# Interval Temporal Logic Model Checking Based on Track Bisimilarity and Prefix Sampling

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**Abstract.** Since the late 80s, LTL and CTL model checking have been extensively applied in various areas of computer science and AI. Even though they proved themselves to be quite successful in many application domains, there are some relevant temporal conditions which are inherently "interval based" (this is the case, for instance, with telic statements like "the astronaut must walk home in an hour" and temporal aggregations like "the average speed of the rover cannot exceed the established threshold") and thus cannot be properly modelled by point-based temporal logics. In general, to check interval properties of the behavior of a system, one needs to collect information about states into behavior stretches, which amounts to interpreting each finite sequence of states as an interval and to suitably defining its labelling on the basis of the labelling of the states that compose it.

In order to deal with these properties, a model checking framework based on Halpern and Shoham's interval temporal logic (HS for short) and its fragments has been recently proposed and systematically investigated in the literature. In this paper, we give an original proof of **EXPSPACE** membership of the model checking problem for the HS fragment $\mathsf{A\overline{A}B\overline{B}\overline{E}}$ (resp., $\mathsf{A\overline{A}E\overline{B}\overline{E}}$) of Allen's interval relations *meets*, *met-by*, *started-by* (resp., *finished-by*), *starts*, and *finishes*. The proof exploits *track bisimilarity* and *prefix sampling*, and it turns out to be much simpler than the previously known one. In addition, it improves some upper bounds.



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## 1 Introduction

Interval temporal logics (ITLs) have been proposed as an alternative setting for reasoning about time [7, 17, 20] with respect to standard, point-based logics such as LTL [18] and CTL [6]. ITLs take intervals, rather than points, as their primitive entities, and their expressiveness enables them to specify, for instance, actions with duration, accomplishments, and temporal aggregations, which are inherently "interval-based" and cannot be expressed by point-based logics.

In this paper, we make use of ITLs as the specification language in model checking (MC), one of the most successful techniques in the area of formal methods, which allows a user to automatically check whether some desired properties of a system, specified by a temporal logic formula, hold over a model of it (usually a Kripke structure). In order to verify interval properties of computations, one needs to collect information about states into computation stretches: each finite path in a Kripke structure is interpreted as an interval, whose labelling is defined on the basis of the labelling of the component states. We focus our attention on *Halpern and Shoham's modal logic of time intervals* (HS) [7] which features one modality for each of the 13 possible ordering relations between pairs of intervals (the so-called Allen's relations [1]), apart from equality. Its *satisfiability problem* turns out to be undecidable for all relevant (classes of) linear orders [7]. The same holds for most fragments of HS [3, 8, 12]; however, some exceptions exist, e.g., the *logic of temporal neighbourhood* and the *logic of sub-intervals* [4, 5].

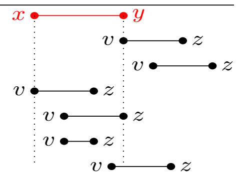
The *MC problem* for HS has been considered only very recently [2, 9, 10, 11, 13, 14, 15, 16]. In [13], Molinari et al. study MC for full HS (under the homogeneity assumption [19]). They introduce the problem and prove its non-elementary decidability. In [2], the authors prove its **EXPSPACE**-hardness. Since then, the attention was also brought to the fragments of HS, which, similarly to what happens with satisfiability, are often computationally better. The MC problem for *epistemic extensions* of some HS fragments has been investigated by Lomuscio and Michaliszyn [9, 10, 11] (a detailed account of their results can be found in [13]). However, their semantic assumptions differ from those of [13] (we make the same assumptions here), thus making it difficult to compare the two research lines.

In this paper, we study the MC problem for the HS fragment $\mathsf{A\overline{A}B\overline{B}\overline{E}}$ (resp., $\mathsf{A\overline{A}E\overline{B}\overline{E}}$), whose modalities allow one to access intervals which are met by/meet the current one, or are prefixes (resp., suffixes) or right/left-extensions of it. In [15], the authors show that the problem is in **EXPSPACE**. The MC algorithm they describe exploits the possibility of finding, for each track of a Kripke structure, a satisfiability-preserving track of bounded length, called a *track representative*. Thus, the algorithm needs to check only tracks with a bounded maximum length. In [14], they prove the problem to be **PSPACE**-hard. The proof of membership to **EXPSPACE** is rather involved, and two very technical notions, namely, the notions of *scan function* and *configuration*, are introduced in order to determine the aforementioned bound to the length of representatives. Here, we provide a much easier proof, which leads to another class of track representatives, with the same purpose of those of [15], but shorter in general.

