On a Local Search for Hexamatrix Games

Andrei Orlov and Alexander Strekalovsky

Matrosov Institute for System Dynamics and Control
Theory of Siberian Branch of RAS, Irkutsk, Russia
{anor,strekal}@icc.ru
http://nonconvex.isc.irk.ru

Abstract. The problem of finding a Nash equilibrium in polymatrix game of three players (hexamatrix game) is considered. For the equivalent nonconvex optimization problem an issue of local search is investigated. First, we study the special local search method (SLSM), based on the linearization of the original problem with respect to basic nonconvexity. Then the new local search procedure is elaborated. This procedure uses the bilinear structure of the objective function and is based on the consecutive solving of auxiliary linear programs. At the end of the paper the results of comparative computational simulation for two local search methods are presented and analyzed.

Keywords: Polymatrix games, hexamatrix games, global search theory, local search, computational simulation

1 Introduction

As known, the efficient numerical methods for solving problems of Game Theory is the important issue of the contemporary mathematical optimization [9]. One of the conventional concept of a solution in Game Theory is a Nash equilibrium. In such a point for none of the players it is not profitable to change own strategy alone. Unfortunately, there are no efficient algorithms for finding Nash equilibria in the games of a general form with a large number of proper players' strategies. From the optimization viewpoint even classical bimatrix game has a nonconvex structure [7, 16], because a problem of finding a Nash equilibrium in such a game is equivalent to some nonconvex bilinear programming problem. Therefore classical optimization methods are not directly applicable for solving bimatrix games.

The new nonconvex optimization approach to numerical finding of Nash equilibria in bimatrix games, based on Mills theorem [5] and Global Search Theory (GST) [10, 12], demonstrated efficiency for large-scale problems [7, 16, 19]. Recently, the theoretical foundation for finding Nash equilibrium in polymatrix games [20] (generalization of Mills theorem) was developed [14]. In particular, polymatrix game of three players (hexamatrix game) can be reduced to the nonconvex mathematical optimization problem with triple bilinear structure in the objective function [14].

In: A. Kononov et al. (eds.): DOOR 2016, Vladivostok, Russia, published at http://ceur-ws.org

The reduced problem belongs to the class of nonconvex optimization problems with functions of A.D. Alexandrov, since it is easy to show that a bilinear function can be represented as the difference of two convex functions [7, 16]. Such functions are called d.c. functions [1, 10, 12]. Therefore, in order to solve this problem the Global Search Theory for d.c. minimization (maximization) problems [10, 12] can be applied. The first results of its application to hexamatrix games can be found in [8].

According to GST, one of the key elements of global search algorithms in the reduced problems is specialized local search methods, taking into account the structure of the problem under investigation [11, 13, 15, 17]. Our experience shows that the efficiency of the whole Global Search Algorithm depends heavily on the efficiency of Local Search.

In this paper two local search methods for hexamatrix games are studied. The first method employs the d.c. structure of the problem in question and applies the linearization of the problem with respect to the basic nonconvexity [10, 13, 18]. The second method is based on the consecutive solving of auxiliary linear programs and employs the bilinearity of the objective function [7, 16, 17]. The results of comparative computational simulation for these methods are presented and analyzed.

2 Problem Formulation

Consider the following hexamatrix game with mixed strategies

$$\left.\begin{aligned} F_1(x,y,z) &\triangleq \langle x, A_1 y + A_2 z\rangle \uparrow \max_x, \quad x \in S_m \ , \\ F_2(x,y,z) &\triangleq \langle y, B_1 x + B_2 z\rangle \uparrow \max_y, \quad y \in S_n \ , \\ F_3(x,y,z) &\triangleq \langle z, C_1 x + C_2 y\rangle \uparrow \max_z, \quad z \in S_l \ , \end{aligned}\right\}$$

where $S_p = \{u = (u_1, \ldots, u_p)^T \in I\!R^p \mid u_i \geq 0, \ \sum_{i=1}^{p} u_i = 1\}, \quad p = m, n, l.$

Definition 1. *a) The triple $(x^*, y^*, z^*) \in S_m \times S_n \times S_l$ satisfying the inequalities*

$$\left.\begin{aligned} v_1^* = v_1(x^*,y^*,z^*) &\triangleq F_1(x^*,y^*,z^*) \geq F_1(x,y^*,z^*) \quad \forall x \in S_m \ , \\ v_2^* = v_2(x^*,y^*,z^*) &\triangleq F_2(x^*,y^*,z^*) \geq F_2(x^*,y,z^*) \quad \forall y \in S_n \ , \\ v_3^* = v_3(x^*,y^*,z^*) &\triangleq F_3(x^*,y^*,z^*) \geq F_3(x^*,y^*,z) \quad \forall z \in S_l \end{aligned}\right\}$$

will be henceforth called a Nash equilibrium point in the game $\Gamma_3 = \Gamma(A,B,C)$ $(A = (A_1, A_2),\ B = (B_1, B_2),\ C = (C_1, C_2))$. Herewith, the strategies x^*, y^*, and z^* will be called the equilibrium strategies.

b) The numbers v_1^*, v_2^*, and v_3^* will be called the payoffs of players 1, 2, and 3, respectively, at the equilibrium point (x^*, y^*, z^*).

c) Denote the set of all Nash equilibrium points of the game $\Gamma_3 = \Gamma(A,B,C)$ by $NE = NE(\Gamma_3) = NE(\Gamma(A,B,C))$.

