On the Equivalence of Optimality Principles in the Two-Criteria Problem of the Investment Portfolio Choice

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Abstract. In this paper, we examine the problem of finding an optimal portfolio of securities by using the probability function of portfolio risk as a constraint. We obtain the value of the risk coefficient for which the problem of maximizing the expectation of the portfolio return with a probabilistic risk function constraint is equivalent to the maximizing the linear convolution of the criteria "expectation – variance". The positive correlation of portfolios returns that are solutions of different optimization problems is proved.

Keywords: optimality principles, expectation, variance, correlation, convolution of the criteria

1 Introduction

The problem on the choice of optimality principles of investor behavior in the stock market has been discussed in an extensive literature (for example, [1], [3], [5-7], [9], [12]). In this case, the development of optimality criteria for a securities portfolio involves solving the issue on the relationship between the return and risk of the portfolio. A static formulation of the problem on the portfolio selection initially was proposed by Markowitz [9]. In his studies, Markowitz used for the risk assessment a risk function defined in the metric l_2^2 (variance). Then Markowitz [10] stated the problem on the selection of an optimal portfolio as the problem of minimizing the difference between the variance and the expectation of the portfolio return. In addition, in the same book the problem of maximizing the expected return under a constraint on the variance is considered. The problem of minimizing the variance under the constraint on the return was also considered. Solutions of all these problems are efficient portfolios.

Sharpe et al. [12] has proposed to consider the probability risk functions (VAR) for finding the optimal portfolio of securities. This trend has been developed in recent papers ([1], [4], [8], [13]).

In the papers of Gorelik and Zolotova ([1], [2]) the problem on portfolio selection was considered as the problem of maximizing a linear convolution of criteria "expectation—variance" with a weight factor (risk coefficient). By the convexity of the set of attainable

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values for the expectation and variance of portfolios (in the "north-west" direction) it gives necessary and sufficient conditions for the Pareto optimality, i.e., any problem whose solution is an effective portfolio is equivalent to a given problem at a certain value of risk coefficient.

In this paper, we consider two statements of the problem on portfolio selection: the problem of maximizing the expectation with probabilistic risk function in a constraint and the problem of maximizing the linear convolution of the criteria "expectation—variance" with a weight factor. We show that the optimal choice in the problem with probabilistic risk function in a constraint leads to one of the efficient portfolios corresponding to a definite value of the risk coefficient at the variance in the problem of maximizing the linear convolution of the criteria "expectation—variance". We explicitly find also the value of weight factor in terms of the known initial parameters of the problem. An example of finding the optimal portfolio of shares of Russian companies is given. It illustrates the obtained results and demonstrates some of the characteristics of methods for optimal portfolio selection. In addition, a positive correlation of portfolios returns as the decisions of the various productions of optimization problems is proved. That allows us to speak not only about formal equivalence of models under fixed initial data but also on their uniform response to market fluctuations.

2 The equivalence of the problem of maximizing the expectation with probabilistic risk function in a constraint and the problem of maximizing the convolution "expectation—standard deviation"

We assume that the stock market is characterized by the vector of expectations of financial instruments $\bar{r} = (\bar{r}_1, ..., \bar{r}_i, ..., \bar{r}_n)$ and the covariance matrix K. We assume that the behavior of an investor whose control is the vector x (a portfolio of securities) is based on this information. Components of the portfolio are proportions of funds invested in financial instruments of the final list $(i = 1, ..., n)$.

Define an optimal portfolio as a solution of the problem of maximizing the expectation of the portfolio return, provided that the probability of a negative random value of the portfolio return does not exceed a given, sufficiently small value:

$$\max_x \bar{r}x, \quad P(rx \leq 0) \leq \varepsilon,\ xe = 1, \tag{1}$$

where ε is a given sufficiently small positive value, $e = (1, ..., 1)$, and P is the probability. Hereinafter, there is no distinction in notation of row vectors and column vectors; we assume that these vectors comply with the requirements of multiplication of matrices and vectors. The problem of selecting an optimal portfolio in (1) and problems discussed below implies the presence of short sales that take place through securities lending, which would then be repaid by the same securities (this is reflected in the absence of the nonnegative conditions for the components of the vector x).