**Table 1.** Allen's relations and corresponding HS modalities.

| Allen relation | HS | Definition w.r.t. interval structures |
|---|---|---|
| MEETS | $\langle \mathrm{A} \rangle$ | $[x,y]\mathcal{R}_A[v,z] \iff y = v$ |
| BEFORE | $\langle \mathrm{L} \rangle$ | $[x,y]\mathcal{R}_L[v,z] \iff y < v$ |
| STARTED-BY | $\langle \mathrm{B} \rangle$ | $[x,y]\mathcal{R}_B[v,z] \iff x = v \wedge z < y$ |
| FINISHED-BY | $\langle \mathrm{E} \rangle$ | $[x,y]\mathcal{R}_E[v,z] \iff y = z \wedge x < v$ |
| CONTAINS | $\langle \mathrm{D} \rangle$ | $[x,y]\mathcal{R}_D[v,z] \iff x < v \wedge z < y$ |
| OVERLAPS | $\langle \mathrm{O} \rangle$ | $[x,y]\mathcal{R}_O[v,z] \iff x < v < y < z$ |

Example




The paper is organized as follows. In the next section, we introduce the fundamental elements of the MC problem for HS, and we give a short account of the known complexity results about MC for HS fragments. In Sect. 3, we introduce the notion of *bisimilarity* among tracks, that is exploited in Sect. 4, along with *prefix samplings*, to build, given a (generic) track $\rho$, a track $\rho'$ of bounded length, and indistinguishable from $\rho$ with respect to satisfiability of $\mathsf{A\overline{A}B\overline{B}\overline{E}}$ formulas, having nesting depth of modality $\langle \mathrm{B} \rangle$ up to some $k \geq 0$.

## 2 Preliminaries

*The interval temporal logic HS.* An interval algebra to reason about intervals and their relative order was proposed by Allen in [1], while a systematic logical study of interval representation and reasoning was done a few years later by Halpern and Shoham, who introduced the interval temporal logic HS featuring one modality for each Allen relation, but equality [7]. Table 1 depicts 6 of the 13 Allen's relations, together with the corresponding HS (existential) modalities. The other 7 relations are the 6 inverse relations (given a binary relation $\mathcal{R}$, the inverse relation $\overline{\mathcal{R}}$ is such that $b\overline{\mathcal{R}}a$ if and only if $a\mathcal{R}b$) and equality.

The language of HS consists of a set of proposition letters $\mathcal{AP}$, the Boolean connectives $\neg$ and $\wedge$, and a temporal modality for each of the (non trivial) Allen's relations, i.e., $\langle \mathrm{A} \rangle$, $\langle \mathrm{L} \rangle$, $\langle \mathrm{B} \rangle$, $\langle \mathrm{E} \rangle$, $\langle \mathrm{D} \rangle$, $\langle \mathrm{O} \rangle$, $\langle \overline{\mathrm{A}} \rangle$, $\langle \overline{\mathrm{L}} \rangle$, $\langle \overline{\mathrm{B}} \rangle$, $\langle \overline{\mathrm{E}} \rangle$, $\langle \overline{\mathrm{D}} \rangle$, and $\langle \overline{\mathrm{O}} \rangle$. HS formulas are defined by the grammar $\psi ::= p \mid \neg\psi \mid \psi \wedge \psi \mid \langle X \rangle\psi \mid \langle \overline{X} \rangle\psi$, where $p \in \mathcal{AP}$ and $X \in \{A, L, B, E, D, O\}$. In the following, we will also exploit the other usual logical connectives (disjunction $\vee$, implication $\rightarrow$, and double implication $\leftrightarrow$) as abbreviations. Furthermore, for any modality $X$, the dual universal modalities $[X]\psi$ and $[\overline{X}]\psi$ are defined as $\neg\langle X \rangle\neg\psi$ and $\neg\langle \overline{X} \rangle\neg\psi$, respectively.

