Stochastic Optimal Growth Model with S-Shaped Utility Function

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Abstract. In this paper we examine the problem of finding the optimal solution of the stochastic dynamic growth problem with infinite time horizon for a decision maker described in the prospect theory (PT– decision maker) that has asymmetric attitude with respect to a reference level of consumption. We compare the optimal solution of the PT– decision maker with the optimal solution of the decision maker with the classic power utility. Numerical results show that the optimal behavior of PT– decision maker is quite different from the behavior of the power utility investor: the value functions of PT– decision maker are still monotone increasing, but non-concave; the optimal investment strategies of PT– decision maker are non-monotone.

Keywords: stochastic dynamic programming; prospect theory; optimal growth model.

1 Introduction

Let us consider the following stochastic optimal growth model [1]

$$V(k_0, \theta_0) = \max_{\{c_t\}} \mathbb{E}\left\{\sum_{t=0}^{\infty} \beta u(c_t)\right\}, \; c_t = F(k_t, \theta_t) - k_{t+1}, \tag{1}$$

where c_t is consumption, k_t is the capital invested at time t, $(\theta_t)_{t\geq 0}$ is stochastic technology process, $F(\cdot,\cdot)$ is a net production function, $\mathbb{E}(\cdot)$ is the expectation operator, V is the value function, k_0, θ_0 are initial states of capital and technology processes respectively, $0 < \beta < 1$.

The process $(k_t)_{t\geq 0}$ is the discrete time continuous state process. We will suppose that the process $(\theta_t)_{t\geq 0}$ is a discrete-time Markov chain, i.e. is the stochastic process satisfying the Markov property with the state space $S = \{s_1, \dots, s_d\}$ and transition matrix $\Pi = (\pi_{jk})$, $\pi_{jk} = \mathbb{P}(s_{t+1} = s_k | s_t = s_j)$.

The classical theory of investment considers a decision maker with a concave utility function u. However, the works [2], [3], [4] provide examples and demonstrations showing that under the conditions of laboratory experiments the predictions of expected utility theory are regularly disturbed. Furthermore, a new theory was proposed [2]

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— the prospect theory (PT) which accounts for people's behavior in decision-making under risk in the experiments where the traditional theory of expected utility failed.

We will assume that PT– decision maker has a reference level of consumption X. Then the deviation of consumption from reference point X on the date t is equal to the difference between a real consumption c_t and the reference level X (decision maker compares the real consumption c_t with X at the moment t and if $c_t > X$, he/she considers the deviation $c_t - X$ as a gain; in the case $c_t < X$, decision maker thinks it to be a loss $X - c_t$. In this paper we consider the stochastic dynamic problem for PT– decision maker assuming that the utility function is s-shaped, i.e. is convex over losses and is strictly convex over gains. The problem of finding the optimal path for PT– decision maker is to maximize the unconditional expectation of the infinite sum of discounted values of the s-shaped utility function at discrete time moments.

In the beginning, we briefly present the summary of main ideas of the prospect theory, and then we proceed to the problem of finding the optimal solution of the stochastic dynamic problem with infinite time horizon for PT– decision maker. We compare the optimal solution of PT– decision maker with the optimal solution of the decision maker with the classic power utility.

While such a problem has not been considered by other authors, there are research papers examining similar ones. For example, M. Dempster [5] considers the problem of asset liability management for individual households with S-shaped utility functions as the multistage stochastic programming problem rather than stochastic control problem. The work [6] studies stochastic control problems using performance measures derived from the cumulative prospect theory. This thesis solves the problem of evaluating Markov decision processes (MDPs) using CPT-based performance measures. The paper [7] introduces the concept of a Markov risk measure and the author uses it to formulate risk-averse control problems for a finite horizon model and a discounted infinite horizon model. For both Markov models the paper [7] derives risk-averse dynamic programming equations and a value iteration method.

2 PT– decision maker

Prospect theory (PT) has three essential features:

- decision maker makes investment decisions based on deviation of his/her final consumption from a reference point X and not according to his/her final consumption, i.e. PT-decision maker concerned with deviation of his/her final consumption from a reference level, whereas Expected Utility decision maker takes into account only the final value of his/her consumption.
- utility function is S-shaped with turning point in the origin, i.e. decision maker reacts asymmetrical towards gains and losses;
- decision maker evaluates gains and losses not according to the real probability distribution per ce but on the basis of the transformation of this real probability distribution, so that decision maker's estimates of probability are transformed in the way that small probability (close to 0) is overvalued and high probability (close to 1) is undervalued.