It is well known that in the case of the game $\Gamma_3 = \Gamma(A,B,C)$ due to Nash's Theorem [14], there exists a Nash equilibrium point in mixed strategies.

From the viewpoint of a real numerical search the following definition of approximate Nash equilibrium point is appropriated.

Definition 2. *The triple* $(x^0, y^0, z^0) \in S_m \times S_n \times S_l =: S$ *satisfying the inequalities*

$$\left.\begin{array}{l} F_1(x^0, y^0, z^0) + \varepsilon \geq F_1(x, y^0, z^0) \;\forall x \in S_m \,, \\ F_2(x^0, y^0, z^0) + \varepsilon \geq F_2(x^0, y, z^0) \;\forall y \in S_n \,, \\ F_3(x^0, y^0, z^0) + \varepsilon \geq F_3(x^0, y^0, z) \;\forall z \in S_l \end{array}\right\}$$

will be called an ε*-Nash equilibrium point for the game* Γ_3 $((x^0, y^0, z^0) \in NE(\Gamma_3, \varepsilon))$.

Further consider the following optimization problem $(\sigma \triangleq (x, y, z, \alpha, \beta, \gamma))$:

$$\left.\begin{array}{c} \Phi(\sigma) \triangleq \langle x, A_1 y + A_2 z\rangle + \langle y, B_1 x + B_2 z\rangle + \\ +\langle z, C_1 x + C_2 y\rangle - \alpha - \beta - \gamma \uparrow \max\limits_{\sigma} \,, \\ \sigma \in D \triangleq \{(x, y, z, \alpha, \beta, \gamma) \in I\!R^{m+n+l+3} \mid x \in S_m, \;\; y \in S_n, \;\; z \in S_l \,, \\ A_1 y + A_2 z \leq \alpha e_m, \quad B_1 x + B_2 z \leq \beta e_n, \quad C_1 x + C_2 y \leq \gamma e_l\} \,, \end{array}\right\} \qquad (\mathcal{P})$$

where $e_p = (1, 1, ..., 1) \in I\!R^p, p = m, n, l$.

Lemma 1. [14] *The objective function of Problem* $(\mathcal{P})$ *is nonpositive at each feasible point:*

$$\Phi(\sigma) = \Phi(x, y, z, \alpha, \beta, \gamma) \leq 0 \quad \forall \sigma \in D \,.$$

The following equivalence theorem for Problem $(\mathcal{P})$ and the problem of finding the Nash equilibrium in the game Γ_3 takes place.

Theorem 1. [14] *The point* (x^*, y^*, z^*) *is a Nash equilibrium point in the hexamatrix game* $\Gamma(A, B, C) = \Gamma_3$ *if and only if it is the part of the global solution* $\sigma_* \triangleq (x^*, y^*, z^*, \alpha_*, \beta_*, \gamma_*) \in I\!R^{m+n+l+3}$ *to Problem* $(\mathcal{P})$*. At the same time, the numbers* α_**,* β_**, and* γ_* *are the payoffs of the first, the second, and the third players, respectively, in the game* Γ_3*:* $\alpha_* = v_1(x^*, y^*, z^*)$*,* $\beta_* = v_2(x^*, y^*, z^*)$*,* $\gamma_* = v_3(x^*, y^*, z^*)$*. In addition, the optimal value* $\mathcal{V}(\mathcal{P})$ *of Problem* $(\mathcal{P})$ *is equal to zero:*

$$\mathcal{V}(\mathcal{P}) = \Phi(x^*, y^*, z^*, \alpha_*, \beta_*, \gamma_*) = 0 \,.$$

Furthermore, it is easy to prove the result of the relationship between an approximate solution of Problem $(\mathcal{P})$ and an approximate Nash equilibrium.

Proposition 1. [14] *Suppose, the point* $(x^0, y^0, z^0, \alpha_0, \beta_0, \gamma_0)$ *is an* ε*-solution to Problem* $(\mathcal{P})$*. Then the triple* (x^0, y^0, z^0) *is the* ε*-Nash equilibrium point for the game* $\Gamma_3 = \Gamma(A, B, C)$*:* $(x^0, y^0, z^0) \in NE(\Gamma_3, \varepsilon)$*.*

Thus, the seeking for a Nash equilibrium can be carried out by approximate solving Problem $(\mathcal{P})$. To this end, we propose to use the Global Search Theory mentioned above [10, 12]. According to this Theory the global search consists of two principal stages: 1) a local search, which takes into account the structure of the problem under scrutiny; 2) the procedures based on Global Optimality Conditions (GOC) [10, 12], allow improving the point provided the local search method, in other words, to escape a local pit.

