Modified Duality Method for Obstacle Problem

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Abstract. In this paper we proposed a modified Lagrangian functional for obstacle problem, investigated its properties. Then we construct Uzawa method for finding a saddle point, proved the convergence theorems. Some numerical examples are provided.

Keywords: Obstacle problem, sensitivity functional, duality scheme, modified Lagrangian functional, saddle point, Uzawa method

1 Introduction

We consider a modified duality method for solving the obstacle problem. The obstacle problem is a typical example of the elliptic variational inequality. Many important problems ranging from contact problems in continuum mechanics to option pricing in computational finance can be formulated as the obstacle problem. See for instance the book [1] where many of these applications are described, as well as the classical literature on this problem. Finally, apart from their practical relevance, obstacle problems are fascinating mathematical objects of their own value. The basic properties of the solution, including existence and uniqueness, were established by Lions and Stampacchia [2]. Many approaches for the numerical solution of obstacle problems have been suggested and pursued [3–5]. The main existing numerical methods for the solution of obstacle problems in particular, are mathematical programming approach, and schemes based on penalty formulations and Lagrangian multiplier formulations [6–8].

The modified Lagrangian functional for the first time were developed and investigated for solving the problem of finite-dimensional optimization. Their emergence was related to the fact that classical Lagrangian functionals that are linear functions of the dual variables are not suitable for solving the singular optimization problems. The construction of modified Lagrangian function (functional) actually comprises regularization of dual variables. In last time the Lagrangian multiplier method is successfully applied to the solution of infinite-dimensional variational inequalities in mechanics [9–11]. In this paper the duality scheme based on the modified Lagrangian functional is examined for the obstacle problem.

The paper is structures as follows. In Sect. 2, we introduce the obstacle problem. In Sect. 3, we present the sensitivity functional for the obstacle problem and we investigate

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its properties. In Sect. 4, we propose the modified Lagrangian functional and construct the Uzawa method for finding a saddle point, prove the convergence theorems. Finally, in Sect. 5 we present the numerical examples. Sect. 6 contains some concluding remarks.

2 The Obstacle Problem

Let us consider a simple model for a problem with obstacle. There are horizontal circular wire and a membrane hanging on this wire (see fig. 1(a)). We assume that this membrane is horizontal and above a plate. This plate is a obstacle for plate's deflection. When we load the membrane with a force f in the vertical direction, it undergoes deflection (fig. 1(b) — the case without the plate). If there is the place we get a contact area between the membrane and the obstacle which is the plate. This contact area is called the coincidence set (fig 1(c)).

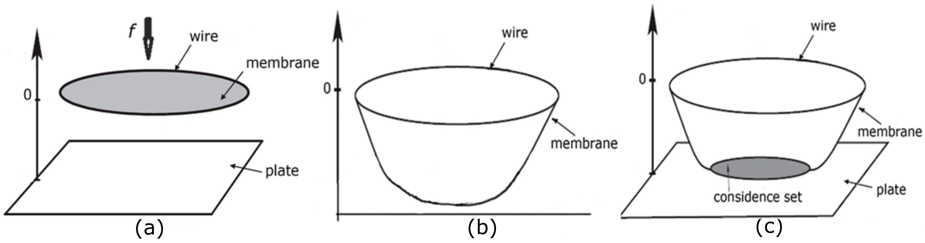

Fig. 1. Membrane over a plate

The obstacle problem can be described as follows: find the equilibrium position $u(x)$, $x \in \Omega \subset R^2$ of an elastic membrane constrained to lie above a given obstacle $\psi(x)$ under an external force $f(x)$. Then $u(x)$ is the formal solution of the boundary problem

$$-\Delta u(x) = f(x) \quad \text{a.e. in } N = \{x \in \Omega : \; u(x) > \psi(x)\},$$
$$u(x) = \psi(x) \quad \text{in } \Lambda = \{x \in \Omega : \; u(x) = \psi(x)\},$$
$$u(x) = 0, \quad x \in \Gamma.$$

Here $\Omega \subset R^2$ be a bounded and open domain with smooth boundary Γ; where f is an element $L_2(\Omega)$; ψ is an element of $H_0^1(\Omega)$ with $\psi \leq 0$ on Γ.

