On Formulation and Software Implementation of Some Financial Management and Barter Transactions Problems

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Abstract. The article presents some mathematical problems, reducible to the classical search problem of minimum cost circulation in the network. We also provide information about the software implementation of the problems on computers of the latest generation. Processing time on data with 100,000 customers and 1,000,000 transactions did not exceed 1.5 minutes.

Keywords: minimum cost network circulation, polynomial algorithm, reduction, software implementation

1 Introduction

The high level of mutual non-payments on debt obligations was one of the most serious problems that companies encountered throughout Russia and other CIS countries in the 90-ies years of the last century. A huge number of companies were involved into complex schemes of credit-debt relationship. The majority of enterprises could hardly been considered bankrupt, as long as the amount of debt that they were due matched to the amount of debt they were attributable from other counterparties. Companies were unable to pay off their debts until they received respective payments from their debtors. In practice, a considerable amount of settlements were made through barter transactions and transactions which involved various types of securities and their derivatives. However, such method of payment provide significantly less flexibility, and lack of centralized supply and distribution systems as well as undeveloped securities market created serious difficulties in management of supply and procurement for a standalone enterprise. Every company knows some their potential suppliers of the feedstock and equipment as well as some direct purchasers of their end-products. However, such information is not necessarily sufficient to organize suitable barter transactions in the face

of lack of working capital. One of the possible ways for an enterprise to resolve such situation is to use complex barter schemes based on a comprehensive analysis of resource requirements and the corresponding aggregate production capacity of enterprises.

Note that the specific problem remains quite topical at the moment, at least in a number of former Soviet republics (Ukraine, Belarus, and others).

This article focuses on the use of mathematical models and methods for solving research and operational problems that arise in the analysis of so-called institutional traps: barter, non-payments, tax evasion and corruption, leading to inefficient forms of organization of single companies and the financial system as a whole.

These problems are connected with the problems of mutual debtness of business on one hand, as well as with performing appropriate barter transactions on the other side. These problems include:

• the problem of maximization of repayment of debts by their directed-*redistribution* (taking into account importance of a piece of debt);

• the problem of the maximization of non-monetary repayment of debts by *circuit* offset method;

• the problem of optimal use of *credit and financial* resources for the implementation of the largest possible matching. In addition to bounding the debt limit to each counterparty, the aggregate amount of debt for the company can be set. Direct limit-purpose lending of individual enterprises is allowed.

• the problem to find appropriate *barter* set of transactions under which companies can obtain the required resources for the production process (within enterprise's demand) and realise entire output (within enterprise's capacity).

• the problem of maximization of debt offsets by utilising appropriate *credit and barter transactions* (also taking into account transportation costs).

• optimisation of mutual offset and barter transactions, which account for financial impact of the implementation of each specific operation and its priority (which could be important for example for tax purposes).

Significant relevance of listed above fiscal infrastructure problems inspired a number of research papers focused on obtaining adequate mathematical models of the mutual offset and barter transactions, so as led to development of effective computational algorithms for problem solving. We would note papers [5]–[7] which consider problem setting which offsets mutual debts of enterprises, where functional is set as a sum of squares of debts of enterprises remaining indebted after redistribution. Described choice of the objective function seems to be artificial because it does not express the merits and could be explained by the authors bias towards the method of Lagrange multipliers. The papers [8], [9] deal with a more natural functional equal to the sum of debts of all businesses. From a mathematical point of view, the abovementioned problem could be trivially reduced to the problem to find an arbitrary feasible solution of the transport problem (ie, the cost of transportation is not included). Methods proposed in [8], [9] to resolve the derived transportation problem are already well known to the operations research society for a long time. For example, algorithm used in [8] basicly is a variation of the method of "North-Western" corner. In [9] redistribution of debt payments is made in proportion to the aggregate balance sheets of enterprises. At the

same time consideration of less trivial problems by staging proposed in [8], [9] methods do not provide a exact solution.

On the basis of works [1] – [3], which explored reduction of problems listed above to find circulation of minimum cost, efficient (polynomial time complexity) software was developed. Abovementioned software allows one to find the exact solution of the problems listed above as well as to do efficient analysis and if needed manage complex financial and commodity flows between enterprises at national, regional and enterprise levels. In these papers authors have proposed mathematical model of debt repayment, based on the reduction to the minimum cost circulation problem.