The reminder of the paper will be devoted to the first stage of the approach.

3 Local Search Method for D.C. Formulation of Problem $(\mathcal{P})$

It can be readily seen that the objective function of Problem $(\mathcal{P})$ (which is the special kind of quadratic functions with bilinear structure) is a d.c. function, since it can be represented as a difference of two convex functions, for example, in the following way:

$$\Phi(x,y,z,\alpha,\beta,\gamma) = h_0(x,y,z) - g_0(x,y,z,\alpha,\beta,\gamma)\ , \tag{1}$$

where

$$\left.\begin{aligned} h_0(x,y,z) &= \frac{1}{4}\Big(\|x+A_1y\|^2 + \|x+A_2z\|^2 + \\ &+\|B_1x+y\|^2 + \|y+B_2z\|^2 + \|C_1x+z\|^2 + \|C_2y+z\|^2\Big), \\ g_0(\sigma) &= \frac{1}{4}\Big(\|x-A_1y\|^2 + \|x-A_2z\|^2 + \|y-B_1x\|^2 + \|y-B_2z\|^2 + \\ &+\|C_1x-z\|^2 + \|C_2y-z\|^2\Big) + \alpha+\beta+\gamma\ . \end{aligned}\right\} \tag{2}$$

Besides, one can show that these functions are convex on (x,y,z) and σ, respectively.

Thus, the first method of local search, which can be suggested here, is the Special Local Search Method (SLSM) for d.c. programming [10, 12]. This method, well known as the DCA [18], is based on a solving the sequence of problems linearized with respect to basic nonconvexity and exploits the d.c. structure of Problem $(\mathcal{P})$.

In terms of Problem $(\mathcal{P})$ the scheme of the SLSM can be written in the following way. Let $\sigma^0 = (x^0, y^0, z^0, \alpha_0, \beta_0, \gamma_0) \in D$ be a starting point. Further if one has the point $\sigma^s \in D$ (at a current iteration), then the next point $\sigma^{s+1} \in D$ is an approximate solution of the linearized (at the point $\sigma^s \in D$) problem:

$$\Psi_s(\sigma) \triangleq g_0(\sigma) + \langle a^s, x\rangle + \langle b^s, y\rangle + \langle c^s, z\rangle \downarrow \min_{\sigma}, \qquad \sigma \in D\ , \tag{$\mathcal{PL}(\sigma^s)$}$$

where

$$a^s \triangleq -\nabla_x h_0(x^s,y^s,z^s) = -\frac{1}{2}[2x^s + A_1y^s + (B_1x^s+y^s)B_1 + A_2z^s + (C_1x^s+z^s)C_1]\ ;$$

$$b^s \triangleq -\nabla_y h_0(x^s,y^s,z^s) = -\frac{1}{2}[2y^s + B_1x^s + (x^s+A_1y^s)A_1 + B_2z^s + (C_1y^s+z^s)C_2]\ ;$$

$$c^s \triangleq -\nabla_z h_0(x^s,y^s,z^s) = -\frac{1}{2}[2z^s + C_1x^s + (x^s+A_2z^s)A_2 + C_2y^s + (y^s+B_2z^s)B_2]\ .$$

Further, let the numerical sequence $\{\rho_s\}$ such that

$$\rho_s > 0,\ s = 0,1,2,...,\ \sum_{s=0}^{\infty}\rho_s \le +\infty\ ,$$

be given. Then the next iteration $\sigma^{s+1} \in D$ is constructed to satisfy the following inequality:

$$\Psi_s(\sigma^{s+1}) \le \inf_{\sigma}\{\Psi_s(\sigma) \mid \sigma \in D\} + \rho_s\ . \tag{3}$$

The convergence theorem will be presented here in terms of Problem $(\mathcal{P})$. Note that the objective function of Problem $(\mathcal{P})$ is bounded below due to Theorem 1 and Nash's Theorem.

Theorem 2. [10, 12, 16] *The sequence* $\{\sigma^s\}$ *generated by the rule (3) satisfies the following conditions:*

a) $\lim_{s\to\infty}[\mathcal{V}(\mathcal{PL}(\sigma^s)) - \Psi_s(\sigma^{s+1})] = 0$, *where* $\mathcal{V}(\mathcal{PL}(\sigma^s))$ *is the optimal value of Problem* $(\mathcal{PL}(\sigma^s))$.

b) If the function $h_0(\cdot)$ *is strongly convex, then we have* $\lim_{s\to\infty}\|\sigma^s - \sigma^{s+1}\| = 0$.

c) Any limit point $\hat{\sigma}$ *of the sequence* $\{\sigma^s\}$ *generated by the SLSM is a solution of the linearized problem* $(\mathcal{PL}(\hat{\sigma}))$.

The point $\hat{\sigma}$ be called *a critical (with respect to the SLSM) point to Problem* $(\mathcal{P})$. The resulting critical point will also satisfy classical first order stationarity condition [10, 12, 16].