Therefore the obstacle problem can be posed as a variation problem. Set

$$J(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 \, d\Omega - \int_{\Omega} f \, v \, d\Omega \tag{1}$$

and

$$\mathcal{K} = \{w : \; w \in H_0^1(\Omega), \; w \geq \psi \text{ a.e. in } \; \Omega\}.$$

The set $\mathcal{K}$ is not empty.

We consider the following variational inequality:

$$\begin{cases} \text{Find } v \in \mathcal{K} \text{ such that} \\ J(v) - \min\,. \end{cases} \tag{2}$$

The problem (2) is called the obstacle problem, and the set $\mathcal{K}$ is called the set of constraints [1].

The functional $J(v)$ is strongly coercive in $H_0^1(\Omega)$, it means $J(v) \to +\infty$ for $\|v\|_{H^1(\Omega)} \to \infty$. Hence the problem (2) has a unique solution u. It is known if $\Delta\psi(x) \in L_2(\Omega)$, then u is an element $H^2(\Omega)$ (see [1, 12]).

In [13] the duality method with classical Lagrangian functional

$$L(v,l) = J(v) + \int_\Omega l\ (\psi - v)\ d\Omega, \quad v \in H_0^1(\Omega),\ l \in L_2(\Omega)$$

is consider, and it shows that a point $(v^*, l^*) = (u, -\Delta u - f)$ is a unique saddle point of the Lagrangian functional $L(v,l)$,

$$L(u,l) \le L(u, -\Delta u - f) \le L(v, -\Delta u - f), \quad \forall\ v \in H_0^1(\Omega),\ l \in (L_2(\Omega))^+$$

where

$$(L_2(\Omega))^+ = \{w \in L_2(\Omega) :\ w \ge 0 \text{ a.e. in } \Omega\}.$$

It is known that duality methods based on classical Lagrangian functionals don't guarantee the convergence of the available methods for finding saddle points in variational inequalities in mechanics. The convergence of duality methods can be established according to direct variable only. In addition the step according to dual variable must be sufficiently small.

3 The Sensitivity Functional

For arbitrary $m \in L_2(\Omega)$ we introduce the set

$$K_m = \{v \in H_0^1(\Omega) : \psi - v \le m \text{ a.e. on } \Omega\}$$

and for all functions $m \in L_2(\Omega)$ define the sensitivity functional

$$\chi(m) = \begin{cases} \inf\limits_{v \in K_m} J(v), & \text{if } K_m \ne \varnothing, \\ +\infty, & \text{otherwise.} \end{cases}$$

It is easy to see that if a function $m \in L_2(\Omega)$ is lower bounded on Ω, the corresponding set K_m is not empty and $\inf\limits_{v \in K_m} J(v) > -\infty$ [14]. The set K_m can be empty if $m \in L_2(\Omega) \setminus H^1(\Omega)$ and not lower bounded on Ω. Then $\chi(m)$ is a proper convex functional on $L_2(\Omega)$, but it's effective domain $\text{dom}\chi = \{m \in L_2(\Omega) :\ \chi(m) < +\infty\}$ does not coincide with $L_2(\Omega)$. Notice that $\text{dom}\chi$ is a convex but not closed set. In this case, $\overline{\text{dom}}\chi = L_2(\Omega)$.

Since the functional $J(v)$ is coercive then the problem

$$\begin{cases} \text{Find } v \in K_m \text{ such that} \\ J(v) - \min\,. \end{cases}$$

has a unique solution u_m for any $m \in dom\chi$. Hence $\chi(m) = J(u_m)$ and $\chi(0) = \inf\limits_{v \geq \psi} J(v) = J(u)$.

