P_q-König Extended Forests and Cycles

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Abstract. A graph is König for a q-path if every its induced subgraph has the following property. The maximum number of pairwise vertex-disjoint induced paths each on q vertices is equal to the minimum number of vertices, such that removing all the vertices produces a graph having no an induced path on q vertices. In this paper, for every $q \geq 5$, we describe all König graphs for a q-path obtained from forests and simple sycles by replacing some vertices into graphs not containing induced paths on q vertices.

1 Introduction

Let $\mathcal{F}$ be a set of graphs. A set of pairwise vertex-disjoint induced subgraphs of a graph G each isomorphic to a graph in $\mathcal{F}$ is called a *$\mathcal{F}$-matching* of G. The *$\mathcal{F}$-matching problem* is to find a maximum $\mathcal{F}$-matching in a graph. A subset of vertices of a graph G which covers all induced subgraphs of G each isomorphic to a graph in $\mathcal{F}$ is called a *vertex cover of G with respect to $\mathcal{F}$* or simply its *$\mathcal{F}$-cover*. In other words, removing all vertices of any $\mathcal{F}$-cover of G produces a graph that does not contain none of the graphs in $\mathcal{F}$ as an induced subgraph. The *$\mathcal{F}$-cover* problem is to find a minimum $\mathcal{F}$-cover in a graph. A König graph for $\mathcal{F}$ is a graph in which every induced subgraph has the property that the maximum cardinality of its $\mathcal{F}$-matching is equal to the maximum cardinality of its $\mathcal{F}$-cover [1]. The class of all König graphs for a set $\mathcal{F}$ is denoted as $\mathcal{K}(\mathcal{F})$. If $\mathcal{F}$ consists of a single graph H, then we will talk about H-matchings, H-covers, and König graphs for H, respectively.

One can find some similar terms in the literature: "König-Egervary graph" [2], "a graph with the König property" [3], "König graph" [4] . They all mean a graph in which the cardinalities of a maximum matching and a minimum vertex cover are equal. The known König Theorem claims that the class of bipartite graphs is exactly the class of all graphs whose cardinalities of a maximum matching and a minimum vertex cover are equal not only for a graph but also for all its induced subgraphs. Note that our definition of a König graph is not a generalization of the notion in [2,3,4], because we

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require the equality of the parameters in all induced subgraphs of a graph. Thus, the class of bipartite graphs coincides with the class of all König graphs for P_2 in this sense.

A lot of papers on the $\mathcal{F}$-matching problem are devoted to algorithmic aspects (see [5,6,7,8]). It is known that the Matching problem (i.e., P_2-matching problem) can be solved in polynomial time [9], but the H-matching problem is NP-complete for any graph H having a connected component with three or more vertices.

It seems perspective to find new polynomially solvable cases for the $\mathcal{F}$-matching and $\mathcal{F}$-cover problems in the context of the method of critical graph classes developed in the papers of V. E. Alekseev and D. S. Malyshev [10]–[18].

Being formulated as the integer linear programming problems, the $\mathcal{F}$-matching and $\mathcal{F}$-cover problems form a pair of dual problems. So, König graphs are graphs, such that for any their induced subgraph there is no a duality gap for the problems above. In this regard, König graphs are similar to perfect graphs having the same property with respect to another pair of dual problems (vertex coloring and maximum clique) [19], which helps to solve efficiently these problems on perfect graphs [20].

Every hereditary class $\mathcal{X}$ can be described by a set of its minimal forbidden induced subgraphs, i.e. minimal by the relation "to be an induced subgraph" graphs not belonging to $\mathcal{X}$. A class $\mathcal{K}(\mathcal{F})$ is hereditary for any $\mathcal{F}$ and therefore it can be described by a set of forbidden induced subgraphs. Such a characterization for $\mathcal{K}(P_2)$ is given by the König theorem. In addition to this classical theorem, the following results are known. All minimal forbidden induced subgraphs for the class $\mathcal{K}(\mathcal{C})$ are described in [21], where $\mathcal{C}$ is the set of all simple cycles. All minimal forbidden induced subgraphs for the class $\mathcal{K}(P_3)$ are described, and full structural description of this class is given in [1,22]. Several families of forbidden induced subgraphs for $\mathcal{K}(P_4)$ are found, and it was conjectured that these families form a complete set of minimal forbidden induced graphs for this class in [23]. A structural description for one of the subclasses of $\mathcal{K}(P_4)$ is given in [24]. Graphs of this subclass can be obtained from graphs of a special type by replacing vertices with *cographs*, i.e., graphs not containing induced paths on 4 vertices.