In the implementation of the SLSM the following inequality can be used as a stopping criterion:

$$\Psi_s(\sigma^s) - \Psi_s(\sigma^{s+1}) \le \tau/2\ ,$$

where $\tau > 0$ is the given accuracy. If $\rho_s \le \tau/2$ then point σ^s will be τ-critical point to Problem $(\mathcal{P})$ with respect to the SLSM [10, 12, 16].

4 "Mountain Climbing" Procedure

The second idea of a local search in Problem $(\mathcal{P})$ finds its roots in the "mountain climbing" procedure, which is proposed by H. Konno [3] for the bilinear programming problems. To implement this idea for Problem $(\mathcal{P})$, first, let us split variables in several groups, and, after that, solve a sequence of specially constructed linear programming (LP) problems with respect to these groups of variables. Wherein, in contrast to results of H. Konno, and analogous early publications, the auxiliary linear programs can be solved approximately.

Similar idea of a local search have previously demonstrated its efficiency in bimatrix games [7, 16, 19], bilinear programming problems [6, 16], and bilevel problems [17].

In order to describe the method, consider the following LP problems:

$$\left.\begin{array}{c} f_1^{(v,w)}(x,\beta) \triangleq \langle x, (A_1 + B_1^T)v + (A_2 + C_1^T)w\rangle - \beta \uparrow \max\limits_{(x,\beta)}\ , \\ (x,\beta) \in X(v,w,\bar{\gamma}) \triangleq \{(x,\beta) \mid x \in S_m, \\ B_1 x - \beta e_n \le -B_2 w,\ C_1 x \le \bar{\gamma} e_l - C_2 v\}\ ; \end{array}\right\} \qquad (\mathcal{LP}_x(v,w,\bar{\gamma}))$$

$$\left.\begin{array}{c} f_2^{(u,w)}(y,\gamma) \triangleq \langle y, (B_1 + A_1^T)u + (B_2 + C_2^T)w\rangle - \gamma \uparrow \max\limits_{(y,\gamma)}\ , \\ (y,\gamma) \in Y(u,w,\bar{\alpha}) \triangleq \{(y,\gamma) \mid y \in S_n, \\ A_1 y \le \bar{\alpha} e_m - A_2 w,\ C_2 y - \gamma e_l \le -C_1 u\}\ ; \end{array}\right\} \qquad (\mathcal{LP}_y(u,w,\bar{\alpha}))$$

$$\left.\begin{array}{c} f_3^{(u,v)}(z,\alpha) \triangleq \langle z, (C_1 + A_2^T)u + (C_2 + B_2^T)v\rangle - \alpha \uparrow \max\limits_{(z,\alpha)}\ , \\ (z,\alpha) \in Z(u,v,\bar{\beta}) \triangleq \{(z,\alpha) \mid z \in S_l, \\ A_2 z - \alpha e_m \le -A_1 v,\ B_2 z \le \bar{\beta} e_n - B_1 u\}\ . \end{array}\right\} \qquad (\mathcal{LP}_z(u,v,\bar{\beta}))$$

Here $(u, v, w, \bar{\alpha}, \bar{\beta}, \bar{\gamma}) \in D$ is a feasible solution to Problem $(\mathcal{P})$.

It is easy to see that these problems will be feasible and bounded due to the boundedness from above of the function $\Phi(\cdot)$ on the set D by Lemma 1.

Let $(x^0, y^0, z^0, \alpha_0, \beta_0, \gamma_0) \in D$ be a starting point. For example, one can use the barycenters of standard simplexes as follows:

$$\begin{gathered} x_i^0 = \frac{1}{m},\ i = 1, ..., m; \quad y_j^0 = \frac{1}{n},\ j = 1, ..., n; \quad z_t^0 = \frac{1}{l},\ t = 1, ..., l\ ; \\ \alpha_0 = \max_i (A_1 y^0 + A_2 z^0)_i; \quad \beta_0 = \max_j (B_1 x^0 + B_2 z^0)_j\ ; \\ \gamma_0 = \max_t (C_1 x^0 + C_2 y^0)_t\ . \end{gathered} \tag{4}$$

YZ_γ-procedure

Step 0. Set $s := 1$, $y^s := y^0$, $z^s := z^0$, $\gamma_s := \gamma_0$.