We show that problem (1) is reduced to a problem of convex programming and its solution coincides with the solution of the problem of maximizing the linear convolution of the criteria of the expectation and the standard deviation of the random portfolio return for some weight coefficient of the standard deviation. Consider the problem

$$\max_x \bar{r}x, \quad k\bar{r}x \geq (xKx)^{1/2}, \, xe = 1, \tag{2}$$

for which the Lagrange function

$$L(x, \lambda) = \bar{r}x + \lambda(k\bar{r}x - (xKx)^{1/2}) \tag{3}$$

is defined on the set $X = \{x|xe = 1\}$, λ is the Lagrange multiplier, k is a positive coefficient.

Lemma 1. *If the convex programming problem (2) has a solution x^0 and the corresponding Lagrange multiplier is positive, $\lambda^0 > 0$, i.e., (x^0, λ^0) is a saddle point of the function (3), then x^0 is a solution of the problem*

$$\max_x [\bar{r}x - \frac{\lambda^0}{1 + \lambda^0 k}(xKx)^{1/2}], \quad xe = 1. \tag{4}$$

Proof. By the convexity, problem (2) is equivalent to the maximizing the Lagrange function $L(x, \lambda^0) = \bar{r}x + \lambda^0(k\bar{r}x - (xKx)^{1/2})$ on the set X, where the Lagrange multiplier λ^0 provides a minimum of the function $L(x^0, \lambda)$. Since $\lambda^0 > 0$, we see that the inequality constraint in problem (2) at the optimal point becomes active: $k\bar{r}x^0 = (x^0Kx^0)^{1/2}$ (otherwise, this constraint would be insignificant). After a transformation we obtain $\frac{L(x,\lambda^0)}{1+\lambda^0 k} = \bar{r}x - \frac{\lambda^0}{1+\lambda^0 k}(xKx)^{1/2}$. Then the equivalent problem takes the form (4), which was required.

Theorem 1. *Let $\{r_i\}$ be a system of random variables each of which has a normal distribution, $\bar{r}_i$ be the expectations, $K = (\sigma_{ij})_{n \times n}$ be the covariance matrix, and let the conditions of the lemma hold. Then the solution of problem (1) coincides with the solution of the problem of maximizing the linear convolution of the criteria of the expectation and the standard deviation of the random portfolio return:*

$$\max_{x \in X} [\bar{r}x - \alpha_1 (xKx)^{1/2}], \tag{5}$$

where $\alpha_1 = \frac{\lambda^0}{1+\lambda^0 d}$, $d = (\Phi^{-1}(1-2\varepsilon))^{-1}$, $\Phi(\cdot)$ is the Laplace function, λ^0 is the value of the Lagrange multiplier in problem (2).

Proof. We prove that problem (1) is reduced to problem (2) under these assumptions. The random variable rx is normally distributed, i.e., $P(rx \leq 0) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{0} e^{\frac{-(t-m)^2}{2\sigma^2}} dt$, where $m = \bar{r}x$ is the expectation of the random portfolio return and $\sigma = (xKx)^{1/2}$ is the standard deviation of the random variable rx. We transform the expression for$P(rx \leq 0)$ by using the Laplace function $\Phi(y) = \frac{2}{\sqrt{2\pi}} \int_0^y e^{\frac{-t^2}{2}} dt$: $P(rx \leq 0) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{0} e^{\frac{-(t-m)^2}{2\sigma^2}} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{-m}{\sigma}} e^{\frac{-z^2}{2}} dz$, where $z = \frac{t-m}{\sigma}$ or $t = m + \sigma z$. We have $P(rx \leq 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{\frac{-z^2}{2}} dz - \frac{1}{\sqrt{2\pi}} \int_0^{\frac{m}{\sigma}} e^{\frac{-z^2}{2}} dz = \frac{1}{2} + \Phi(0) - \Phi(\frac{m}{\sigma}) = \frac{1}{2} - \frac{1}{2}\Phi(\frac{m}{\sigma})$. By the condition, $\frac{1}{2} - \frac{1}{2}\Phi(\frac{m}{\sigma}) \leq \varepsilon$, therefore, $\frac{m}{\sigma} \geq \Phi^{-1}(1-2\varepsilon)$ or $\frac{\sigma}{m} \leq (\Phi^{-1}(1-2\varepsilon))^{-1}$. Introducing the notation $(\Phi^{-1}(1-2\varepsilon))^{-1} = d$, we arrive at the problem of convex programming (2), in which $k = d$:

$$\max_x \bar{r}x, \quad d\bar{r}x \geq (xKx)^{1/2}, \, xe = 1. \tag{6}$$

By the lemma 1, problem (6) is equivalent to problem (4), i.e., problem (1) is equivalent to (4).