The aim of this paper is to extend the structural description in [1,22,24] to some simple subclasses of $\mathcal{K}(P_q)$ for any $q \geq 5$. In section 2, we define the procedure of a R-F_q-extention of graphs. In section 3, we show how to obtain König graphs for P_q applying this procedure to forests. Moreover, in section 4, we find some forbidden induced subgraphs and show what graphs obtained from simple cycles by the R-F_q-extention are König for P_q.

In what follows we consider induced P_q with $q \geq 5$. The maximum number of subgraphs in P_q-matchings of G is denoted as $\mu_{P_q}(G)$, and the minimum number of vertices in its P_q-covers as $\beta_{P_q}(G)$.

A *q-path* is an induced subgraph isomorphic to P_q. We denote by $(v_1, v_2, \ldots, v_q)$ a q-path that consists of vertices $v_1, v_2, \ldots, v_q$. We denote by F_q the class of graphs not containing q-paths.

We denote by $|G|$ the number of vertices in G. We denote by $N(x)$ the neighbourhood of a vertex x in a graph.

We denote by $G \cup H$ the graph obtained from graphs G and H with non-intersected sets of vertices by their union. We denote by $G - v$ the graph obtained from G by deleting a vertex v with all incident edges.

Considering a cycle C_n, we assume that its vertices are clockwisely numbered as $0, 1, \ ..., \ n-1$. The arithmetic operations with the vertex numbers are performed modulo n. Every residue class of vertex numbers for modulo q is called a *q-class*.

2 R-$\boldsymbol{F_q}$-extention of graphs

In this section, we define the R-F_q-extention of graphs and describe the basic properties of graphs obtained by this procedure.

Definition 1. *The operation of replacement of a vertex x in a graph G with a graph H, where $V(G) \cap V(H) = \emptyset$, consists of the following. We take a graph $(G - x) \cup H$ and add all edges connecting $V(H)$ with $N(x)$.*

A *homogeneous set* in a graph G is a set $A \subseteq V(G)$, such that every vertex of $V(G) \backslash A$ is either adjacent to all vertices of A or to none of them. A homogeneous set is *trivial* if it is equal to $V(G)$ or consists of one vertex and *non-trivial*, otherwise.

Definition 2. *Let $\mathcal{X}$ be a class of graphs. The operation of a R-$\mathcal{X}$-extention of G consists of the following. Every vertex of degree 1 or 2 is replaced with an arbitrary graph of $\mathcal{X}$.*

Definition 3. *A section with a base x denoted by $S(x)$ is the set of vertices of the graph by which we replaced x. Every vertex not replaced with any graph is considered as a separate section.*

Obviously, any section is a homogeneous set. A section is *trivial* if it consists of one vertex and *non-trivial*, otherwise.

We say that two sections $S(x)$ and $S(y)$ are *adjacent* in a R-F_q-extention of a graph G if x and y are adjacent in G. Obviously, if $S(x)$ and $S(y)$ are adjacent, then each vertex of $S(x)$ is adjacent to each vertex of $S(y)$.