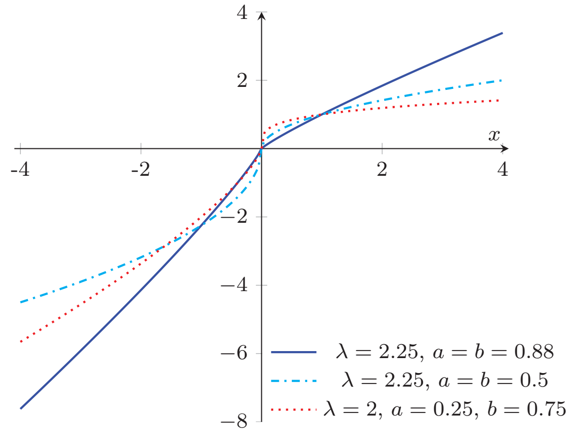

Fig. 1. The plot of the value function $v(\cdot - X)$ for different a, b, λ and $X = 0$

Let $\mathbb{X} = \{x_1, \dots, x_n\}$ is the set of all possible outcomes, X be the reference point and x refers to the element of $\mathbb{X}$. The deviation $x - X$ can be negative (loss) and positive (gain). PT includes three important parts: a PT– value function over outcomes, $v(\cdot - X)$; a weighting function over probabilities, $w(\cdot)$; PT-utility as unconditional expectation of the PT– value function v under probability distortion w.

Definition 1. *The PT– value function derives utility from gains and losses and is defined as follows [3]:*

$$v(x - X) = \begin{cases} (x - X)^a, & \text{if } x \geq X, \\ -\lambda(X - x)^b, & \text{if } x < X. \end{cases} \tag{2}$$

Fig. 1 plots the value function for different values of α, β, λ. Note that the value function is convex over losses if $0 \leq b \leq 1$ and it is strictly convex if $0 < b < 1$. The PT– decision maker dislikes losses with a factor of $\lambda > 1$ as compared to his/hers liking of gains. It reflects the fact that individual decision makers are more sensitive to losses than to gains. Moreover, the value function reflects loss aversion when $\lambda > 1$. Kahneman and Tversky estimated the parameters of the value function $a = b = 0.88$, $\lambda = 2.25$ based on experiments with gamblers [2].

Definition 2. *The PT-probability weighting function* $w : [0, 1] \to [0, 1]$ *is defined by*

$$w\left(p\right) = \frac{p^d}{\left(p^d + (1-p)^d\right)^{1/d}}, \quad d \leq 1 \tag{3}$$

The function $w(\cdot)$ is well-defined when a, b are less than $2d$. In the following we will assume that $0.28 < d \leq 1$ and $a < 2d$, $b < 2d$.

Fig. 2 presents the plots of the probability weighting function for different values of d.

Let $P = \{p_1, \dots, p_n\}$ denote probabilities of outcomes $\{x_1, \dots, x_n\}$ respectively.

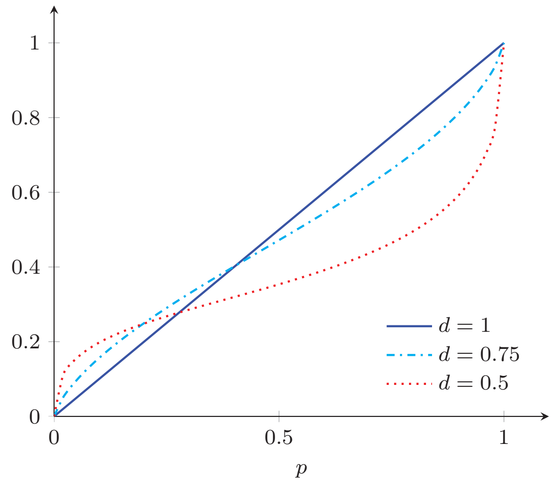

Fig. 2. The plot of the probability weighting function $w(x)$ for different d

Definition 3. *The PT-utility of a prospect $G = (\mathbb{X}, P)$ with the reference point X is defined as [2]*