We have the next result.

Theorem 1. *The sensitivity functional $\chi(m)$ is weakly lower semicontinuous on $L_2(\Omega)$.*

Since $\chi(m)$ is convex functional, it is suffices to show that $\chi(m)$ is lower semicontinuous on $L_2(\Omega)$ (in norm space $L_2(\Omega)$).

We take an arbitrary sequence $\{m_i\} \subset L_2(\Omega)$ such that $\bar{m} = \lim\limits_{i\to\infty} m_i$. We can show that the conditions
1) $\lim\limits_{i\to\infty} \chi(m_i) = +\infty$ if $\bar{m} \notin dom\chi$;
2) $\underline{\lim}_{i\to\infty} \chi(m_i) \geq \chi(\bar{m})$ if $\bar{m} \in dom\chi$
are satisfied. Desirable property of lower semicontinuous for $\chi(m)$ follows from these conditions. The proof is complete.

For an arbitrary $l \in L_2(\Omega)$, we consider the functional

$$F_l(m) = \chi(m) + \int\limits_{\Omega} l\, m\, d\Omega + \frac{r}{2} \int\limits_{\Omega} m^2\, d\Omega,$$

where $r > 0$ is a constant. The functional $F_l(m)$ is very important for constructing the duality methods based on modified Lagrangian functionals [11].

For a fixed $l \in L_2(\Omega)$, we examined the functional $F_l(m)$ for $m \in L_2(\Omega)$. From theorem 1 follows that $F_l(m)$ is a weakly semicontinuous functional on $L_2(\Omega)$.

Theorem 2. *The functional $F_l(m)$ is coercive in $L_2(\Omega)$.*

Since $\chi(m)$ is a lower semicontinuous functional, then the epigraph of sensitivity functional

$$epi\chi \equiv \{(v, a) \in L_2(\Omega) \times R\colon \chi(v) \leq a\}$$

is a convex closed set in $L_2(\Omega) \times R$. According Mazur separation theorem [15, p. 164] there are $\alpha \in L_2(\Omega)$ and $\gamma \in R$, such that

$$\int\limits_{\Omega} \alpha\, m\, d\Omega + \chi(m) + \gamma \geq 0 \quad \forall\, m \in dom\chi.$$

Hence the estimate

$$F_l(m) \geq -\int\limits_{\Omega} \alpha\, m\, d\Omega + \int\limits_{\Omega} l\, m\, d\Omega + \frac{r}{2} \int\limits_{\Omega} m^2\, d\Omega + \gamma \quad \forall\, m \in L_2(\Omega)$$

is satisfied and $F_l(m) \to +\infty$ if $\|m\|_{L_2(\Omega)} \to \infty$.
The proof is complete.

Therefore for any $l \in L_2(\Omega)$ there exists a unique element

$$m(l) = \arg \min_{m \in L_2(\Omega)} F_l(m).$$

It is obvious that $m(l) \in \mathrm{dom}\chi$.

Let us introduce the function

$$\Phi(l) = \chi(m(l)), \qquad \forall l \in L_2(\Omega).$$

Theorem 3. *The function $\Phi(l)$ is continuous in $L_2(\Omega)$.*

Since $F_l(m)$ is a strongly convex functional, then the inequality

$$\chi(m(l)) + \int_\Omega l\, m(l)\, d\Omega + \frac{r}{2}\int_\Omega (m(l))^2\, d\Omega + \frac{r}{2}\|m(l) - m\|^2_{L_2(\Omega)} \leq$$
$$\leq \chi(m) + \int_\Omega l\, m\, d\Omega + \frac{r}{2}\int_\Omega m^2\, d\Omega$$

is fulfilled for a given element $l \in L_2(\Omega)$ and every $m \in L_2(\Omega)$.