The joint nesting depth of $B$ and $E$ in a formula $\psi$, denoted by $\mathrm{d_{BE}}(\psi)$, is defined as: $(i)$ $\mathrm{d_{BE}}(p) = 0$, for any $p \in \mathcal{AP}$; $(ii)$ $\mathrm{d_{BE}}(\neg\psi) = \mathrm{d_{BE}}(\psi)$; $(iii)$ $\mathrm{d_{BE}}(\psi \wedge \phi) = \max\{\mathrm{d_{BE}}(\psi), \mathrm{d_{BE}}(\phi)\}$; $(iv)$ $\mathrm{d_{BE}}(\langle \mathrm{X} \rangle\, \psi) = 1 + \mathrm{d_{BE}}(\psi)$, when $X = B$ or $X = E$; $(v)$ $\mathrm{d_{BE}}(\langle \mathrm{X} \rangle\, \psi) = \mathrm{d_{BE}}(\psi)$, when both $X \neq B$ and $X \neq E$. If we consider formulas $\psi$ of HS fragments devoid of $E$ (resp., $B$), the nesting depth of modality $B$ (resp., $E$) in $\psi$, denoted as $\mathrm{d_B}(\psi)$ (resp., $\mathrm{d_E}(\psi)$), accounts for modality $B$ (resp., $E$) only, and $\mathrm{d_B}(\psi) = \mathrm{d_{BE}}(\psi)$ (resp., $\mathrm{d_E}(\psi) = \mathrm{d_{BE}}(\psi)$).

Given any subset of Allen's relations $\{X_1, .., X_n\}$, we denote by $\mathsf{X_1 \cdots X_n}$ the HS fragment featuring existential (and universal) modalities for $X_1, .., X_n$ only.

W.l.o.g., we assume the *non-strict semantics of HS*, which admits intervals consisting of a single point[5]. Under such an assumption, all HS modalities can be expressed in terms of modalities $\langle \mathrm{B} \rangle, \langle \mathrm{E} \rangle, \langle \overline{\mathrm{B}} \rangle$, and $\langle \overline{\mathrm{E}} \rangle$ [20]. HS can thus be regarded as a multi-modal logic with these 4 primitive modalities and its semantics can be defined over a multi-modal Kripke structure, called *abstract interval model*, where intervals are treated as atomic objects and Allen's relations as binary relations between pairs of intervals. Since later we will focus on the HS fragments $\mathsf{A\overline{A}E\overline{B}\overline{E}}$ and $\mathsf{A\overline{A}B\overline{B}\overline{E}}$—which do not feature $\langle \mathrm{B} \rangle$ and $\langle \mathrm{E} \rangle$ respectively—we add both $\langle \mathrm{A} \rangle$ and $\langle \overline{\mathrm{A}} \rangle$ to the considered set of HS modalities.

**Definition 1.** *[13] An* abstract interval model *is a tuple* $\mathcal{A} = (\mathcal{AP}, \mathbb{I}, A_{\mathbb{I}}, B_{\mathbb{I}}, E_{\mathbb{I}}, \sigma)$*, where* $\mathcal{AP}$ *is a set of proposition letters,* $\mathbb{I}$ *is a possibly infinite set of atomic objects (worlds),* $A_{\mathbb{I}}$*,* $B_{\mathbb{I}}$*, and* $E_{\mathbb{I}}$ *are three binary relations over* $\mathbb{I}$*, and* $\sigma : \mathbb{I} \mapsto 2^{\mathcal{AP}}$ *is a (total) labeling function, assigning a set of proposition letters to each world.*

In the interval setting, $\mathbb{I}$ is interpreted as a set of intervals and $A_{\mathbb{I}}$, $B_{\mathbb{I}}$, and $E_{\mathbb{I}}$ as Allen's relations $A$ (*meets*), $B$ (*started-by*), and $E$ (*finished-by*), respectively; $\sigma$ assigns to each interval in $\mathbb{I}$ the set of proposition letters that hold over it.

