 Illustrating Example

In order to illustrate our methodology, let us describe a simple example coming from the domain of train traffic control. We consider the following situation: a train B waits at a certain station for a train A to arrive in order to allow some passengers to change train A to train B. Now, a conflict arises when the train A is late. In this situation, the following alternatives can be taken into account:

- train B departs in time, and an additional train is employed for the train A passengers;
- train B departs in time. In this case, passengers disembarking train A have to wait for a later train;
- train B waits for train A to arrive. In this case, train B will depart with delay.

In order to describe the traffic conflict, we propose to consider the following four IF-THEN fuzzy rules:

- r_1: IF s_2 THEN s_6 (CF = 0.6)
- r_2: IF s_3 THEN s_6 (CF = 0.6)
- r_3: IF s_1 AND s_4 AND s_6 THEN s_7 (CF = 0.5)
- r_4: IF s_4 AND s_5 THEN s_8 (CF = 0.8)

where:

- s_1: 'Train B is the last train in this direction today',
- s_2: 'The delay of train A is huge',
- s_3: 'There is an urgent need for the track of train B',
- s_4: 'Many passengers would like to change for train B',
- s_5: 'The delay of train A is short',
- s_6: '(Let) train B depart according to schedule',
- s_7: 'Employ an additional train C (in the same direction as train B)',
- s_8: 'Let train B wait for train A'.

In the further considerations we accept the following assumptions:

- the logical operator AND we interpret as min fuzzy operator;
- to the statements s_7 and s_8 we assign the fuzzy values 0.6 and 0.4, respectively;
- each of rules r_1, r_2, r_3, and r_4 we interpret firstly as the Łukasiewicz implication;
- the truth degrees of rules r_1, r_2, r_3, and r_4 are equal to 0.6, 0.6, 0.5, 0.8, respectively.

Assume that the user wants, for example, to know for which the truth degree of statements s_4 and s_5 the truth degree of the statement s_8 (i.e., the conclusion of the rule r_4) is equal to 0.6. Observe that in this situation the rule r_4 can be considered. Taking into account the dependency $I_{LK}(x, a) = b$ from Example 3 with $a = 0.4$ (the truth degree of the statement s_8) and $b = 0.8$ (the truth degree of the rule r_4) we get the truth degree of statements s_4 and s_5 equal to $x = a - b + 1 = 0.6$. However, if we interpret these four rules as the Reichenbach implications ($I_{RC}(x_1, a) = b$), and if we choose the same rule as above we obtain the truth degree of the statements s_4 and s_5 equal to $x_1 = \frac{b-1}{a-1} \simeq 0.33$. At last, if we execute the similar simulation of backward fuzzy reasoning for the rule r_4 considered above and, if we interpret these rules as the Kleene-Dienes implications we obtain the truth degree of the statements s_4 and s_5 equal to 0.2. Hence, we have $I_{opt} = I_{KD}$ for considered three interpretations of the rule r_4, and $x_{opt} = 0.2$. In analogous way one can analyze the situation in which the user wants to know the truth degree of the statements s_1, s_2, s_3, s_6 knowing the truth degree of the statement s_7.

This example shows clearly that different interpretations for the rules may lead to quite different truth degree of starting statements (corresponding to premises of given production rules). Choosing a suitable interpretation for fuzzy implications we may apply the theorem and the two algorithms presented in Sects. 4 and 5, respectively. The rest in this case certainly depends on the experience of the decision support system designer to a significant degree.

7 Concluding Remarks

In the paper, we have presented a methodology for selecting relevant fuzzy implication in backward reasoning, which has for example the least truth value of the premise when the truth value of the conclusion and the truth value of the implication are given. This methodology takes full advantage of the functional representation of fuzzy implications and the algebraic properties of the family of all fuzzy implications.

We know that there are a lot of implication functions in the field of fuzzy logic, and the nature of the inference changes variously depending on the implication function to be used. However, it is very difficult to select a suitable implication function for actual applications. But taking into account the methodology proposed in this paper we can reduce the efforts related to a selection of a suitable implication function.

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