If we define an optimal portfolio from the solution of the problem of maximizing the linear convolution of the criteria of the expectation and the standard deviation of the random portfolio return,

$$\max_{x \in X}[\bar{r}x - \alpha_1 (xKx)^{1/2}], \tag{7}$$

where $\alpha_1 > 0$ is the risk coefficient, then for $\alpha_1 = \frac{\lambda^0}{1+\lambda^0 d}$ problem (7) coincides with problem(4). By the lemma, $\lambda^0 > 0$, and the Laplace function takes positive values, and hence $\frac{\lambda^0}{1+\lambda^0 d} > 0$. Therefore, problem (7) for $\alpha_1 = \frac{\lambda^0}{1+\lambda^0 d}$ is equivalent to the initial problem (1). The theorem is proved.

The optimality conditions for the problem(7) due to the presence of the square root in the objective function are complicated. It is therefore desirable to find a connection of the problem (1) with a suitable convolution "expectation - variance".

3 The equivalence of the problem of maximizing the expectation with probabilistic risk function in a constraint and the problem of maximizing the linear convolution of the criteria "expectation — variance"

Now we find an optimal portfolio as a solution of the problem of maximizing the linear convolution of the expectation and variance criteria for the portfolio return with the weight coefficient $\alpha > 0$:

$$\max_x [\bar{r}x - \alpha(xKx)], \quad xe = 1. \tag{8}$$

As it is known, the covariance matrix K is nonnegative definite, and therefore (1) is a convex programming problem. Let assume that the matrix K is no degenerate ($\det K \neq 0$). We examine the following question: in which case solutions of problems (1) and (8) coincide.

Theorem 2. *Let x^0 be a solution of problem (1), the optimal value of the Lagrange multiplier in problem (2) is positive, $\lambda^0 > 0$, and the covariance matrix $K = (\sigma_{ij})_{n \times n}$ is strongly positive definite. Then there exists a value of the weight coefficient α in problem (8) such that the solutions of problems (1) and (8) coincide.*

Proof. We denote the expected return of a portfolio $\bar{r}x^0$ at the solution point of problem (1) by r_p^0, i.e., $\bar{r}x^0 = r_p^0$, and consider two equivalent problems

$$\min_x xKx, \quad \bar{r}x \geq r_p^0, \; xe = 1, \tag{9}$$

and

$$\min_x (xKx)^{1/2}, \quad \bar{r}x \geq r_p^0, \; xe = 1. \tag{10}$$

Note that problems (9) and (10) are equivalent for any r_p. By the convexity of the Pareto set in the space "expectation–standard deviation", the point x^0 as a solution of problem (5), i.e., the maximum of the linear convolution of these criteria, is Pareto-optimal. Therefore, the minimum in problem (10) cannot be less than $(x^0Kx^0)^{1/2}$. However, it satisfies the constraint in problem (10); therefore, x^0 is a solution of problem (10) and the equivalent problem (9). By the convexity, problem (9) is equivalent to the problem of minimizing the Lagrange function $L_1(x, \lambda_1^0) = xKx + \lambda_1^0(r_p^0 - \bar{r}x)$ on the set X for some value of the Lagrange multiplier λ_1^0, and this value λ_1^0 provides the maximum of the function $L_1(x^0, \lambda_1)$. This problem is equivalent to the problem of minimizing the function $xKx - \lambda_1^0\bar{r}x$ on the set X. Obviously, $\lambda_1^0 > 0$, since in the opposite case the problem is reduced to the minimizing of the variance, i.e., the constraint $\bar{r}x \geq r_p^0$ becomes insignificant. We set $\alpha = \frac{1}{\lambda_1^0}$, then $\alpha > 0$ and the solutions of problems (8) and (9) coincide and hence x^0 is a solution of problem (8), which was required.

Theorem 2 proves the existence of a value of the risk coefficient α in problem (8) for which solutions of problems (1) and (8) coincide. However, Theorem 2 allows one to find the risk coefficient only by solving problem (9). In the following assertion (Theorem 3), we obtain a value of the risk coefficient α.