Choose two elements l', l'' on $L_2(\Omega)$. Let $m' = m(l')$ and $m'' = m(l'')$. Last inequality implies the relations

$$\chi(m') + \int_\Omega l'\, m'\, d\Omega + \frac{r}{2}\int_\Omega (m')^2\, d\Omega + \frac{r}{2}\|m' - m''\|^2_{L_2(\Omega)} \leq$$
$$\leq \chi(m'') + \int_\Omega l'\, m''\, d\Omega + \frac{r}{2}\int_\Omega (m'')^2\, d\Omega, \tag{3}$$

$$\chi(m'') + \int_\Omega l''\, m''\, d\Omega + \frac{r}{2}\int_\Omega (m'')^2\, d\Omega + \frac{r}{2}\|m'' - m'\|^2_{L_2(\Omega)} \leq$$
$$\leq \chi(m') + \int_\Omega l''\, m'\, d\Omega + \frac{r}{2}\int_\Omega (m')^2\, d\Omega. \tag{4}$$

Combining (3) and (4), we find that

$$r\|m' - m''\|^2_{L_2(\Omega)} \leq \int_\Omega (l' - l'')(m'' - m')\, d\Omega, \tag{5}$$

From (5), we derive

$$\|m' - m''\|_{L_2(\Omega)} \leq \frac{1}{r}\|l' - l''\|_{L_2(\Omega)}. \tag{6}$$

Relations (3) and (4) also imply the two-sided inequality

$$\int_\Omega l''(m''-m')\,d\Omega+\frac{r}{2}\int_\Omega((m'')^2-(m')^2)\,d\Omega\le\chi(m')-\chi(m'')\le$$
$$\le\int_\Omega l'(m''-m')\,d\Omega+\frac{r}{2}\int_\Omega((m'')^2-(m')^2)\,d\Omega.$$

Let l'' approaches l' in $L_2(\Omega)$. The above two-sided inequality and relation (6) lead to the equality $\lim_{l''\to l'}\Phi(l'')=\Phi(l')$. Hence the theorem is proved.

4 Modified Duality Method

We define the modified Lagrangian functional on the set $H_0^1(\Omega)\times L_2(\Omega)$ as

$$M(v,l)=J(v)+\frac{1}{2r}\int_\Omega\left(\left((l+r(\psi-v))^+\right)^2-l^2\right)\,d\Omega,$$

where $w^+=\max\{w,0\}$, $r>0$ — constant.

A point (v^*,l^*) is called a saddle point of functional $M(v,l)$ if two-sided inequality

$$M(u^*,l)\le M(u^*,l^*)\le M(v,l^*)$$

is fulfilled for any $v\in H_0^1(\Omega)$, $l\in L_2(\Omega)$. The sets of saddle points of modified and classical Lagrangian functionals are equal[16].

Let us introduce the dual functional

$$\underline{\mathrm{M}}(l)=\inf_{v\in H_0^1(\Omega)}\mathrm{M}(v,l)=$$
$$=\inf_{v\in H_0^1(\Omega)}\left\{J(v)+\frac{1}{2r}\int_\Omega\left(\left((l+r(\psi-v))^+\right)^2-l^2\right)\,d\Omega\right\}. \tag{7}$$

We can write the another presentation the functional $\underline{\mathrm{M}}(l)$ with using the sensitivity functional:

$$\underline{\mathrm{M}}(l)=\inf_{m\in L_2(\Omega)}\left\{\chi(m)+\int_\Omega l\,m\,d\Omega+\frac{r}{2}\int_\Omega m^2\,d\Omega\right\}=\inf_{m\in L_2(\Omega)}F_m(l). \tag{8}$$

If (v^*,l^*) is a saddle point of $M(v,l)$, then v^* is a solution of problem (2) and l^* is a solution of dual problem

$$\begin{cases}\text{Find } l\in L_2(\Omega)\text{ such that}\\ \underline{\mathrm{M}}(l)-\max.\end{cases} \tag{9}$$

From theorems 2, 3 and inequality (6) it follows that convex functional $(-\underline{\mathrm{M}}(l))$ is a continuous functional in $L_2(\Omega)$.