Step 1. Using an LP technique, find a $\rho_s/3$-solution (x^{s+1}, β_{s+1}) to Problem $(\mathcal{LP}_x(y^s, z^s, \gamma_s))$, so that the following inequality holds:

$$\begin{gathered} f_1^s(x^{s+1}, \beta_{s+1}) + \rho_s/3 := f_1^{(y^s, z^s)}(x^{s+1}, \beta_{s+1}) + \rho_s/3 \triangleq \\ = \langle x^{s+1}, (A_1 + B_1^T) y^s + (A_2 + C_1^T) z^s \rangle - \beta_{s+1} + \rho_s/3 \geq \\ \geq \sup_{(x,\beta)} \{ f_1^s(x, \beta) \mid (x, \beta) \in X(y^s, z^s, \gamma_s) \}\ . \end{gathered}$$

Step 2. Find a $\rho_s/3$-solution (y^{s+1}, γ_{s+1}) to Problem $(\mathcal{LP}_y(x^{s+1}, z^s, \alpha_s))$, so that the following inequality takes place:

$$\begin{gathered} f_2^s(y^{s+1}, \gamma_{s+1}) + \rho_s/3 := f_2^{(x^{s+1}, z^s)}(y^{s+1}, \gamma_{s+1}) + \rho_s/3 \triangleq \\ = \langle y^{s+1}, (B_1 + A_1^T) x^{s+1} + (B_2 + C_2^T) z^s \rangle - \gamma_{s+1} + \rho_s/3 \geq \\ \geq \sup_{(y,\gamma)} \{ f_2^s(y, \gamma) \mid (y, \gamma) \in Y(x^{s+1}, z^s, \alpha_s) \}\ . \end{gathered}$$

Step 3. Find a $\rho_s/3$-solution (z^{s+1}, α_{s+1}) to Problem $(\mathcal{LP}_z(x^{s+1}, y^{s+1}, \beta_{s+1}))$, so that the following inequality holds:

$$\begin{gathered} f_3^s(z^{s+1}, \alpha_{s+1}) + \rho_s/3 := f_3^{(x^{s+1}, y^{s+1})}(z^{s+1}, \alpha_{s+1}) + \rho_s/3 \triangleq \\ = \langle z^{s+1}, (C_1 + A_2^T) x^{s+1} + (C_2 + B_2^T) y^{s+1} \rangle - \alpha_{s+1} + \rho_s/3 \geq \\ \geq \sup_{(z,\alpha)} \{ f_3^s(z, \alpha) \mid (z, \alpha) \in Z(x^{s+1}, y^{s+1}, \beta_{s+1}) \}\ . \end{gathered}$$

Step 4. Set $s := s + 1$, and loop to Step 1.

Note that the whole point $(x^0, y^0, z^0, \alpha_0, \beta_0, \gamma_0)$ is not required to start the YZ_γ-procedure, only the part (y^0, z^0, γ_0) is sufficient.

The following convergence result of the local search method described above takes place.

Theorem 3. *Suppose,* $\rho_s > 0,\ s = 0, 1, 2, \ldots,\ \sum_{s=0}^{\infty} \rho_s < +\infty$. *Then the sequence of vectors* $\sigma^s \triangleq (x^s, y^s, z^s, \alpha_s, \beta_s, \gamma_s)$*, produced by the* YZ_γ*-procedure, converges to the point* $\hat{\sigma} \triangleq (\hat{x}, \hat{y}, \hat{z}, \hat{\alpha}, \hat{\beta}, \hat{\gamma})$*, which satisfies the following inequalities:*

$$\Phi(\hat{\sigma}) \geq \Phi(x, \hat{y}, \hat{z}, \hat{\alpha}, \beta, \hat{\gamma}) \quad \forall (x, \beta) \in X(\hat{y}, \hat{z}, \hat{\gamma})\ , \tag{5}$$

$$\Phi(\hat{\sigma}) \geq \Phi(\hat{x}, y, \hat{z}, \hat{\alpha}, \hat{\beta}, \gamma) \quad \forall (y, \gamma) \in Y(\hat{x}, \hat{z}, \hat{\alpha}) \ , \tag{6}$$

$$\Phi(\hat{\sigma}) \geq \Phi(\hat{x}, \hat{y}, z, \alpha, \hat{\beta}, \hat{\gamma}) \quad \forall (z, \alpha) \in Z(\hat{x}, \hat{y}, \hat{\beta}) \ . \tag{7}$$

Note, here we present the convergence result for sequence $\{\sigma^s\}$, in contrast to the theorem from [8], where the convergence of numerical sequence $\{\Phi_s\}$ was studied only.

Definition 3. *Any 6-tuple $\hat{\sigma}$, satisfying the inequalities (5), (6), and (7), will henceforth be called a critical point to Problem ($\mathcal{P}$). If the inequalities (5), (6), and (7) are satisfied with certain accuracy at some point, then this point is called the approximate critical point.*

It can be readily seen, that this concept of a critical point to Problem ($\mathcal{P}$) is due to the structure of the proposed local search method. Moreover, the structure of the YZ_γ-procedure suggests using the following inequalities as the stopping criterion:

$$f_1^s(x^{s+1}, \beta_{s+1}) - f_1^s(x^s, \beta_s) \leq \frac{\tau}{3} \ , \tag{8}$$

$$f_2^s(y^{s+1}, \gamma_{s+1}) - f_2^s(y^s, \gamma_s) \leq \frac{\tau}{3} \ , \tag{9}$$

$$f_3^s(z^{s+1}, \alpha_{s+1}) - f_3^s(z^s, \alpha_s) \leq \frac{\tau}{3} \ , \tag{10}$$

where $\tau > 0$ is the given accuracy.