$$U_{PT}(G) = \sum_{j=1}^{n} w(p_j)v(x_j - X). \tag{4}$$

3 Stochastic Optimal Growth Model

3.1 Model with Power Utility Function

Bellman equation for the stochastic optimal growth model (1) has the following form

$$V(k, \theta) = \max_{k' \in \mathcal{K}(\theta)} (u(F(k, \theta) - k') + \beta \mathbb{E}\{V(k', \theta')|k, \theta\},$$

where k and θ are states of (discrete time continuous) capital and (stochastic discrete state) technology processes at the present time moment respectively, k', θ' are states of the same processes at the next time moment, $F(\cdot, \theta)$ is the net production function at the technology state θ, $u(\cdot)$ is a utility function, $\mathcal{K}(\theta)$ is a feasible set of k' at the technology state θ, $c = F(k, \theta) - k'$ is consumption at the technology state θ.

Let us suppose that utility function is the power utility: $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$, where $c = F(k, \theta) - k'$ is consumption at the present time and at technology state θ. The control variable $k' = F(k, \theta) - c$ is the amount of capital invested in production.

We will assume that $F(k, \theta) = \exp(\theta)k^{\alpha} + (1-\delta)k$, where $0 < \alpha \leq 1$ is the parameter of the model. Then Bellman equation is

$$V(k, s_i) = \max_{(1-\delta)k \leq k' \leq \exp(s_i)k^{\alpha} + (1-\delta)k} \left(u(F(k, \theta) - k') + \beta \sum_{j=1}^{d} \pi_{ij} V(k', s_j) \right).$$

3.2 Model with S-Shaped Utility Function

For PT– decision maker we will assume that reference level of consumption at any time moment is X. Then the deviation of consumption from reference point X is equal to $F(k,\theta) - k' - X$ and Bellman equation is

$$V(k, s_i) = \max_{k' \in \mathcal{K}(\theta)} \left(v(F(k,\theta) - k' - X) + \beta \sum_{j=1}^{d} w(\pi_{ij}) V(k', s_j) \right), \tag{5}$$

where v and w are defined in (2) and (2) respectively. Note that the existence of dynamic programming equations for non-convex performance criteria was proved in the work [9].

Lemma 1. *Blackwell's sufficiency conditions are fulfilled for the operator defined in* (5).

3.3 Numerical Results

We use the value function iteration algorithm with high-precision discretization [8] to solve both the stochastic optimal growth problem with power utility function and the problem with S-shaped utility function, and the algorithm iterates until it converges under the stopping criterion. The convergence of the value function iteration procedure for the model with power utility function is ensured by the contraction mapping theorem. It follows from Lemma 1 that the operator defined in (5) is also contraction mapping.

We assume that θ has two states s_1, s_2 with transition matrix $\Pi = (\pi_{ij})$, $\pi_{11} = \pi_{22} = \pi$, $\pi_{12} = \pi_{21} = 1 - \pi$. We chose values s_1, s_2, π so that the process reproduces the conditional first and second order moments ρ, σ_ε^2 of the AR(1) process. The code is written in MatLab. To obtain numerical solution we use values $\alpha = 0.3$, $\beta = 0.95$, $\delta = 0.1$, $\sigma = 1.5$, $\rho = 0.8$, $\sigma_\varepsilon = 0.12$. We use 1000 equally-spaced interpolation nodes on the range of the continuous state, [0.2,6], for each discrete state θ. The algorithm then converges in less than 250 iterations when the stopping criterion is $e = 10^{-6}$.

Optimal value functions, policy functions, as well as consumption paths both for PT-investor and the investor with power utility function, obtained for the stochastic optimal growth model, are shown in Figures 3, 4 and 5, respectively.

Numerical results show that the optimal behavior of PT– decision maker is quite different from the behavior of the power utility investor. For example, while the value function of the PT-investor is increasing, it has an inflexion point, where convexity changes to concavity.