We have the next theorem [16].

Theorem 4. *The dual functional* $\underline{\mathrm{M}}(l)$ *is Gáteaux differentiable in* $L_2(\Omega)$ *and its derivative* $\nabla\underline{\mathrm{M}}(l)$ *satisfies the Lipschitz condition with the constant* $1/r$*; that is, for all* l'*,* $l'' \in L_2(\Omega)$*, it holds that*

$$\|\nabla\underline{\mathrm{M}}(l') - \nabla\underline{\mathrm{M}}(l'')\|_{L_2(\Omega)} \leq \frac{1}{r}\|l' - l''\|_{L_2(\Omega)}.$$

It is easy to show [16] that $\nabla\underline{\mathrm{M}}(l) = m(l) = \max\{-l/r, \psi - v\}$.

Since the gradient of the functional $\underline{\mathrm{M}}(l)$ satisfies the Lipschitz condition, the dual problem (9) can be solved by using the gradient method for maximizing a functional (see [17])

$$l^{k+1} = l^k + r\nabla\underline{\mathrm{M}}(l^k), \quad k = 0, 1, 2, \dots \quad (l^0 \in L_2(\Omega) \text{ is given}).$$

The gradient method can be written as

$$l^{k+1} = l^k + r\max\{-l^k/r, \psi - v\} = (l^k + r(\psi - v))^+. \tag{10}$$

Theorem 5. *The sequence* $\{l^k\}$ *constructed by the gradient method (10) satisfies the limit equality* $\lim\limits_{k\to\infty} \|m(l^k)\|_{L_2(\Omega)} = 0$.

The proof of this theorem can be found in [16], it is analogous to that of the theorem [17, p. 31].

The gradient method (10) can be used for construct an algorithm for solving problem (2) :

$$\begin{aligned} (i) \qquad & u^{k+1} = \arg\min_{v\in H_0^1(\Omega)} \mathrm{M}(v, l^k); \\ (ii) \qquad & l^{k+1} = l^k + r\nabla\underline{\mathrm{M}}(l^k) = (l^k + r(\psi - u^{k+1}))^+, \end{aligned} \tag{11}$$

i=0,1,2,...; $l^0 \in L_2(\Omega)$ is given.

We can show that the sequence $\{u^k, l^k\}$ is bounded sequence in $H^1(\Omega) \times L_2(\Omega)$.

Theorem 6. *The algorithm (11) converges with respect to the functional; that is,*

$$\lim_{k\to\infty} J(u^k) = \min_{v\in\mathcal{K}} J(v) = J(u).$$

As before, u *is a solution of the problem (2).*

Indeed, the sequence $\{l^k\}$ is bounded sequence in $L_2(\Omega)$, and the functional $\chi(m)$ is weakly lower semicontinuous on $L_2(\Omega)$, which yields

$$\varliminf_{k\to\infty} \left\{ \chi(m(l^k)) + \int_\Omega l^k\, m(l^k)\, d\Omega + \frac{r}{2}\int_\Omega (m(l^k))^2\, d\Omega \right\} =$$
$$= \varliminf_{k\to\infty} \chi(m(l^k)) \geq \chi(0) = J(u^*).$$

On the other hand, we have from the definition $\underline{\mathrm{M}}(l)$

$$\underline{\mathrm{M}}(l^k) = \chi(m(l^k)) + \int\limits_{\Omega} l^k\, m(l^k)\, d\Omega + \frac{r}{2}\int\limits_{\Omega} (m(l^k))^2\, d\Omega =$$

$$= \inf_{m \in L_2(\Omega)} \left\{ \chi(m) + \int\limits_{\Omega} l^k\, m\, d\Omega + \frac{r}{2}\int\limits_{\Omega} m^2\, d\Omega \right\} \leq \chi(0), \quad k = 0, 1, 2, \ldots.$$