2 Mimimum-cost circulations problem (MCCP)

The papers [1] – [3] used a model of the Mimimum-cost circulations problem (MCCP) in the network.

Let $G = (V, U)$ — a directed graph with vertex set A and the set of arcs $U \subset V \times V$.

For any function w, defined on the set of arcs, we denote by $\mathrm{div}_w(x)$ divergence function w in the vertex $x \in V$, is given by:

$$\mathrm{div}_w(x) = \sum_{u \in A(x)} w(u) - \sum_{u \in B(x)} w(u) \ ,$$

where $A(x), B(x)$ — sets of edges in the graph G, that go from a vertex x out, and enter into a vertex x, respectively.

The function f, defined on the arcs of the graph $G = (V, U)$, called the circulation flow, or *circulation*, if $\mathrm{div}_f(v) = 0$ for every $v \in V$.

The mimimum-cost circulation problem (MCCP) on the graph $G = (V, U)$ can be set as follows:

$$\text{minimize} \sum_{u \in U} c(u)f(u) \tag{1}$$

subject to:

$$\mathrm{div}_f(v) = 0 \ , \ v \in V \ , \tag{2}$$

$$\check{a}(u) \leq f(u) \leq \hat{a}(u), \ u \in U \ , \tag{3}$$

where $c(u)$ — the price of a single passage of the flow along the arc u; $\check{a}(u)$ and $\hat{a}(u)$ — lower and upper bounds on the quantity of flow on the arc $u \in U$.

This MCCP form was formulated back by Ford and Fulkerson [4].

Extensive material on the characteristics of the implementation of effective methods MCCP solutions and comparative analysis of different algorithms, contained, for example, in [10] – [12].

3 Mathematical models of financial and barter operations, reducible to MCCP

3.1 Reallocation companies debts problem

Let $N = \{1, \ldots, n\}$ — a set of enterprises. On the set of pairs of enterprises $E = \{(i, j) \mid i \neq j, \ 1 \leq i, j \leq n\}$ functions $d(e), c(e)$ and $a(e)$ are given for each $e = (i, j)$, are equal to debt, debt unit price and the maximum permissible value of the debt company i owes the company j respectively. It is natural to assume that $d(e) \geq 0, \ a(e) \geq 0, e \in E$.

The redistribution of debts of enterprises problem (RDEP) is formulated as follows: for given functions $d(e)$, $c(e)$, $a(e)$ find such a redistribution of debt $f(e)$, which minimizes the functional

$$\sum_{e \in E} c(e) f(e)$$

under the following conditions:
  • divergence functions d and f are the same:

$$\operatorname{div}_f(x) = \operatorname{div}_d(x), \ x \in N;$$

  • fulfilled restrictions on the value of debts:

$$0 \leq f(e) \leq a(e), \ e \in E \ .$$

Proposition 1. *[1]-[3] The redistribution of debts of enterprises problem is reduced to the MCCP (1)–(3).*

3.2 The problem of appropriate credit to pay off debts of enterprises

Let $N = \{1, \ldots, n\}$ — a set of enterprises, and $\mathcal{A} \subset E = \{(i, j) \mid i \neq j, \ 1 \leq i, j \leq n\}$ — a subset of pairs (i, j), on which debt d_{ij} of enterprise i to enterprise j are given. We also know the value of the total loan Q, provided by enterprises to repay debt.

Initially, the company $i \in N$ has *negative balance* at $\operatorname{div}_d(i) > 0$ and *surplus* at $\operatorname{div}_d(i) < 0$.

For each company $i \in N$, in principle, the opportunity to obtain a private loan q_i^0 of the total (all enterprises) credit the amount of Q^0 and the target credit q_i, $0 \leq q_i \leq Q_i$ (intended only for the enterprise), and as a result of further settlements some amount of p_i may be on "cash" account of the enterprise i.