The Lagrange function for problem (8) is $L(x, \lambda) = \bar{r}x - \alpha(xKx) + \lambda(1 - xe)$. Optimality conditions of portfolio lead to a system of linear algebraic equations: $\bar{r} - 2\alpha(Kx^0) - \lambda e = 0, \quad x^0e = 1$, which solution gives the optimal portfolio for the problem (8):

$$x^0(\alpha) = \frac{K^{-1}e}{eK^{-1}e} + (K^{-1}\bar{r} - \frac{eK^{-1}\bar{r}}{eK^{-1}e}K^{-1}e)\frac{1}{2\alpha}. \tag{11}$$

We show that the weight coefficient α can be expressed through the parameters of the problem (1).

Theorem 3. *Let the conditions of Theorem 2 be satisfied. If in problem (8) the weight coefficient α satisfies the equation*

$$\begin{aligned} &4(1 - \left(\tfrac{\bar{r}K^{-1}e}{eK^{-1}e}\right)^2 d^2)\alpha^2 - 4d^2\left(\tfrac{\bar{r}K^{-1}e}{eK^{-1}e}\right)\left(\bar{r}K^{-1}\bar{r} - \tfrac{(eK^{-1}\bar{r})^2}{eK^{-1}e}\right)\alpha + \\ &+ \left(\bar{r}K^{-1}\bar{r} - \tfrac{(eK^{-1}\bar{r})^2}{eK^{-1}e}\right) - d^2\left(\bar{r}K^{-1}\bar{r} - \tfrac{(eK^{-1}\bar{r})^2}{eK^{-1}e}\right)^2 = 0, \end{aligned} \tag{12}$$

where $d = (\Phi^{-1}(1-2\varepsilon))^{-1}$, $\varepsilon > 0$, then solutions of problems (1) and (8) coincide.

Proof. By Theorem 1, solutions of problems (1) and (5) coincide for $\alpha_1 = \frac{\lambda^0}{1+\lambda^0 d}$, moreover, by the lemma, the condition $\lambda^0 > 0$ leads to $d\bar{r}x^0 = (x^0 K x^0)^{1/2}$ in problem (2), to which problem (1) is reduced. Thus, a solution of problem (8) defining a portfolio (11) is also a solution of problem (1) if the following relation holds $d\bar{r}x^0(\alpha) = (x^0(\alpha)Kx^0(\alpha))^{1/2}$. Using (11) we have

$$(d\bar{r}x^0(\alpha))^2 = \left(\frac{\bar{r}K^{-1}e}{eK^{-1}e}d + \left(\bar{r}K^{-1}\bar{r} - \frac{(eK^{-1}\bar{r})^2}{eK^{-1}e}\right)\frac{d}{2\alpha}\right)^2 = \left(\frac{\bar{r}K^{-1}e}{eK^{-1}e}\right)^2 d^2 + \\ + \left(\frac{\bar{r}K^{-1}e}{eK^{-1}e}\right)\left(\bar{r}K^{-1}\bar{r} - \frac{(eK^{-1}\bar{r})^2}{eK^{-1}e}\right)\frac{d^2}{\alpha} + \left(\bar{r}K^{-1}\bar{r} - \frac{(eK^{-1}\bar{r})^2}{eK^{-1}e}\right)^2 \frac{d^2}{4\alpha^2},$$

$$x^0(\alpha)Kx^0(\alpha) = \\ = \left(\frac{K^{-1}e}{eK^{-1}e} + (K^{-1}\bar{r} - \frac{eK^{-1}\bar{r}}{eK^{-1}e}K^{-1}e)\frac{1}{2\alpha}\right) K \left(\frac{K^{-1}e}{eK^{-1}e} + (K^{-1}\bar{r} - \frac{eK^{-1}\bar{r}}{eK^{-1}e}K^{-1}e)\frac{1}{2\alpha}\right) = \\ = \left(\frac{e}{eK^{-1}e} + (\bar{r} - \frac{eK^{-1}\bar{r}}{eK^{-1}e}e)\frac{1}{2\alpha}\right) K^{-1} \left(\frac{e}{eK^{-1}e} + (\bar{r} - \frac{eK^{-1}\bar{r}}{eK^{-1}e}e)\frac{1}{2\alpha}\right) = \\ = 1 + \frac{K^{-1}e}{eK^{-1}e}(\bar{r} - \frac{eK^{-1}\bar{r}}{eK^{-1}e}e)\frac{1}{\alpha} + (\bar{r} - \frac{eK^{-1}\bar{r}}{eK^{-1}e}e)(K^{-1}\bar{r} - \frac{eK^{-1}\bar{r}}{eK^{-1}e}K^{-1}e)\frac{1}{4\alpha^2} = \\ = 1 + \left(\bar{r}K^{-1}\bar{r} - \frac{(eK^{-1}\bar{r})^2}{eK^{-1}e}\right)\frac{1}{4\alpha^2}.$$