Therefore,

$$\overline{\lim_{k\to\infty}} \left\{ \chi(m(l^k)) + \int\limits_{\Omega} l^k\, m(l^k)\, d\Omega + \frac{r}{2}\int\limits_{\Omega} (m(l^k))^2\, d\Omega \right\} \leq \chi(0).$$

Consequently, there exists the limit

$$\lim_{k\to\infty} \left\{ \chi(m(l^k)) + \int\limits_{\Omega} l^k\, m(l^k)\, d\Omega + \frac{r}{2}\int\limits_{\Omega} (m(l^k))^2\, d\Omega \right\} = \chi(0) = J(u^*).$$

Now, theorem 5 implies that

$$\lim_{k\to\infty} J(u^k) = \lim_{k\to\infty} \chi(m(l^k)) = \chi(0) = J(u).$$

The proof is complete.

By using the convexity of $(-\underline{\mathrm{M}}(l))$ and the theorem 5 we get the following estimate

$$\underline{\mathrm{M}}(l^k) - \underline{\mathrm{M}}(l^*) \leq \|m(l^k)\|_{L_2(\Omega)} \|l^k - l^*\|_{L_2(\Omega)}.$$

Hence, $\lim_{k\to+\infty} \underline{\mathrm{M}}(l^k) = \underline{\mathrm{M}}(l^*)$. It means that algorithm (11) converges with respect to dual functional. This fact can not be shown for the classical Lagrangian functional. The convergence of this algorithm with respect to the argument u^k was examined in [16].

5 A Numerical Example

In this section we present some numerical experiments in solving an obstacle problem by using the algorithm (11).

For the numerical realization (11) we use the finite element method. Suppose that the boundary Γ is polygonal. For a triangulation T of Ω, let $h = h(T)$ be the max of the lengths of the edges. Then T satisfies the shape regularity and the maximum angle condition if

(a) there is a positive constant ρ such that for any $\tau \in T$, there is a disk B of radius r with $B \subset \tau$ and $\rho h \leq r < h$,

(b) maximum angle $\leq \pi/2$.

We call a family of triangulations regular if each triangulation in it satisfies (a) and (b) with ρ uniform for the family. Given a triangulation T_h, let $\bar{V}_h = V_h(T_h)$ denote the collection of all $H^1(\Omega)$ functions which are affine on each triangle in T_h; $\bar{V}_h$ is the space of continuous piecewise linear functions over T_h. Take $V_h = \bar{V}_h \cap H_0^1(\Omega)$. For $v \in C^0(\bar{\Omega})$, let $v_h \in \bar{V}_h$ be the interpolant of v; $v = v_h$ at each vertex in T_h. Define $\mathcal{K}_h = \{v_h \in V_h\colon v_h \geq \psi_h\}$.

The discrete approximation of u is given by $u_h \in \mathcal{K}_h$, where u_h is a solution the next problem

$$\begin{cases} \text{Find } v_h \in \mathcal{K}_h \text{ such that} \\ J(v_h) - \min . \end{cases}$$

Let k be an integer number denoting the iteration parameter. The algorithm presented in Sect. 4, to solve the obstacle problem, can be expressed as follows.
Step 0 (Initialization). Given an element l_h^0 in V_h.
Step 1. Find a solution $u_h^{k+1} \in V_h$ of the problem

$$\begin{cases} \mathrm{M}(v_h, l^k) = J(v_h) + \dfrac{1}{2r} \displaystyle\int\limits_{\Omega} \left((l^k + r(\psi - v_h))^+ \right)^2 \, d\Omega \to \min, \\ v_h \in V_h . \end{cases}$$

Step 2. Compute

$$l_h^{k+1} = (l_h^k + r([\psi]_h - u_h^{k+1}))^+ .$$

Step 3. Repeat steps 1 and 2 until a stopping criterion is satisfied.