Lemma 1. *If a set of vertices A induce P_q in a R-F_q-extention of a graph G, then $|A \cap S(x)| = 1$ for all $x \in V(G)$.*

Proof. Every section induces a subgraph from F_q. Then the set A can not be entirely contained in any section. Since any section is a homogeneous set, if a section $S(x)$ contains two or more vertices of A, then the set $A \cap S(x)$ is a non-trivial homogeneous set in the subgraph induced by A. But a q-path does not contain a non-trivial homogeneous set for any $q > 3$. ■

Lemma 2. *Every minimum P_q-cover of any R-F_q-extention of any graph consists of whole sections.*

Proof. Let C be a minimum P_q-cover of an R-F_q-extention of some graph G. Suppose that there exists a section which contains vertices $x \in C$ and $y \notin C$. Since C is a minimum P_q-cover, there exists a set $A \subseteq V(G)$ inducing P_q, such that $A \cap C = \{x\}$. Otherwise, the vertex x can be deleted from C. By Lemma 1, $y \notin A$. But then $A \backslash \{x\} \cup \{y\}$ induce P_q, but it does not contain vertices of C. ■

3 R-$\boldsymbol{F_q}$-extended forests

In this section, we consider graphs obtained by a R-F_q-extention of forests. We will prove that all such graphs are König for P_q.

Theorem 1. *For any q, every R-F_q-extention $\tilde{F}$ of a forest F is a König graph for P_q.*

Proof. The proof is by induction on the number of q-paths in $\tilde{F}$. If there is no a q-path, then $\mu_{P_q}(\tilde{F}) = \beta_{P_q}(\tilde{F}) = 0$. Now consider $\tilde{F}$ which contains at least one q-path. By Lemma 1, all vertices of this q-path are contained in different sections. Obviously, they induce P_q in F.

Let T be a connected component of F containing a q-path. Let us select an arbitrary root r of T having degree three or more, if one exists. If T is a simple path, then r is one of its leaves.

Speaking about trees, we consider that they are drawn in a bottom-up manner from roots to leaves. For every q-path in T, a *bottom vertex* is the closest to r vertex of this q-path. Let a be the farther from r bottom vertex for all q-paths. There exists a q-path X which consists of a and descendants of a. Let $T^{'}$ be a subtree of T with the root a. Then in $T^{'}$ every q-path contains the vertex a.

Let $X = \{x_1, \ldots, x_k\}$ be a set of vertices inducing P_q in T with the bottom vertex a. Obviously, a is a member of this set. Let us select an arbitrary vertex y_i in every section $S(x_i)$. It is easy to see that $Y = \{y_1, \ldots, y_k\}$ induce P_q in $\tilde{F}$. Delete all vertices of Y from $\tilde{F}$. Denote obtained graph as $\tilde{F}^{'}$. It contains less q-paths than $\tilde{F}$. By the induction hypothesis, there exists a P_q-matching M of $\tilde{F}^{'}$ and its P_q-cover C, such that $|P| = |C|$. Then $M \cup \{Y\}$ is the P_q-matching of $\tilde{F}$ of the cardinality $|M| + 1$.

For a P_q-cover of $\tilde{F}$, we have two cases:

1. C contains at least one of the sections $S^{'}(x_1), \ldots, S^{'}(x_k)$, where $S^{'}(x_i) = S(x_i)\backslash\{y_i\}$. Suppose that C contains sections $S^{'}(x_i)$ and $S^{'}(x_j)$, where $i \neq j$. If the vertex x_i is an ancestor of x_j in the tree T, then every q-path, intersected with $S^{'}(x_j)$ intersects $S^{'}(x_i)$ as well. Therefore, $C\backslash S^{'}(x_j)$ is a P_q-cover of $\tilde{F}'$.
   Now suppose that x_i, x_j are not ancestors of each other. Then they have a common ancestor x_l in T. Obviously, x_l is a descendant of a or equals to a. Since degree of x_l is three or more, $|S\left(x_l\right)| = 1$. In this case, every q-path intersected with $S^{'}(x_i)$ or $S^{'}(x_j)$ contains x_l. Therefore, $C\backslash\left(S^{'}\left(x_i\right) \cup S^{'}\left(x_j\right)\right) \cup \{x_l\}$ is a P_q-cover of the graph $\tilde{F}^{'}$.
   Thus, if C is a minimum P_q-cover of the graph $\tilde{F}^{'}$, then C contain exactly one of the sections $S^{'}(x_1), \ldots, S^{'}(x_k)$. Let $S^{'}(x_i)$ be a such section. Then $C \cup \{y_i\}$ is a P_q-cover of $\tilde{F}$.
2. None of the sections $S^{'}(x_1), \ldots, S^{'}(x_k)$ is a subset of C. Then at least one of $S^{'}(x_1), \ldots, S^{'}(x_k)$ is empty. Let b denote the highest common ancestor of x_i with $S^{'}(x_i) = \emptyset$. Then either $b = a$ or b is a descendant of a and every section between a and b consists of more than one vertex. In the both cases, every q-path of $\tilde{F}$ with bottom vertex in $S(a)$ contains the vertex b and other q-paths are covered by the set C. Thus, $C \cup \{b\}$ is a P_q-cover of $\tilde{F}$.