Given an abstract interval model $\mathcal{A} = (\mathcal{AP}, \mathbb{I}, A_{\mathbb{I}}, B_{\mathbb{I}}, E_{\mathbb{I}}, \sigma)$ and an interval $I \in \mathbb{I}$, the truth of an HS formula over $I$ is inductively defined as follows:

- $\mathcal{A}, I \models p$ iff $p \in \sigma(I)$, for any $p \in \mathcal{AP}$;
- $\mathcal{A}, I \models \neg\psi$ iff it is not true that $\mathcal{A}, I \models \psi$ (also denoted as $\mathcal{A}, I \not\models \psi$);
- $\mathcal{A}, I \models \psi \wedge \phi$ iff $\mathcal{A}, I \models \psi$ and $\mathcal{A}, I \models \phi$;
- $\mathcal{A}, I \models \langle X \rangle \psi$, for $X \in \{A, B, E\}$, iff there is $J \in \mathbb{I}$ s.t. $I \, X_{\mathbb{I}} \, J$ and $\mathcal{A}, J \models \psi$;
- $\mathcal{A}, I \models \langle \overline{X} \rangle \psi$, for $\overline{X} \in \{\overline{A}, \overline{B}, \overline{E}\}$, iff there is $J \in \mathbb{I}$ s.t. $J \, X_{\mathbb{I}} \, I$ and $\mathcal{A}, J \models \psi$.

*Kripke structures and abstract interval models.* In the context of MC, finite state systems are usually modelled as finite Kripke structures. In [13], the authors define a mapping from Kripke structures to abstract interval models, that allows one to specify interval properties of computations by means of HS formulas.

**Definition 2.** *A* finite Kripke structure *is a tuple* $\mathcal{K} = (\mathcal{AP}, W, \delta, \mu, w_0)$*, where* $\mathcal{AP}$ *is a set of proposition letters,* $W$ *is a finite set of states,* $\delta \subseteq W \times W$ *is a left-total relation between pairs of states,* $\mu : W \mapsto 2^{\mathcal{AP}}$ *is a total labelling function, and* $w_0 \in W$ *is the initial state.*

For all $w \in W$, $\mu(w)$ is the set of proposition letters that hold at $w$, while $\delta$ is the transition relation that describes the evolution of the system over time.

Fig. 1 depicts the finite Kripke structure $\mathcal{K}_a = (\{p, q\}, \{v_0, v_1\}, \delta, \mu, v_0)$, where $\delta = \{(v_0, v_0), (v_0, v_1), (v_1, v_0), (v_1, v_1)\}$, $\mu(v_0) = \{p\}$, and $\mu(v_1) = \{q\}$. The initial state $v_0$ is identified by a double circle.

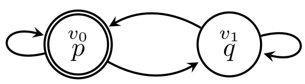



**Fig. 1.** The Kripke structure $\mathcal{K}_a$.

**Definition 3.** *A* track $\rho$ *of a finite Kripke structure* $\mathcal{K} = (\mathcal{AP}, W, \delta, \mu, w_0)$ *is a finite sequence of states* $v_1 \cdots v_n$*, with* $n \geq 1$*, s.t.* $(v_i, v_{i+1}) \in \delta$ *for* $i \in [1, n-1]$.

[5] All the results we prove in the paper hold for the strict semantics as well.

Let $\mathrm{Trk}_{\mathcal{K}}$ be the (infinite) set of all tracks over a finite Kripke structure $\mathcal{K}$. For any track $\rho = v_1 \cdots v_n \in \mathrm{Trk}_{\mathcal{K}}$, we define:

- $|\rho| = n$, $\mathrm{fst}(\rho) = v_1$, and $\mathrm{lst}(\rho) = v_n$;
- any index $i \in [1, |\rho|]$ is called a *$\rho$-position* and $\rho(i) = v_i$;
- $\mathrm{states}(\rho) = \{v_1, \cdots, v_n\} \subseteq W$;
- $\rho(i, j) = v_i \cdots v_j$, for $1 \leq i \leq j \leq |\rho|$, is the subtrack of $\rho$ bounded by $i$, $j$;
- $\mathrm{Pref}(\rho) = \{\rho(1, i) \mid 1 \leq i \leq |\rho| - 1\}$ and $\mathrm{Suff}(\rho) = \{\rho(i, |\rho|) \mid 2 \leq i \leq |\rho|\}$ are the sets of all proper prefixes and suffixes of $\rho$, respectively.