Optimal consumption for the PT-investor (with reference point less than 0.5) is increasing and is close to the optimal consumption for the power utility investor. For small values of k the optimal consumption of the PT-investor is greater than the optimal consumption of the power utility investor. If the reference point is bigger than 0.5 then the optimal consumption of the PT-investor is not monotonic and much greater than the optimal consumption of the power utility investor for small values of k. Starting

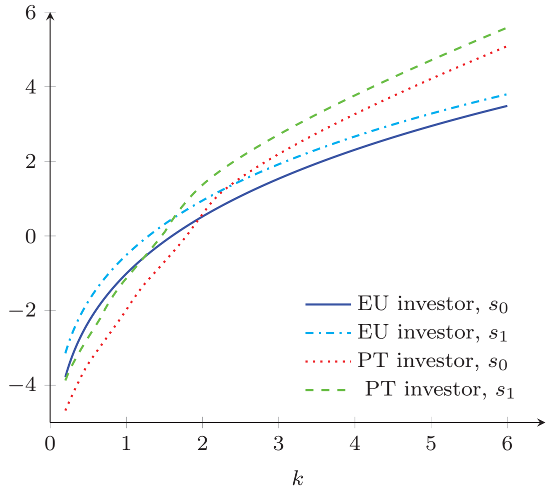

Fig. 3. Value functions for EU-investor and PT-investor with the reference point $X = 1$

with the inflexion point of optimal consumption for the PT-investor, the values of consumption for both types of investors are nearly identical.

While the policy function of the power utility investor is linear, the policy function for PT-investor (with reference point less than 0.5) is monotonic and convex. For small values of k the policy function of the PT-investor is greater than the policy function of the power utility investor. The policy functions for PT-investors are also non-monotonic for reference points bigger than 0.5 and for small values of k are much bigger than the policy function of the power utility investor.

Table 1 shows that the number of iterations needed for solving the problem for the power utility investor by means of the value function iteration algorithm is essentially smaller than the number of iterations needed for solving the problem for the PT-investor. It is not surprisingly, since the PT-utility function is S-shaped and non convex. It should be noted that if the reference point of PT-investor is close to one, the number of iterations needed to achieve a desired accuracy are quite close both for the power utility investor and for PT-investors.

e, accuracy	EU	PT, $X = 0.4$	PT, $X = 0.7$	PT, $X = 1$
10^{-3}	62	126	113	69
10^{-4}	106	171	158	113
10^{-5}	151	216	203	158
10^{-6}	196	261	248	203

Table 1. The number of iterations

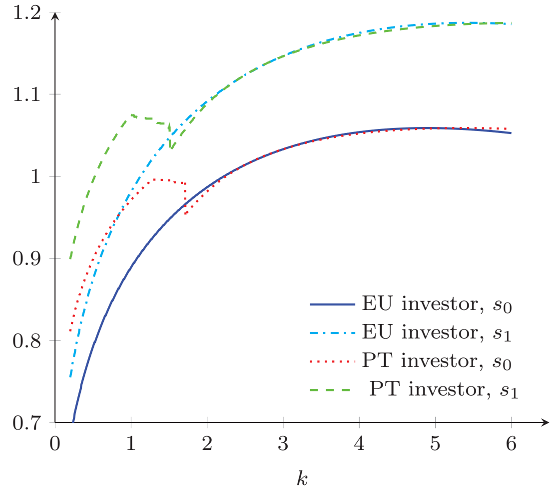

Fig. 4. Optimal consumptions for EU-investor and PT-investor with the reference point $X = 1$

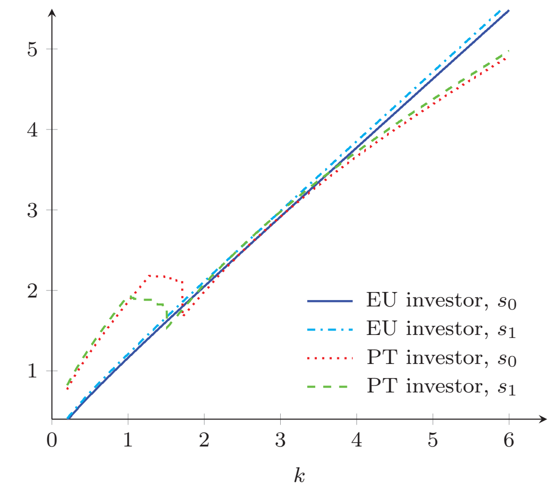

Fig. 5. Optimal policy functions for EU-investor and PT-investor with the reference point $X = 1$

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