Note that summing three inequalities (8)–(10), the conventional for classical numerical methods criterion $\Phi_{s+1} - \Phi_s \leq \tau$ is obtained. Wherein usage of the system (8)–(10) instead of this inequality agrees with the goal of the local search (finding a critical point which is a partial global solution to Problem ($\mathcal{P}$) with respect to three pairs of variables).

It can be shown that when at the iteration s of the YZ_γ-procedure the system (8)–(10) is fulfilled, we obtain an approximate critical point to Problem ($\mathcal{P}$) in the sense of Definition 3.

Now let us move on to a description of a numerical experiment.

5 Computational Simulations

The both local search methods have been implemented with the help of MATLAB 7.11 R2010b [4]. Auxiliary LP problems and convex quadratic problems have been solved by IBM ILOG CPLEX 12.6.2 subroutines "cplexlp" and "cplexqp" [2], respectively, with settings on default. This package shows the considerable advantages with respect to standard MATLAB subroutines "linprog" and "quadprog". The computer with Intel Core i5-2400 CPU (3.1 GHz), 4 Gb RAM has been used.

Test hexamatrix games were randomly generated by the relevant subroutines of MATLAB by the following rules. Pseudorandom elements of matrices A_1, A_2, B_1, B_2, C_1, and C_2 have an uniform distribution and were choosing with the accuracy 10^{-3} from the interval $]-10, 10[$, when $\min\{m, n, l\} \leq 10$, and from the interval $]-\min\{m, n, l\}, \min\{m, n, l\}[$, when $\min\{m, n, l\} > 10$.

The starting point for both methods was chosen according to the formula (4). The accuracy of the stopping criteria was $\tau = 10^{-3}$.

Firstly, let us present the results of the SLSM testing on the problem series of dimension $(m + n + l)$ from $(5 + 5 + 5)$ up to $(50 + 50 + 50)$, that specified in Table 1. There was 1000, 100, and 10 problems in each series depending on the dimension of the problems (see Table 1).

In addition, the following notations have been used in Table 1: QP is the total number of convex quadratic problems solved in course of the SLSM, T stands for the total CPU time for all problems of series (in seconds); Φ^0_{avg} is the average value of the objective function of Problem $(\mathcal{P})$ at the starting point; $\hat{\Phi}_{avg}$ is the average value of the function Φ at the obtained approximately critical point; $\hat{\Phi}_{avg}$ is the worst value of the function Φ at critical points; $Failed$ is the number of problems where an approximate critical point has not been obtained.

Table 1. The SLSM testing with d.c. representation (1)–(2)

$m+n+l$	$Series$	QP	T	Φ^0_{avg}	$\hat{\Phi}_{avg}$	$\hat{\Phi}_{worst}$	$Failed$
5+5+5	1000	166907	547.85	-12.1064	-1.8737	-10.4290	8
10+10+10	1000	317630	1593.99	-11.3223	-2.1403	-6.9423	1
20+20+20	100	110261	1431.38	-20.1273	-4.1621	-8.0592	1
30+30+30	100	212920	6863.76	-26.7482	-5.5145	-10.6234	3
40+40+40	10	43440	3545.94	-33.9384	-6.9155	-14.4621	0
50+50+50	10	33877	2802.37	-37.8681	-8.2039	-10.7426	0

Note that in several problems an approximate critical point has not been obtained because the maximum number of iterations $100(m+n+l)$ has been attained. As a whole, this variant of the SLSM shows itself very slow and inefficient. It can be explained by the bad properties of the Hessian of Problem $(\mathcal{PL}(\sigma^s))$ at each iteration of the method. In particular, function $h_0(\cdot)$ is not strongly convex (see Theorem 2). In order to improve these properties let us consider another regularized d.c. representation of the function Φ:

$$\Phi(\sigma) = h_1(\sigma) - g_1(\sigma) \ , \tag{11}$$

where

$$h_1(\sigma) = h_0(x, y, z) + \mu\|\sigma\|^2, \quad g_1(\sigma) = g_0(\sigma) + \mu\|\sigma\|^2 \tag{12}$$

with a regularizing parameter $\mu > 0$ and Euclidean norm.

Thus, the functions $h_1(\sigma)$ and $g_1(\sigma)$ are strongly convex with respect to variable σ, and in accordance with convex optimization theory, the convergence of numerical methods for solving Problem $(\mathcal{PL}(\sigma^s))$ is going to be faster.

The results of the regularized SLSM testing, where $\mu = 10(m+n+l)$ are presented in Table 2 with the same notations.