Expedient lending problem (ELP) is set to find a system of payments (f_{ij}), lending (q_i^0) and (q_i), as well as "pure" returns debt p_i, which minimizes the total value of the remaining debt (accounting for loans granted):

$$\sum_{j=1}^n (q_j^0 + q_j) + \sum_{(i,j) \in \mathcal{A}} (d_{ij} - f_{ij})$$

subject to:
• amount of the total credit of the consumed resource does not exceed the total limit of the resource:

$$\sum_{j=1}^{n} q_j^0 \leq Q^0, \ q_j^0 \geq 0, \ j \in N;$$

• for each enterprise i value of the consumed credit does not exceed the specified limit of this resource:

$$0 \leq q_i \leq Q_i, \ i \in N :$$

• and payment of the enterprise i to the enterprise j does not exceed the value of its debt:

$$0 \leq f_{ij} \leq d_{ij}, \ (i,j) \in \mathcal{A};$$

• the total amount of funds coming to the enterprise in the form of loans and payments from other companies, is equal to the amount of money left on "cash" account of the enterprise and its payments to creditors:

$$q_i^0 + q_i + \sum_{j \in B(i)} f_{ji} = p_i + \sum_{j \in A(i)} f_{ij}, \ i \in N.$$

Here are some features of ELP (see [3]).
1) The amount of money received for "cash" accounts, equal to the amount of loans received by enterprises

$$\sum_{i \in N} (q_i^0 + q_i) = \sum_{i \in N} p_i.$$

2) For optimal solution ELP the following conclusions are valid

$$q_i^0 + q_i > 0 \Rightarrow p_i = 0, \ p_i > 0 \Rightarrow q_i^0 + q_i = 0.$$

3) For the optimal solution of ELP for every enterprise j with the initial value of the surplus received loans – following sum $q_j^0 + q_j$ is equal to 0.
4) For the optimal solution of ELP for every enterprise i with initially negative balance value of the available balance p_i is equal to 0.

Proposition 2. *[1]-[3] Expedient lending problem is reduced to the MCCP (1)–(3).*

Two fictitious vertices i_1^0, i_2^0 and the arc (i_1^0, i_2^0) between them, as well as fictitious arcs (i_1^0, i) are introduced to account for the reduction of total lending (to all vertices with a negative balance) and (i, i_2^0) of all vertices with a positive balance. The capacity of these arcs to be equal to Q^0.

To account for lending for each vertex i (negative balance) is entered duplicate to its fictitious company i' with arc (i', i), as well as fictitious arcs (j, i') of each vertex j with surplus. Capacity of these arcs are equal to the values of Q_i.

3.3   Expedient barter problem

Let $N = \{1, \ldots, n\}$ — a set of enterprises, $R = \{1, \ldots, m\}$ — a set of types of goods (resources), produced and / or consumed by the enterprises from the set N. In the future, we will not distinguish between "goods" and "resource" as the same product (service) may act as a resource for one enterprise, and as produced goods for another.

For each enterprise $i \in N$ and goods $r \in R$ values $s_i^r \geq 0$ and $p_i^r \geq 0$ are sets, which are equal to the volume of supply and demand (calculated in terms of value). Without loss of generality we can assume that $s_i^r p_i^r = 0$ for all $i \in N$ and $r \in R$. The substantial meaning of this condition lies in the fact that no enterprise can be both a supplier and a buyer of the same goods.

Under the scheme (graph) of the commodity exchange we shall mean the set of values of $f_{ij}^r \geq 0$, $i, j \in N$, $r \in R$. The value of f_{ij}^r equal to the volume (by value) of the goods r, which is transmitted from the enterprise i to the enterprise j. Furthermore, we assume that $f_{ij}^r f_{ji}^r = 0$, i.e there are no counter deliveries of the same goods between a pair of enterprises.

The problem to find expedient barter (PFEB): for given values of s_i^r and p_i^r to find a barter scheme (f_{ij}^r), which would maximize the total volume of barter transactions:

$$\sum_{(i,j) \in N \times N} \sum_{r \in R} f_{ij}^r \tag{4}$$

subject to: • the total volume of goods r, now transferred i, must not exceed its capacity, t. e. for any $r \in R$, $i \in N$

$$\sum_{j \in N} f_{ij}^r \leq p_i^r \; ; \tag{5}$$

• the total volume of goods r, now received j, should not exceed the requirements of the latter, i.e. for any $r \in R$, $j \in N$

$$\sum_{i \in N} f_{ij}^r \leq s_j^r \; ; \tag{6}$$

• the total amount of goods purchased by enterprise, equal to the total volume of goods sold by this company, i.e. for any $i \in N$

$$\sum_{r \in R} \sum_{j \in N} f_{ji}^r = \sum_{r \in R} \sum_{j \in N} f_{ij}^r \; . \tag{7}$$