Given $\rho, \rho' \in \mathrm{Trk}_{\mathcal{K}}$, we denote by $\rho \cdot \rho'$ the concatenation of the tracks $\rho$ and $\rho'$. Moreover, if $\mathrm{lst}(\rho) = \mathrm{fst}(\rho')$, we denote by $\rho \star \rho'$ the track $\rho(1, |\rho| - 1) \cdot \rho'$. In particular, when $|\rho| = 1$, $\rho \star \rho' = \rho'$. In the following, when we write $\rho \star \rho'$, we implicitly assume that $\mathrm{lst}(\rho) = \mathrm{fst}(\rho')$. Finally, if $\mathrm{fst}(\rho) = w_0$ (the initial state of $\mathcal{K}$), $\rho$ is called an *initial track*.

An abstract interval model (over $\mathrm{Trk}_{\mathcal{K}}$) can be naturally associated with a finite Kripke structure $\mathcal{K}$ by considering the set of intervals as the set of tracks of $\mathcal{K}$. Since $\mathcal{K}$ has loops ($\delta$ is left-total), the number of tracks in $\mathrm{Trk}_{\mathcal{K}}$, and thus the number of intervals, is infinite.

**Definition 4.** *The* abstract interval model induced by a finite Kripke structure $\mathcal{K} = (\mathcal{AP}, W, \delta, \mu, w_0)$ *is* $\mathcal{A}_{\mathcal{K}} = (\mathcal{AP}, \mathbb{I}, A_{\mathbb{I}}, B_{\mathbb{I}}, E_{\mathbb{I}}, \sigma)$*, where* $\mathbb{I} = \mathrm{Trk}_{\mathcal{K}}$*,* $A_{\mathbb{I}} = \{(\rho, \rho') \in \mathbb{I} \times \mathbb{I} \mid \mathrm{lst}(\rho) = \mathrm{fst}(\rho')\}$*,* $B_{\mathbb{I}} = \{(\rho, \rho') \in \mathbb{I} \times \mathbb{I} \mid \rho' \in \mathrm{Pref}(\rho)\}$*,* $E_{\mathbb{I}} = \{(\rho, \rho') \in \mathbb{I} \times \mathbb{I} \mid \rho' \in \mathrm{Suff}(\rho)\}$*, and* $\sigma : \mathbb{I} \mapsto 2^{\mathcal{AP}}$ *is such that* $\sigma(\rho) = \bigcap_{w \in \mathrm{states}(\rho)} \mu(w)$*, for all* $\rho \in \mathbb{I}$*.*

Relations $A_{\mathbb{I}}, B_{\mathbb{I}}$, and $E_{\mathbb{I}}$ are interpreted as the Allen's relations $A, B$, and $E$, respectively. Moreover, according to the definition of $\sigma$, $p \in \mathcal{AP}$ holds over $\rho = v_1 \cdots v_n$ if and only if it holds over all the states $v_1, \cdots, v_n$ of $\rho$. This conforms to the *homogeneity principle* [19], according to which a proposition letter holds over an interval if and only if it holds over all its subintervals.

**Definition 5.** *Let* $\mathcal{K}$ *be a finite Kripke structure and* $\psi$ *be an HS formula; we say that a track* $\rho \in \mathrm{Trk}_{\mathcal{K}}$ *satisfies* $\psi$*, denoted as* $\mathcal{K}, \rho \models \psi$*, iff it holds that* $\mathcal{A}_{\mathcal{K}}, \rho \models \psi$*. Moreover, we say that* $\mathcal{K}$ *models* $\psi$*, denoted as* $\mathcal{K} \models \psi$*, iff for all* initial *tracks* $\rho' \in \mathrm{Trk}_{\mathcal{K}}$ *it holds that* $\mathcal{K}, \rho' \models \psi$*. The* model checking problem *for HS over finite Kripke structures is the problem of deciding whether* $\mathcal{K} \models \psi$*.*