So, for the both cases, there exists a P_q-cover of the graph $\tilde{F}$ of the cardinality $|M|+1$. But, every induced subgraph of $\tilde{F}$ is an R-F_q-extention of some forest. Therefore, $\tilde{F}$ is a König graph for P_q. ■

4 R-F_q-extended cycles

4.1 Common cases for forbidden induced graphs

Now we consider the graphs obtained by applying a R-F_q-extention to simple cycles. We call them *R-F_q-extended cycles*.

Considering a R-F_q-extention of a cycle C_n, we assume that the sections are clockwisely numbered as $0, 1, \ ..., \ n-1$. Every residue class of sections numbers for modulo q is called a *q-class*.

At first, we define several infinite families of forbidden induced subgraphs for $\mathcal{K}(P_q)$ for every $q \geq 5$.

Obviously, for any $k > 1$

$$pack\,(C_{qk}) = \mu_{P_q}\,(C_{qk+1}) = \cdots = \mu_{P_q}\,(C_{qk+q-1}) = k;$$

$$cover\,(C_{qk}) = \beta_{P_q}\,(C_{qk-1}) = \cdots = \beta_{P_q}\,(C_{qk-q+1}) = k.$$

Note that every proper induced subgraph of a simple cycle is a forest. By Theorem 1, it is a König graph for P_q. Hence, the following lemma is valid.

Lemma 3. *A cycle C_n belongs to $\mathcal{K}(P_q)$ if n is divisible by q, and C_n is a minimal forbidden induced subgraph for $\mathcal{K}(P_q)$ if n is not divisible by q.*

Thus, a R-F_q-extention of a simple cycle can be a König graph for P_q only if the basic cycle has a number of vertices divisible by q.

Let $k_1, k_2, \ldots, k_q$ be arbitrary naturals, such that $k_1 + k_2 + \cdots + k_q = qn$. Let us chose q vertices in a cycle C_{qn} in such a way that its paths containing no the chosen vertices except their endpoints have lengths $k_1, k_2, \ldots, k_q$. We replace any chosen vertex into an arbitrary two-vertex graph. The set of all resultant graphs is denoted by $\tilde{D_q}\,(k_1, k_2, \ldots, k_q)$. Let $D_q\,(k_1, k_2, \ldots, k_q)$ denote any graph of the set $\tilde{D_q}\,(k_1, k_2, \ldots, k_q)$.

Let denote $r_i = \sum_{j=1}^{i} k_j$. Obviously, one can enumerate vertices along the cycle in such a way that the replaced vertices have numbers $0, r_1, \ldots r_{q-1}$.