The following statement holds

Proposition 3. *[1]-[3] PFEB is reduced to the MCCP (1)–(3).*

3.4   The problem of debt repayment to the use of barter transactions

Suppose that in addition to the original definition of PFEB, described in the previous section, for each pair of enterprises $(i, j) \in N \times N$ the value of $d_{ij} \geq 0$ is given, which

characterizes the debts from enterprise i to enterprise j. Without loss of generality we can assume that $d_{ij} \cdot d_{ji} = 0$ for any $i, j \in N$.

Along with the values of $f_{ij}^r \geq 0$ ($i, j \in N$, $r \in R$), equal to the volume (by value) of the goods r, transmitted from the enterprise i to the enterprise j, we introduce the value of $f_{ij} \geq 0$, $i, j \in N$. The value of f_{ij} is defined as the payment for debt which the enterprise i received from the enterprise j.

Under the graph (scheme) of commodity exchange, allowable under the repayment of debt problems, we mean a set of values (f_{ij}^r, f_{ij}), satisfying the following conditions:
• the total volume of goods r, transferred i, must not exceed its capacity, ie for all $r \in R$, $i \in N$

$$\sum_{j \in N} f_{ij}^r \leq p_i^r \; ; \tag{8}$$

• the total volume of goods r, received j, should not exceed the requirements of the latter, ie. for any $r \in R$, $j \in N$

$$\sum_{i \in N} f_{ij}^r \leq s_j^r \; ; \tag{9}$$

• the sum of the volume of goods, purchased by the enterprise, and debt, credited his creditors, is the sum of the volumes, implemented by the enterprise of goods and debt, credited his debtors, t.e., for any $i \in N$

$$\sum_{r \in R} \sum_{j \in N} f_{ji}^r + \sum_{j \in N} f_{ji} = \sum_{r \in R} \sum_{j \in N} f_{ij}^r + \sum_{j \in N} f_{ij} \; ; \tag{10}$$

• amount of debt, counted i to enterprise j, should not exceed the value of the debt enterprise j to enterprise i, ie for any $i \in N$, $j \in N$

$$f_{ij} \leq d_{ji} \; . \tag{11}$$

Suppose also non-negative values of $\alpha, \beta \geq 0$ are given, meaning of which will be explained later.

The problem of maximum debt repayment using barter transactions (PDR) can be formulated as follows: for given values of $\alpha, \beta, s_i^r, \; p_i^r \quad d_{ij}$ to find a barter scheme (f_{ij}^r, f_{ij}), which would maximize the weighted sum of the volume of barter transactions and offsets:

$$\alpha \sum_{r \in R} \sum_{i \in N} \sum_{j \in N} f_{ij}^r + \beta \sum_{i \in N} \sum_{j \in N} f_{ij} \; , \tag{12}$$

when performing limited tasks purposeful exchange of goods (see Section 4).

Coefficients α and β reflect the degree of importance (priority) of barter and credit operations respectively. In particular, by putting $\alpha = 0$, we get the problem to find the maximum repayment of debts using barter transactions. The case $\beta = 0$ corresponds to the problem of maximizing the total volume of barter transactions. (Note that even in this case, the PDR is not equivalent to the previously discussed PFEB, since the presence of additional connections corresponding to the debt relationship, expanding the set of feasible options for barter).

Proposition 4. *[1]-[3] PDR is reduced to the MCCP (1)–(3).*

Comment. Reducibility to MCCP problem persists in case of weighted objective functions (described in the last two paragraphs), where the value of flows of the respective arcs (relations) are multiplied by predetermined factors assigned to such relations. For such factors one can use (depending on the definition of the problem), the price of the units of flow along the connection, interest tax deductions, the priority of the operation of this connection, etc.

4   Software implementation

Based on the models proposed above, authors developed software, allowing for computational solving of the real problems of mutual offset of debts by enterprises using the barter transactions. The algorithm was coded in C ++ in Visual Studio system, all experiments were carried out on a PC with a 1.7 GHz CPU and 2 Gb RAM with Windows 7 operating system.