In Fig. 2, we provide an example of a finite Kripke structure $\mathcal{K}_{Sched}$ that models the behaviour of a scheduler serving three processes which are continuously requesting the use of a common resource (it is a simplified version of an example given in [13]). The initial state is $v_0$: no process is served in that state. In any other state $v_i$ and $\overline{v}_i$, with $i \in \{1, 2, 3\}$, the $i$-th process is served (this is denoted by the fact that $p_i$ holds in those states). For the sake of readability, edges are marked either by $r_i$, for $request(i)$, or by $u_i$, for $unlock(i)$. Edge labels do not have a semantic value, that is, they are neither part of the structure definition, nor proposition letters; they are simply used to ease reference to edges. Process $i$ is served in state $v_i$, then, after "some time", a transition $u_i$ from $v_i$ to $\overline{v}_i$ is taken; subsequently, process $i$ cannot be served again immediately, as $v_i$ is not directly reachable from $\overline{v}_i$ (the scheduler cannot serve the same process twice in two successive rounds). A transition $r_j$, with $j \neq i$, from $\overline{v}_i$ to $v_j$ is then taken and process $j$ is served.

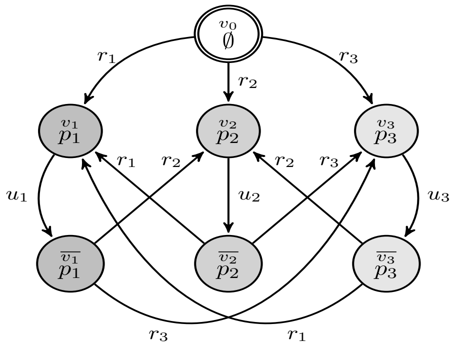



**Fig. 2.** The Kripke structure $\mathcal{K}_{Sched}$.

We now show how some meaningful properties to be checked against $\mathcal{K}_{Sched}$ can be expressed in HS, in particular, by formulas of $\mathsf{A\overline{A}E\overline{B}\overline{E}}$. In all formulas, we force the validity of the considered property over all legal computation sub-intervals by using modality $[E]$ (all computation sub-intervals are suffixes of at least one initial track). The truth of the next statements can easily be checked ($\langle \mathrm{E} \rangle^k$ stands for $k$ occurrences of modality $\langle \mathrm{E} \rangle$):

- $\mathcal{K}_{Sched} \models [E]\big(\langle \mathrm{E} \rangle^3 \top \rightarrow (\chi(p_1, p_2) \vee \chi(p_1, p_3) \vee \chi(p_2, p_3))\big)$, where $\chi(p, q) := \langle \mathrm{E} \rangle \langle \overline{\mathrm{A}} \rangle p \wedge \langle \mathrm{E} \rangle \langle \overline{\mathrm{A}} \rangle q$;
- $\mathcal{K}_{Sched} \not\models [E](\langle \mathrm{E} \rangle^{10} \top \rightarrow \langle \mathrm{E} \rangle \langle \overline{\mathrm{A}} \rangle p_3)$;
- $\mathcal{K}_{Sched} \not\models [E](\langle \mathrm{E} \rangle^5 \rightarrow (\langle \mathrm{E} \rangle \langle \overline{\mathrm{A}} \rangle p_1 \wedge \langle \mathrm{E} \rangle \langle \overline{\mathrm{A}} \rangle p_2 \wedge \langle \mathrm{E} \rangle \langle \overline{\mathrm{A}} \rangle p_3))$.

The first formula states that in any suffix of length at least 4 of an initial track, at least 2 proposition letters are witnessed. $\mathcal{K}_{Sched}$ satisfies the formula since a process cannot be executed twice in a row. The second formula states that in any suffix of length at least 11 of an initial track, process 3 is executed at least once in some internal states (*non starvation*). $\mathcal{K}_{Sched}$ does not satisfy the formula since the scheduler can avoid executing a process ad libitum. The third formula states that in any suffix of length at least 6 of an initial track, $p_1, p_2, p_3$ are all witnessed. The only way to satisfy this property is to constrain the scheduler to execute the 3 processes in a strictly periodic manner, but this is not the case.

*The general picture.* Now we summarize the known complexity results about the MC problem for HS fragments (see Fig. 3 for a graphical account).