Definition 4. *A graph $D(k_1, k_2, \ldots, k_q)$ is crowded if $\forall i, j : r_i \not\equiv r_j (mod\, q)$. It means that exactly one vertex is replaced with a two-vertex graph in every q-class of the basic cycle.*

Definition 5. *A T-array in a R-F_q-extended cycle is a maximal collection of sequentially adjacent trivial sections. Similarly, a N-array in a R-F_q-extended cycle is a maximal collection of sequentially adjacent non-trivial sections. A size of a T-array or a N-array is a number of sections in it.*

Definition 6. *A vector of array sequence (AS-vector for short) of a graph* $D_q(k_1, k_2, \ldots, k_q)$ *is a sequenced collection of numbers* $(u_1, t_1, u_2, t_2, \ldots, u_m, t_m)$*, where for all* $i \in \{1, \ldots, m\}$ t_i *is lenght of T-array and* u_i *is lenght of N-array.*

For example, the AS-vector for the graph $D_9(1, 3, 1, 16, 10, 7, 15, 19, 1)$ is $(3, 2, 2, 15, 1, 9, 1, 6, 1, 14, 1, 18)$.

Note that AS-vector of every graph $D_q(k_1, k_2, \ldots, k_q)$ has the properties:

$$\sum_{i=1}^{m} u_i = q, \qquad \sum_{i=1}^{m} t_i = q(n-1).$$

The arithmetic operations with the indexes in an AS-vector are performed modulo m.

Theorem 2. *A crowded graph* $D_q(k_1, k_2, \ldots, k_q)$ *is a minimal forbidden induced subgraph for* $\mathcal{K}(P_q)$ *if and only if for its AS-vector* $(u_1, t_1, u_2, t_2, \ldots, u_m, t_m)$ *there exists a number* i*, such that* $u_i + t_i + u_{i+1} \not\equiv 0 \,(mod\, q)$.

Proof. For every crowded graph D, $\beta_{P_q}(D) = n + 1$, where qn is length of the basic cycle. We show that for every crowded graph D $\mu_{P_q}(D) = n + 1$ if and only it in its AS-vector for every i $u_i + t_i + u_{i+1}$ is divisible by q and $\mu_{P_q}(D) = n$, otherwise.

Let $D_q(k_1, k_2, \ldots, k_q)$, where $k_1 + k_2 + \cdots + k_q = qn$, be a crowded graph. Its number of vertices equals to $qn + q$. It means that a P_q-matching of the cardinality $n + 1$ must include all vertices of the graph. It is easy to see that every N-array have to begin one of q-paths and end one of q-path of such P_q-matching. In other words, it can not lie in the middle of any q-path of the P_q-matching. Moreover, if the end of a q-path of the P_q-matching belongs to a trivial section, then the beginning of the next q-path belongs strictly to the next section. It is possible only if the number of sections between the beginning of a N-array and the end of the next one is divisible by q, i.e. $\forall i$ $u_i + t_i + u_{i+1} = q$. Otherwise, no one P_q-matching includes all vertices of the graph and the maximum cardinality of P_q-matchings is equal to n.

Every induced subgraph of a crowded graph is a R-F_q-extended forest or a R-F_q-extended cycle, such that one of its q-classes consists of trivial sections only. In the both cases, the graph is König for P_q. So, every crowded graph D with $\mu_{P_q}(D) = n$ is a minimal forbidden induced subgraph for $\mathcal{K}(P_q)$. ■

Notice that for every $q \geq 3$ the minimum number of sections in crowded graphs is equal to $2q$. For example, for any odd q a graph $D_q(2, 2, \ldots, 2)$ (Fig. 1) has the minimum possible number of sections. Similarly, for every even q, a graph $D_q(\underbrace{2, 2, \ldots, 2}_{\frac{q}{2}-1}, 1, \underbrace{2, 2, \ldots, 2}_{\frac{q}{2}-1}, 3)$ is also extremal. By Theorem 2, for any $q \geq 5$, this graphs are minimal forbidden induced subgraphs for $\mathcal{K}(P_q)$.