To demonstrate capabilities of the algorithm, we conducted a series of numerical experiments. Test instances were created as follows. We set a number of clients (enterprises), and randomly create a list of transactions. The payor and the recipient of the bank transfer are selected randomly and equiprobably from the list, payment amount is selected with equal probable distribution out of range (100, 500000). Amount attributed to financing of the project is assigned as 0.01 of the sum of all transactions. The results of test experiments are presented in the table below.

Table 1.

Problems		found cycles	Offset cashless (%)		Processing time (sec)
clients	transactions		total	with credit	
100	1,000	799	80.5	11.3	0.03
1,000	10,000	7871	79.0	25.4	0.25
1,000	100,000	95472	94.3	10.4	2.67
10,000	100,000	77921	77.8	43.4	2.88
10,000	500,000	467489	91.8	22.7	15.4
10,000	1,000,000	956644	94.3	15.2	39.1
50,000	1,000,000	1608960	80.8	14.0	84.0
100,000	1,000,000	1559519	74.6	21.9	94.4

For each of the generated instances we show in the table the number of cycles found; the share of transactions that were offset cashless relative to the total amount of all transactions, including the share of transactions that were offset on the back of use of credit; processing time. As we can see from the table, lending, if allowed, yielded tangible results: one credit led to increase in total amount of offset transactions in the range of 10 to almost 50.

The current version of the software allows one to receive solution of the problem, containing tens of thousands of customers and millions connections (debts) between them. Such dimension corresponds to the average number of enterprises in a large region or even in an area. Unfortunately, at the present time the authors do not have the real data set of similar size to try the algorithm.

Processes of reduction to the MCCP problem that were considered in the models above, as well as processes of subsequent retrieval of solutions of the original problems are structural. It means that the user is allowed to enter some information in hard numbers. For numerical experiment authors use real raw data provided by the Krasnoyarsk Chamber of Accounts. Data contained approximately 1,000 clients and approximately 10,000 bank transactions. Processing time was 0.1 seconds.

The software developed by the authors was used in the National Bank of the Republic Buryatia (Ulan-Ude), JSCB "Kuzbassprombank" (Kemerovo), Krasnoyarsk Chamber of Accounts and APSB "Omskpromstroybank". We believe that wider adoption of the software is constrained by the complexity of gathering the necessary input data and the lack of legislation for cashless offset between counterparties in Russia.

It is worth noting that the developed software provides one-stop shop for getting instant optimum solutions for problems of mutual debts repayment, provided that there is sufficient information on the requirements of customers — both debtors and creditors. To build such database is a standalone problem, which could be resolved by joint efforts of regional administration, banks, tax authority and potentially law enforcement agencies only.

Positive experience of automated cashless offsets is recorded since April 1994, in the Republic of Buryatia, where JSC " Bikombank " (Buryat Innovation Commercial Bank) on a regular basis (twice a month) executed matching of debts between various enterprises within the region. Altai region has experience of creating a database to address the problem of non-payment: regional state committee of the Statistics has created the system of data collection and subsequent computations for cashless repayment of arrears (see "Finance magazine Siberia", 3/17, 1995). The software package developed in the Sobolev Institute of Mathematics, was shared in the past with Tveruniversalbank, the National Bank of the Republic of Buryatia, AKB "Kuzbassprombank" (Kemerovo) and the Chamber of Account of the Sakha Republic.

It is valuable that calculations can be carried out virtually in real time, which is very important in a rapidly changing market sentiment. For example, a conventional personal computer can solve the maximum cashless offset problem while optimising use of credit resources (with tens of thousands customers and hundreds of thousands connections between them) for a time within few minutes. On a number of particular customary data sets it turned out that each unit of credit resources leads to redemption of six to eight units of the debt.

Authors believe that another topical problem at the moment is optimization of barter transactions, whereas different market agents in both groups of sellers (producers) and buyers have different prices for the same goods. The mathematical model implemented for this problem also uses (as a base), minimum cost circulation problem.

5   Conclusion

The article presents some mathematical problems reducible to the classical problem to find in the circulation flow of minimum cost in a network. Computation output that was processed on a latest generation computer is provided. Processing time on data with 100,000 customers and 1,000,000 transactions did not exceed 1.5 minutes.

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