Let Ω be a square, $\Omega = \{(x_1, x_2)\colon 0 < x_1 < 1,\ 0 < x_2 < 1\}$. We define an obstacle in a ring, and the parameters problem (2) so that its solution has the form shown in Fig. 1. As a result, we obtain the numerical solution of obstacle problem shown in Figs. 2.

In Fig. 3 a dashed line (blue) is a graph solution of the problem without obstacle, a thin solid line (green) is a graph of the function that defines an obstacle, and a thick line (red) a graph solution of the obstacle problem.

The numerical experiments demonstrate the possibility to use Uzawa algorithm for modified Lagrange functional. In paper [18] it was found that for problems with constraints on the boundary the best convergence rate is achieved for large values r ($r = 10^6$, 10^8). In this work we we have shown that the optimum value of the parameter r is in the range $20 - 200$.

6 Conclusion

In this paper, we have considered a modified Lagrangian functional for the obstacle problem. The modified Lagrangian functionals considered in the present paper are analogs of the corresponding modified functions constructed to solve the finite-dimensional optimization problems. The duality methods based on the modified Lagrangian functionals offer a convenient and efficient tool to solve the infinite-dimensional variational inequalities of mechanics.

The author makes no attempt to compare the effectiveness of the proposed method with other methods of numerical optimization with constraints. The objective of this

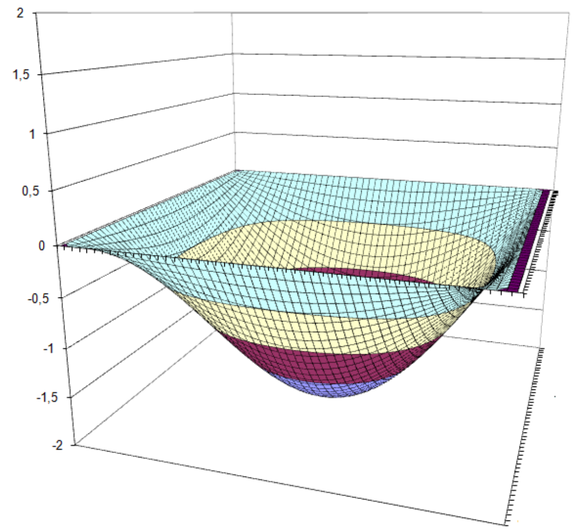

Fig. 2. A solution of the problem without obstacles

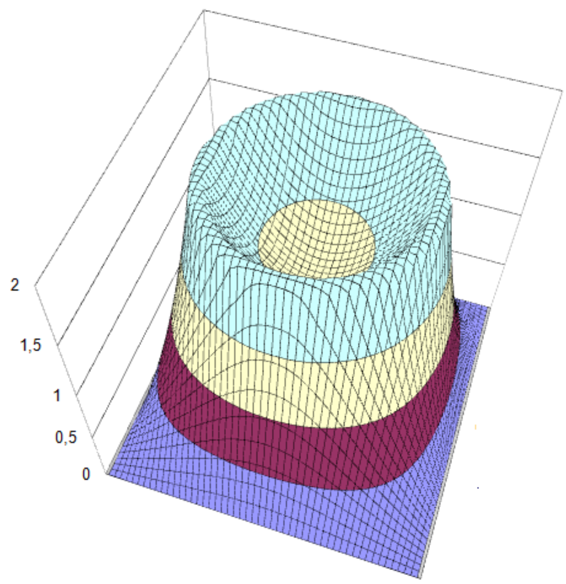

Fig. 3. A solution of the obstacle problem

paper was demonstrate the application of the duality scheme based on the modified Lagrangian functional for the problem with constraints in the domain.