Equating the right-hand sides of these equalities, we obtain:

$$\left(\frac{\bar{r}K^{-1}e}{eK^{-1}e}\right)^2 d^2 + \left(\frac{\bar{r}K^{-1}e}{eK^{-1}e}\right)\left(\bar{r}K^{-1}\bar{r} - \frac{(eK^{-1}\bar{r})^2}{eK^{-1}e}\right)\frac{d^2}{\alpha} + \\ + \left(\bar{r}K^{-1}\bar{r} - \frac{(eK^{-1}\bar{r})^2}{eK^{-1}e}\right)^2 \frac{d^2}{4\alpha^2} = 1 + \left(\bar{r}K^{-1}\bar{r} - \frac{(eK^{-1}\bar{r})^2}{eK^{-1}e}\right)\frac{1}{4\alpha^2}.$$

After a simple transformation we have

$$1 - \left(\frac{\bar{r}K^{-1}e}{eK^{-1}e}\right)^2 d^2 - \left(\frac{\bar{r}K^{-1}e}{eK^{-1}e}\right)\left(\bar{r}K^{-1}\bar{r} - \frac{(eK^{-1}\bar{r})^2}{eK^{-1}e}\right)\frac{d^2}{\alpha} + \\ + \left(\left(\bar{r}K^{-1}\bar{r} - \frac{(eK^{-1}\bar{r})^2}{eK^{-1}e}\right) - d^2\left(\bar{r}K^{-1}\bar{r} - \frac{(eK^{-1}\bar{r})^2}{eK^{-1}e}\right)^2\right)\frac{1}{4\alpha^2} = 0.$$

Multiplying both sides by $4\alpha^2$, we obtain the quadratic equation (12). The theorem is proved.

Example 1. Find the connection between the problems (1) and (8) for a portfolio of shares of "Aeroflot", "MTS" and "Megaphone", using statistical data of stock prices of these companies for the period from January 2013 to January 2014 [15].

To solve these problems we use a specialized program ([11], [14]), written in VB.NET programming language. This program determines the structure of the portfolio, its return mean and standard deviation for the various models and statistics data, selected by user. Shares returns of companies under consideration were determined using the daily closing prices of trading sessions. For shares of "Aeroflot", "MTS" and "Megaphone" we have the vector of expectations of shares returns $\bar{r} = (0,967; 0,189; 0,327)$ and covariance matrix

$$K = \begin{pmatrix} 0,65 & 0,466 & -0,18 \\ 0,466 & 1,678 & -0,189 \\ -0,18 & -0,189 & 0,379 \end{pmatrix}.$$

Let $\varepsilon = 0,2$ in problem (1). Then $1 - 2\varepsilon = 0,6$ and by the table of values of the Laplace function we have $\Phi^{-1}(1-2\varepsilon) = 0,85$ and hence $d = (\Phi^{-1}(1-2\varepsilon))^{-1} = 1,176$. Solving problem (6), which is equivalent to (1) by Theorem 1, for $d = 1,176$ we obtain an optimal portfolio $x^0 = (13,85;\ -7,518;\ -5,332)$. The risk coefficient α in Problem (8) found as the solution of Eq. (12) for $d = 1,176$, is $\alpha = 0,036$. Then by formula (11) we obtain the following composition of the portfolio: $x^0 = (13,042;\ -7,064;\ -4,978)$. The approximate coincidence of the solutions of problems (1) and (8) can be explained by the fact that the solution of problem (6) with nonlinear constraints obtained by using the software Mathcad turns out very inaccurate. Note that the second root of the quadratic equation (12) is $\alpha = 0,792$, according to the (11) it gives the portfolio $(0,923;\ -0,263;\ 0,34)$ which does not coincide with the solution of the problem (6).