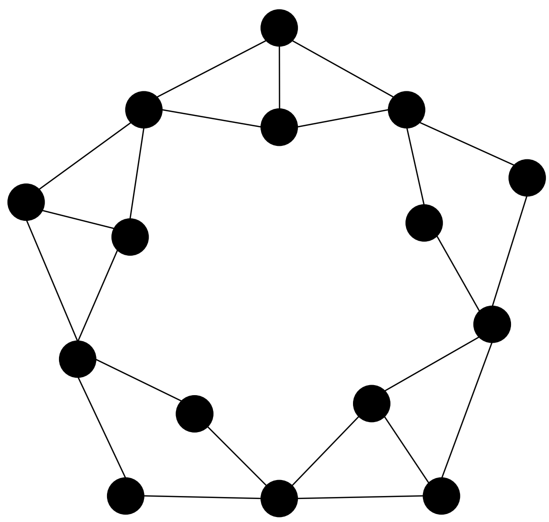

Figure 1. One of the graphs $D_5(2, 2, 2, 2, 2)$

4.2 Main theorem about R-$\boldsymbol{F_q}$-extended cycles

For every P_q, let $\mathcal{D}_q$ denote the set of the crowded graphs that are minimal forbidden induced subgraphs for $\mathcal{K}(P_q)$.

Now we prove that $\mathcal{D}_q$ is a complete set of forbidden induced subgraphs for R-F_q-extended cycles, which are König graphs for P_q.

Theorem 3. *A R-F_q-extended cycle is a König graph for P_q if and only if it does not include induced subgraphs belonging to the set $\mathcal{D}_q$.*

Proof. Let G be obtained from a simple cycle C_{qn} by replacing its vertices with F_q-graphs, and any its induced subgraph does not belong to the set $\mathcal{D}_q$. By Theorem 1, any R-F_q-extended tree it is König for P_4. Every induced subgraph of G is a R-F_q-extended cycle with the same property or an R-F_q-extended tree. Thus, it is enough to prove that there exist a P_q-matching and a P_q-cover of equal cardinalities in G.

The proof is by induction on the number of q-paths in the graph. Let every q-path in G intersect at least one trivial section. It means that deleting a q-path always induce a R-F_q-extended forest. The following cases are possible.

1. G does not contain induced crowded graphs. Then there is at least one q-class consisting of only trivial sections. Obviously, this q-class gives a P_q-cover of G of the cardinality q. A P_q-matching of the same cardinality coincides with one of the maximum P_q-matchings of the basic cycle.
2. G contains N-array of lenght $q - 1$. Let it consist of the sections $S_1, S_2, \ldots, S_{q-1}$. Then we consider a graph $G^{'}$ obtained from G by deletion of sections $S_0, S_1, \ldots, S_q$. The graph $G^{'}$ is a R-F_q-extended tree. Therefore, by Theorem 1, there exist a P_q-matching $M^{'}$ and a P_q-cover $C^{'}$ of $G^{'}$ of equal cardinalities.

In what follows, for each $i \in \{0, nq-1\}$, let u_i denote an arbitary vertiex of a section S_i, and v_i denote an another vertex of the same section, if one exists. It is easy to see that $M = M^{'} \cup \{(u_0, u_1, \dots u_{q-1}), (v_1, v_2, \dots v_{q-1}, u_q)\}$ is the P_q-matching of graph G, and $C = C^{'} \cup S_0 \cup S_q$ is its P_q-cover and $|M| = |C| = \left|M^{'}\right| + 2$