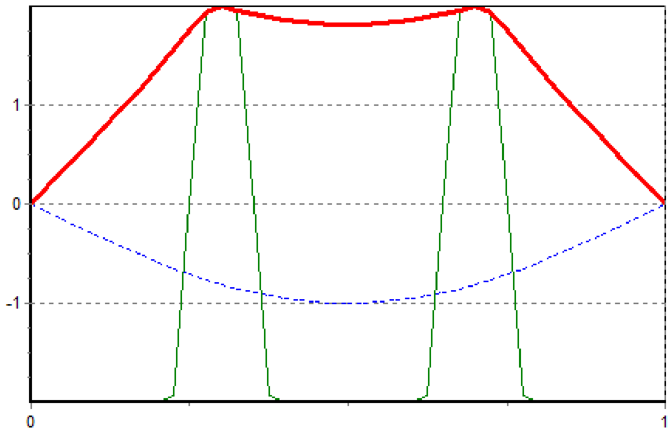

Fig. 4. A solution of the obstacle problem, $x_2 = 0$

References

1. Friedman, A.: Variational Principles and Free-boundary Problems. John Wiley, New York (1982)
2. Lions, J.L., Stampacchia G.: Variational Inequalities. Commun. Pure Appl. Math. 20, 493–519 (1967)
3. Glowinski, R.: Numerical Methods for Nonlinear Variational Problems. Springer- Verlag, Berlin-Heidelberg-New York (1984)
4. Xue, L., Cheng, X. L.: An Algorithm for Solving the Ostacle Problems. Comp. and Math. with Appl., 48, 1651–1657 (2004)
5. Brugnano, L., Sestini, A,: Iterative Solution of Piecewise Linear Systems for the Numerical Solution of Obstacle Problems. J. of Numerical Analysis, Industrial and Applied Mathematics 6, 67–82 (2011)
6. Zulehner, W.: Analysis of iterative Methods for Saddle Point Problems: a Unified Approach. Math. Comp. 71, 479–505 (2002)
7. Karkkainen, T., Kulisch, K., Tarvainen, P.: Augmented Lagrangian Active Set Methods for Obstacle Problems. J. of Opt. Theory and Appl. 119, 499–533 (2003)
8. Graser, C., Kornhuber, R.: Multigrid Methods For Obstacle Problems. Review Article. J. Comp. Math. 27, No.1, 1–44 (2009)
9. Namm, R.V., Vikhtenko, E.M.: Modified Lagrangian Functional for Solving the Signorini Problem with Friction. Advances in Mechanics Research, Nova Science Publishers, New-York, 1, 435–446 (2011)
10. Namm R.V., Woo, Gyungsoo, Shu-Sen Xie, and Yi Sucheol: Solution of Semicoercive Signorini Problem Based on a Duality Scheme with Modified Lagrangian Functional. J. Korean Math. Soc. 49 , No. 4, 843– 854 (2012)
11. Namm, R.V., Vikhtenko, E.M., Woo, G.: Sensitivity Functionals in Contact Problems of Elasticity Theory. J. Comp. Math. and Math. Phys. 54, No. 7, 1190–1200 (2014)
12. Kinderlehrer, D., Stampacchia, G.: An Introduction to Variational Inequalities and Their Applications. Academic Press, New York, (1980)
13. Kaplan, A.A.: Variational inequalities and semi-infinite optimization convex problems. Novosibirsk, Institute of Mathematics, preprint No 27 (1989)

14. Duvaut, G., Lions, J.L.: Inequalities in Mechanics and Physics. Springer-Verlag, Berlin (1976)
15. Fučik, S., Kufner, A.: Nonlinear Differential Equations. Mir, Moscow (1988)
16. Vikhtenko, E.M.: The Lagrangian Multipliers Method for Obstacle Problem. Vestnik TOGU, No 2, 35–46 (2010)
17. Polyak, B.T.: Introduction to Optimization. Nauka, Moscow (1980)
18. Namm, R.V., Sachkov, S.A.: Solving the Quasi-Variational Signorini Inequality by the Method ff Successive Approximations. J. Comp. Math. and Math. 49, No 5, 776–785 (2009)