High volatility (shares of the company "MTS" has dispersion 167.8%) leads to the fact that for small ε constraints of the problem (6) are not satisfied. Suppose that in the problem(1) $\varepsilon = 0,001$, then $1 - 2\varepsilon = 0,998$ and using the table of the Laplace function values we have $\Phi^{-1}(1-2\varepsilon) = 3,09$, $d = (\Phi^{-1}(1-2\varepsilon))^{-1} = 0,324$. The problem (6) at $d = 0,324$ has no solutions.

4 Studying the correlation dependency of return rates of optimal portfolios

Let's calculate the correlation moments of return rates of the optimal portfolios found based on different models. In a view of the above theorems it is enough to select different values of the factor α in the problem (8). First, we'll calculate the covariance matrix $\text{cov}(x^1, x^2)$ of the return rates of two arbitrary portfolios consisting of x^1 and x^2. Let M stand for the mean of a random value, r_x is the return rate of the portfolio taking random values, $\bar{r}_x$ is the expected return rate of the portfolio. According to the definition of covariance we get the following:

$$\begin{aligned}
&cov(x^1, x^2) = M[(r_{x^1} - \bar{r}_{x^1})(r_{x^2} - \bar{r}_{x^2})] = \\
&= M[(\textstyle\sum_{i=1}^n r_i x_i^1 - \sum_{i=1}^n \bar{r}_i x_i^1)(\sum_{i=1}^n r_i x_i^2 - \sum_{i=1}^n \bar{r}_i x_i^2)] = \\
&= M[\textstyle\sum_{i=1}^n (r_i - \bar{r}_i) x_i^1 \sum_{i=1}^n (r_i - \bar{r}_i) x_i^2] M[\sum_{i,j=1}^n (r_i - \bar{r}_i)(r_j - \bar{r}_j) x_i^1 x_j^2] = \\
&= \textstyle\sum_{i,j=1}^n M[(r_i - \bar{r}_i)(r_j - \bar{r}_j)] x_i^1 x_j^2 = \sum_{i,j=1}^n K_{ij} x_i^1 x_j^2.
\end{aligned}$$

Thus, the covariance of random values of return rates of two portfolios is calculated through the components of these portfolios with the help of the following formula:

$$cov(x^1, x^2) = x^1 K x^2. \tag{13}$$

Theorem 4. *Covariance cov(x^{01}, x^{02}) of the two optimal portfolios is positive.*

Proof. Let's demonstrate that if $\det\sigma \neq 0$, then $eK^{-1}e > 0$. $\langle Kx, x\rangle \geq 0$ implies that $\langle K^{-1}x, x\rangle \geq 0\ \forall x$, where $\langle\cdot,\cdot\rangle$ is the inner product of the vectors. Indeed, let's take the equation $Kx = \zeta$. If we multiply the equation by x, we have $\langle Kx, x\rangle = \langle\zeta, x\rangle \geq 0$. On the other hand, $x = K^{-1}\zeta$ and $\langle x, \zeta\rangle = \langle K^{-1}\zeta, \zeta\rangle \geq 0\ \forall\zeta$. As it is commonly known, the minimal eigenvalue of the symmetric matrix K^{-1} equals the minimum of the quadratic form $\langle K^{-1}x, x\rangle$ on a unit sphere $\langle x, x\rangle = 1$. Suppose that $\exists\tilde{x} : \langle K\tilde{x}, \tilde{x}\rangle = 0$, therefore, the minimal eigenvalue $\mu_{min} = 0$. Then, the characteristic equation $\det(K^{-1} - \mu E) = 0$, where E is the diagonal identity matrix, produces $\det K^{-1} = 0$ if $\mu_{min} = 0$. Here we reach a contradiction. It means that $\forall x \ \langle x, x \rangle = 1, \langle K^{-1}x, x \rangle > 0$. For the vector $\tilde{e} = \frac{e}{\sqrt{n}}$ belonging to the unit sphere, it holds true that $\langle K^{-1}\tilde{e}, \tilde{e} \rangle > 0$ or $n^{-1}\langle K^{-1}e, e \rangle > 0$, i.e. $eK^{-1}e > 0$.