It can be readily seen that the second variant of the SLSM finds a critical point much faster than the SLSM with d.c. representation (1)–(2). Especially it appears for

Table 2. The regularized SLSM testing

$m+n+l$	$Series$	QP	T	Φ^0_{avg}	$\hat{\Phi}_{avg}$	$\hat{\Phi}_{worst}$	$Failed$
5+5+5	1000	7327	25.63	-12.1064	-10.8737	-27.0894	0
10+10+10	1000	8056	42.51	-11.3223	-8.4771	-19.0162	0
20+20+20	100	1554	20.89	-20.1273	-13.5700	-20.6861	0
30+30+30	100	3696	124.73	-26.7482	-18.1116	-25.5931	1
40+40+40	10	417	23.37	-33.9384	-19.5362	-25.9790	0
50+50+50	10	583	49.53	-37.8681	-24.0294	-28.7654	0

problems of high dimensions. For example, for problems with $(40 + 40 + 40)$ dimension the time rate is more than 150 times less. Moreover, only for one problem an approximate point has not been obtained because the threshold number of iterations $10(m + n + l)$ was overcome.

On the other hand, the values $\hat{\Phi}_{avg}$ and $\hat{\Phi}_{worst}$ obtained by regularized SLSM are farther from the global value 0 than for the first variant of the method. At the same time, the main goal of the local search stage according to the Global Search Theory is the obtaining a critical point (point with "good" properties, see (5)–(7)) as fast as possible, because in course of a global search it is necessary to carry out a local search many times [10, 12]. Therefore the regularized SLSM is more suitable to apply within a global search. We hope to overcome the remaining gap to the global value by a global search procedure.

Now let us pass to the testing of the YZ_γ-procedure and its comparison with the SLSM. The YZ_γ-procedure shows promising efficiency during the preliminary computational simulation, therefore the series of test problems were increased for each dimension. In Table 3 the results of applying the YZ_γ-procedure to the problem series from $(5 + 5 + 5)$ up to $(200 + 200 + 200)$ dimension are presented. There was 10000, 1000, 100 and 10 problems in each series. In Table 3 the notation LP instead of the notation QP has been used, because the auxiliary problems within the YZ_γ-procedure are linear programs. All the other notations are the same as before.

Further, in Table 4 the results of testing the regularized SLSM with $\mu = \max\{m, n, l\}(m+n+l)$ is presented. Such a value of regularizing parameter μ was chosen for ensuring the diagonal predominance in the matrix of quadratic problem $(\mathcal{PL}(\sigma^s))$, which was built by the random matrices A_1, A_2, B_1, B_2, C_1, and C_2.

First of all, it is worth to emphasize the advantage of the YZ_γ-procedure concerning the speed of finding a critical points in random generated hexamatrix games. This advantage increases from 1.8 times for the problems of (5+5+5) dimension up to approximately 20 times for the problems of (125+125+125) and (200+200+200) dimension. Therefore, the YZ_γ-procedure is much more faster than the regularized SLSM especially for problems of high dimension.

As for $\hat{\Phi}_{avg}$ and $\hat{\Phi}_{worst}$ values for the YZ_γ-procedure they are considerably closer to 0 than for the regularized SLSM and even they are comparable with the corresponding values for the SLSM with the d.c. representation (1)–(2) (see Table 1). In particular, note that the average values of the objective function at critical points provided the YZ_γ-procedure are not far from its global value.

Table 3. The YZ_γ-procedure testing on the large series

$m+n+l$	*Series*	*LP*	*T*	Φ^0_{avg}	$\hat{\Phi}_{avg}$	$\hat{\Phi}_{worst}$	*Failed*
5+5+5	10000	99705	180.75	-12.1048	-3.0153	-13.9799	0
10+10+10	10000	140637	262.82	-11.3247	-1.9221	-7.0783	0
20+20+20	10000	215847	454.73	-19.9186	-2.5504	-7.6788	1
30+30+30	1000	27723	69.76	-26.9311	-3.0342	-8.3432	0
40+40+40	1000	36099	111.82	-32.8441	-3.3164	-7.0914	0
50+50+50	1000	46692	181.52	-38.4491	-3.6867	-8.6850	0
75+75+75	100	5352	39.02	-50.4902	-4.4565	-7.2436	0
100+100+100	100	7779	114.68	-60.8572	-4.8479	-7.3159	0
125+125+125	100	7389	172.97	-70.5397	-5.5732	-7.4939	0
150+150+150	10	870	29.62	-79.3345	-5.0982	-7.6286	0
175+175+175	10	1773	92.55	-85.8656	-6.1977	-7.9071	0
200+200+200	10	1479	109.68	-92.4432	-6.0895	-7.4830	0

Table 4. The regularized SLSM testing on the large series

$m+n+l$	*Series*	*QP*	*T*	Φ^0_{avg}	$\hat{\Phi}_{avg}$	$\hat{\Phi}_{worst}$	*Failed*
5+5+5	10000	95156	330.58	-12.1048	-10.0327	-35.4659	0
10+10+10	10000	82471	423.34	-11.3247	-8.4828	-21.6185	0
20+20+20	10000	106272	1440.18	-19.9186	-13.8200	-30.3405	0
30+30+30	1000	13119	465.79	-26.9311	-18.0002	-29.1439	0
40+40+40	1000	15961	920.04	-32.8441	-21.5384	-35.6769	2
50+50+50	1000	16835	1370.53	-38.4491	-24.8071	-37.1146	1
75+75+75	100	1695	332.06	-50.4902	-31.6184	-40.9689	0
100+100+100	100	2168	966.36	-60.8572	-37.4916	-49.7228	0
125+125+125	100	4764	3909.66	-70.5397	-41.8590	-51.6223	1
150+150+150	10	136	181.03	-79.3345	-47.6004	-54.2623	0
175+175+175	10	210	424.61	-85.8656	-52.5941	-61.2330	0
200+200+200	10	762	2186.90	-92.4432	-53.7768	-59.0646	0

So one can see that the reliability of the YZ_γ-procedure is superior than this one for the regularized SLSM (only 1 failed problem for all series versus 4 problems, respectively).