3. G contains a crowded induced subgraph and the maximum size l of N-arrays is more than 2 and no more than $q-2$. Suppose that a N-array of the maximum size consists of sections $S_1, S_2, \dots, S_l$. Assume that a section S_{qi+2} is non-trivial for some $i \geq 1$. Then G contains a crowded induced subgraph, such that its AS-vector contains a sequence u_j, t_j, u_{j+1}, where $u_j = 1$, $t_j = 1$, $u_{j+1} = l-2$. The sum of this parameters is l. By Theorem 2, such graph must be forbidden. Thus, sections $S_{q+2}, S_{2q+2}, \dots S_{(n-1)q+2}$ are trivial. Note that sections S_0 and S_{l+1} are trivial as well. Let us consider a set
$$C = \bigcup_{i=1}^{n-1} S_{qi+2} \cup S_0 \cup S_{l+1}.$$
It is easy to see that C is a P_q-cover of G and $|C| = n+1$. But none crowded induced subgraph of G is forbidden. Therefore, by proof of Theorem 2, it contains a P_q-matching M of the cardinality $n+1$. Obviously, M is a P_q-matching of graph G.
4. G contains a crowded induced subgraph and every N-array of G consists of no more than 2 sections. Consider two neighbour N-arrays of some crowded subgraph D of G. Let the first of them begin from S_1. It means that the section S_1 of the graph G is non-trivial. By Theorem 2, the next N-array ends in section, which number is divisible by q. In other words, for G there exists l, such that S_{ql} is non-trivial.
Let $l > 1$. Consider a graph $D^{'}$ obtained from D by adding a vertex into the section S_{q+3} and deleting one vertex from the non-trivial section of the same q-class. The AS-vector of the graph $D^{'}$ contains a sequence u_i, t_i, u_{i+1}, such that their sum is equal to $q+3$. By Theorem 2, $D^{'}$ must be forbidden. Thus, the section S_{q+3} is trivial in G. Similarly, all sections $S_{2q+3}, S_{3q+3}, \dots S_{(l-1)q+3}$ are trivial in G. Since every N-array consists of no more than 2 sections, the section S_3 is trivial as well. Similarly, if S_r is the begin section of some N-array of a crowded induced subgraph of G and S_{r+x} is the begin section of its next N-array, then for $0 < x - hq - 2 < q$, all sections $S_{r+2}, S_{r+q+2}, \dots S_{r+hq+2}$ of the graph G are trivial. By Theorem 2, $x \equiv q-1 (mod\, q)$ if the N-array consists of one section or $x \equiv q-2 (mod\, q)$ if it consists of two sections.
Consider an AS-vector $(u_1, t_1, u_2, \dots, u_m, t_m)$ of the graph D. Let u_1 correspond to the N-array, which begins from the section S_1. Denote $r_0 = 1$, $r_i = r_{i-1} + u_i + t_i$ for all $i \in \{1, \dots, m-1\}$, h_i is the number, such that $0 < r_{i+1} - r_i - h_i q - 2 < q$. It is easy to see that
$$C = \bigcup_{i=0}^{m} \bigcup_{j=0}^{h_i} S_{r_i + jq + 2}$$
is a $P-q$-cover of the graph G and $|C| = n+1$. But none crowded induced subgraph of G is forbidden. Therefore, by proof of Theorem 2, it contains a P_q-matching M of the cardinality $n+1$. Obviously, M is a P_q-matching of the graph G.

Thus, if every q-path of the graph G intersects at least one trivial section, then G is a König graph for P_q.

Now let some q-path of the graph G intersect only non-trivial sections. Since for $q \geq 5$, there exists a crowded forbidden graph having $2q$ sections, every R-F_q-extended cycle, which consists of only non-trivial sections, contains a crowded forbidden induced subgraph. Thus, G contains at least one trivial section. Suppose that S_0 is a trivial section and $S_1, S_2, \ldots, S_q$ are non-trivial sections in G.

Let us select one vertex v_i from each section S_i, $i \in \{1, \ldots, q\}$. Let $G^{'}$ denote a graph obtained from G by deleting vertices $v_1, v_2, \ldots, v_q$. By the induction hypothesis, $G^{'}$ contains a P_q-matching $M^{'}$ and a P_q-cover $C^{'}$ of equal cardinalities. Note that $C^{'}$ contains at least one section among $S_i \backslash \{v_i\}, i \in \{1, \ldots, q\}$, but no more than 2. If there are two such sections, then S_0 is not a subset of $C^{'}$. Otherwise $C^{'}$ is not minimum. Then we change section with the minimum number into S_0 in $C^{'}$. The obtained P_q-cover is a minimum P_q-cover, which contains exactly one section among $S_i \backslash \{v_i\}, i \in \{1, \ldots, q\}$. We add to $C^{'}$ the deleted vertex of the corresponding section. We add to $M^{'}$ the q-path $(v_1, v_2, \ldots, v_q)$. The obtained sets are some P_q-cover and some P_q-matching of the graph G of equal cardinalities. ■

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