A structure of an optimal portfolio (11) can be presented as follows

$$x^0(\beta) = C_0 + C_1\beta, \tag{14}$$

where $\beta = \frac{1}{2\alpha}$ and $C_0 = (C_{01}, \ldots, C_{0j}, \ldots, C_{0n})$, $C_1 = (C_{11}, \ldots, C_{1j}, \ldots, C_{1n})$ are determined according to the following formulas

$$C_0 = \frac{K^{-1}e}{eK^{-1}e}, C_1 = K^{-1}\bar{r} - \frac{eK^{-1}\bar{r}}{eK^{-1}e}K^{-1}e. \tag{15}$$

Consider two optimal portfolios x^{01} and x^{02}, which are determined from the solution of the problem (8) at different values of the parameter α or, in accordance with the above notation, the parameter β. According to (14), the structure of the optimal investment portfolios looks as follows: $x^{01} = C_0 + C_1\beta_1$ and $x^{01} = C_0 + C_1\beta_2$ respectively. It follows from (13) and (15) that the covariance of the two portfolios returns is $cov(x^{01}, x^{02}) = x^{01}Kx^{02} = (C_0 + C_1\beta_1)K(C_0 + C_1\beta_2) = C_0KC_0 + (C_1KC_1)\beta_1\beta_2 + (C_0KC_1)\beta_2 + (C_1KC_0)\beta_1$.

Since the matrix K is symmetric, we have the equality $C_0KC_1 = C_1KC_0$ and the following expression for covariance:

$$cov(x^{01}, x^{02}) = C_0KC_0 + (C_1KC_1)\beta_1\beta_2 + (C_0KC_1)(\beta_1 + \beta_2). \tag{16}$$

Using the properties of the inner product and (15), we have

$$\begin{aligned} C_0KC_1 &= \tfrac{K^{-1}e}{eK^{-1}e}K(K^{-1}\bar{r} - \tfrac{eK^{-1}\bar{r}}{eK^{-1}e}K^{-1}e) = \tfrac{K^{-1}e}{eK^{-1}e}(\bar{r} - \tfrac{eK^{-1}\bar{r}}{eK^{-1}e}e) = \\ &= \tfrac{1}{eK^{-1}e}(\langle K^{-1}e, \bar{r} \rangle - \tfrac{eK^{-1}\bar{r}\langle K^{-1}e, e \rangle}{eK^{-1}e}) = \tfrac{1}{eK^{-1}e}(eK^{-1}\bar{r} - \tfrac{(eK^{-1}\bar{r})(eK^{-1}e)}{eK^{-1}e}) = 0. \end{aligned}$$

Then, (16) will look as follows: $cov(x^{01}, x^{02}) = C_0KC_0 + (C_1KC_1)\beta_1\beta_2$.

Since the matrix K is non-negatively defined, then $C_1KC_1 \geq 0$ and $C_0KC_0 = \frac{K^{-1}e}{eK^{-1}e}K\frac{K^{-1}e}{eK^{-1}e} = \frac{\langle K^{-1}e, e \rangle}{(eK^{-1}e)^2} = \frac{1}{eK^{-1}e} > 0$. It means that the inequality $cov(x^{01}, x^{02}) > 0$ holds true for two optimal portfolios x^{01} and x^{02}, Q.E.D.

Comment. In the absence of short selling a similar result holds under the additional assumption of strict positive definiteness of the covariance matrix K.

Example 2. For the data of Example 1 determine the covariance of the two portfolios, which are solutions of problems (1) and (8). It was shown above, that the solution of the problem (1) at $d = 1,176$ is $x^0 = (13,85;\ -7,518;\ -5,332)$. Risk factor α in the problem (8) at $d = 1,176$ is $\alpha = 0,036$, and by (11) the structure of the portfolio is $x^0(0,036) = (13,042;\ -7,064;\ -4,978)$. Let now $\alpha = 1$ (i.e., consider the model of Markowitz [10]), then the solution of the problem (8) is the portfolio $x^0(1) = (0,804;\ -0,196;\ 0,392)$. According to (13) we have a positive correlation of optimal portfolios $cov(x^0, x^0(1)) = x^0Kx^0(1) = 4,991$.