To sum up, based on the results of computational simulation, we can conclude that for the purpose of a local search, within global search algorithms, the using of the YZ_γ-procedure is more preferable.

6 Conclusion

In the paper the problem of finding a Nash equilibrium in hexamatrix games was considered. For the equivalent nonconvex optimization problem an issue of a local search was investigated. We studied two local search methods which are based on different ideas. First, we describe the special local search method (SLSM), based on the linearization of the original problem in d.c. form with respect to basic nonconvexity. Then the new local search procedure (so called the YZ_γ-procedure) was elaborated. This procedure is based on the consecutive solving of auxiliary linear programs and uses the bilinear structure of the objective function.

The results of comparative computational simulation for two local search methods certified the advantage of the YZ_γ-procedure with respect to the SLSM from the perspective of using them within the global search procedures, intended for a seeking a Nash equilibria in hexamatrix games.

Acknowledgments. This work is carried out under financial support of Russian Science Foundation (project no. 15-11-20015).

References

1. Horst, R., Tuy, H.: Global optimization. Deterministic approaches. Springer-Verlag, Berlin (1993)
2. IBM ILOG CPLEX optimization studio, `http://www-03.ibm.com/software/products/ru/ibmilogcpleoptistud`
3. Konno, H.: A cutting plane algorithm for solving bilinear programs. Math. Prog. 11, 14–27 (1976)
4. MATLAB — The language of technical computing, `http://www.mathworks.com/products/matlab/`
5. Mills, H.: Equilibrium points in finite games. J. Soc. Indust. Appl. Math. 8(2), 397–402 (1960)
6. Orlov, A.V.: Numerical solution of bilinear programming problems Comput. Math. Math. Phys. 48(2), 225–241 (2008)
7. Orlov, A.V., Strekalovsky, A.S.: Numerical search for equilibria in bimatrix games. Comput. Math. Math. Phys. 45(6), 947–960 (2005)
8. Orlov, A.V., Strekalovsky, A.S., Batbileg, S.: On computational search for Nash equilibrium in hexamatrix games. Optim. Lett. 10(2), 369–381 (2016)
9. Pang, J.-S.: Three modeling paradigms in mathematical programming. Math. Prog. Ser.B. 125(2), 297–323 (2010)

10. Strekalovsky, A.S.: Elements of nonconvex optimization [in Russian]. Nauka, Novosibirsk (2003)
11. Strekalovsky, A.S.: On a local search for reverse convex problems. In: Liberti, L., Maculan, N. (eds.) Global optimization: from theory to implementation, 33–44. Springer, New York (2006)
12. Strekalovsky, A.S.: On solving optimization problems with hidden nonconvex structures. In: Rassias, T.M., Floudas, C.A., Butenko, S. (eds.) Optimization in science and engineering, 465–502. Springer, New York (2014)
13. Strekalovsky, A.S.: On local search in d.c. optimization problems. Applied Math. and Comput. 255, 73–83 (2015)
14. Strekalovsky, A.S., Enkhbat, R.: Polymatrix games and optimization problems. Autom. Remote Control. 75(4), 632–645 (2014)
15. Strekalovsky, A.S., Gruzdeva, T.V.: Local search in problems with nonconvex constraints. Comput. Math. Math. Phys. 47(3), 381–396 (2007)
16. Strekalovsky, A.S., Orlov, A.V.: Bimatrix games and bilinear programming [in Russian]. FizMatLit, Moscow (2007)
17. Strekalovsky, A.S., Orlov, A.V., Malyshev, A.V.: Local search in a quadratic-linear bilevel programming problem. Numer. Anal. Appl. 3(1), 59–70 (2010)
18. Tao, P.D., Souad, L.B.: Algorithms for solving a class of non convex optimization. Methods of subgradients. In: Hiriart-Urruty J.-B. (ed.) Fermat days 85, 249–271. Elservier Sience Publishers B.V., North Holland (1986)
19. Vasilyev, I.L., Klimentova, K.B., Orlov, A.V.: A parallel search of equilibrium points in bimatrix games [in Russian]. Numer. Methods Prog. 8, 233–243 (2007) `http://num-meth.srcc.msu.ru/english/index.html`
20. Yanovskaya, E.B.: Equilibrium points in polymatrix games [in Russian]. Latv. Math. Collect. 8, 381–384 (1968)