5 Conclusion

We have considered the problem of finding an optimal portfolio of securities using the probabilistic function of portfolio risk. We have found the value of the risk coefficient in the model "expectation–variance" at which the problem of maximizing the expected return with the probabilistic risk function in a constraint is equivalent to the problem of maximizing the linear convolution of criteria "expectation–variance". Thus, if we use the model with a probabilistic risk function for the search of an optimal portfolio, the results of this study make it possible to determine the equivalent ratio of the investor to risk (the risk coefficient). The convex programming problem (6), to which the problem (1) is reduced at first, is inconvenient from the computing point of view; this is related to the type of nonlinear constraints that make it difficult to find an exact solution analytically, whereas numerical methods provide an approximate solution with large errors. The problem (8) is computationally most convenient because it is reduced to a system of linear equations. The results obtained in the present paper allow one to solve problem (8) instead of (1) for certain values of the parameters of these problems.

Positive covariance of different portfolios returns means that a particular investor control has a property of stability in a sense, that using various two-criteria decision-making models, he obtains portfolios which random returns tend to vary in the same direction. So we can talk about the robustness of the two-criteria model of portfolio formation (with the use of sub-optimization or convolution of criteria of mathematical expectation, variance, standard deviation, VAR). This result (Theorem 4), as well as the existence of a weighting factor for which the solutions of problems (1) and (8) coincide (Theorem 2) are valid for any law of return distribution.

Quantitative estimates of the parameters in Theorems 1 and 3 were obtained under the assumption of a normal distribution of returns. It should be noted that in the simulation of some processes in the economy and finance "heavy-tailed" distributions of random variables are used (eg. Pareto). However, the normal distribution is often most convenient for modeling of random processes. Moreover, according to the central limit theorem of probability theory, linear combination of a sufficiently large number of comparable variance random variables with any laws of distribution is approximately normal distributed. In addition, in this study, the formulations of problems exclude right tails.

References

1. Gorelik, V.A., Zolotova, T.V.: Criteria for evaluation and the optimality of risk in complex organizational systems. Moscow. CC RAS (2009)
2. Gorelik, V.A., Zolotova, T.V.: Some problems of the assessment of correlation of returns in investment portfolios. Control Sciences. 3, 36–42 (2011)
3. Gorelik, V.A., Zolotova, T.V.: Stability criteria for the stock market and their relationship with the awareness and the principles of investor behavior. Finance Journal. 3, 17–28 (2013)
4. Fulga, C.: Portfolio optimization under loss aversion. European journal of operational research. 1, 310-322 (2016)
5. Howison, S.D., Kelly, F.P., Willmott, P. (eds.): Mathematical Models in Finance. London Chapman & Hall (1995)

6. Ingersoll, J.E.: Theory of Financial Decision Making. London-Lanham Rowman and Littlefield (1987)
7. Javanmardi, L., Lawryshyn, Y.: A new rank dependent utility approach to model risk averse preferences in portfolio optimization. Annals of operations research. 1-2, 161-176 (2016)
8. Kibzun, A.I., Ignatov, A.N.: The two-step problem of investment portfolio selection from two risk assets via the probability criterion. Automation and Remote Control. 7, 1201-1220 (2015)
9. Markowitz, H.M.: Portfolio selection. Journal of Finance. 7, 77–91 (1952)
10. Markowitz, H.M.: Portfolio Selection: Efficient Diversification of Investment. N.-Y. Wiley (1959)
11. Prohorova, M.S.: Investigation of the relationship of the problems of maximization linear convolution "expectation - variance" and minimization variance under limiting the return. Economics, Statistics and Informatics. 3, 162–166 (2014)
12. Sharpe, W.F., Alexander, G.J., and Bailey, J.V.: Investments. Prentice Hall, Englewood Cliffs. New Jersey (1995)
13. Zhao, P., Xiao, Q.: Portfolio selection problem with Value-at-Risk constraints under non-extensive statistical mechanics. Journal of computational and applied mathematics. 298, 64-71 (2016)
14. Zolotova, T.V., Prohorova M.S.: Information aspects and tools for sustainability assessment in the stock market. Scientific notes of KnAGTU. 18, 28-34 (2014)
15. The quotations from the Moscow Interbank Currency Exchange [electronic resource]. Access mode: http://www.finam.ru/analysis/profile00008/default.asp