In [13], Molinari et al. show that, given a finite Kripke structure $\mathcal{K}$ and a bound $k$ on the structural complexity of HS formulas (nesting depth of $\langle \mathrm{E} \rangle$ and $\langle \mathrm{B} \rangle$ modalities), it is possible to obtain a *finite* representation for $\mathcal{A}_{\mathcal{K}}$, which is equivalent to $\mathcal{A}_{\mathcal{K}}$ w. r. to satisfiability of HS formulas with structural complexity less than or equal to $k$. Then, by exploiting such a representation, they prove that the MC problem for (full) HS is decidable, providing an algorithm with non-elementary complexity. In [2], Bozzelli et al. show that the problem for the fragment $\mathsf{BE}$, and thus for full HS, is **EXPSPACE**-hard. In [15], Molinari et al. study the fragments $\mathsf{A\overline{A}B\overline{B}\overline{E}}$ and $\mathsf{A\overline{A}E\overline{B}\overline{E}}$, devising for each of them an **EXPSPACE** MC algorithm which exploits the possibility of finding, for each track of a Kripke structure, a satisfiability-preserving track of bounded length (*track representative*). In this way, the algorithm needs to check only tracks having a bounded maximum length. In [14], they prove that the problem for

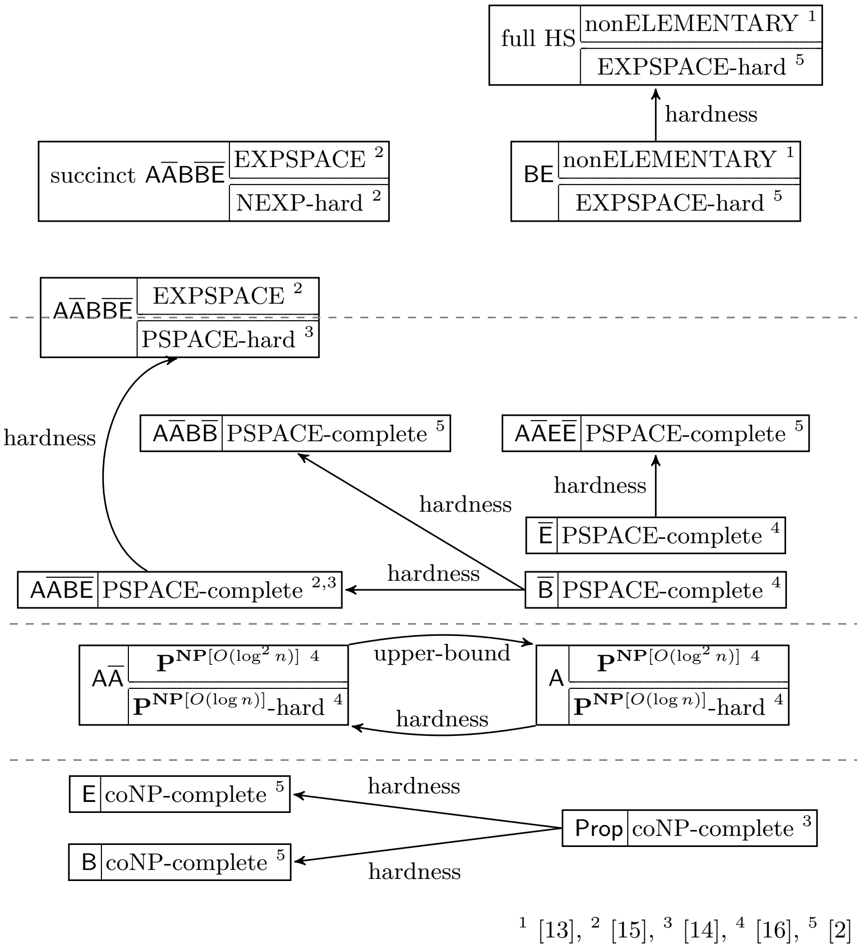



**Fig. 3.** Complexity of the MC problem for HS fragments

$\mathsf{A\overline{A}B\overline{B}\overline{E}}$ and $\mathsf{A\overline{A}E\overline{B}\overline{E}}$ is **PSPACE**-hard (with a succinct encoding of formulas the algorithm remains in **EXPSPACE**, but a **NEXPTIME** lower bound can be given [15]). The MC problem for other HS fragments has been studied in the following papers:

– $\mathsf{A\overline{A}\overline{B}\overline{E}}$, $\mathsf{\overline{B}}$, $\mathsf{\overline{E}}$, $\mathsf{A\overline{A}B\overline{B}}$, and $\mathsf{A\overline{A}E\overline{E}}$ are **PSPACE**-complete [2, 14, 15, 16];
– $\mathsf{A\overline{A}}$, $\mathsf{A}$, and $\mathsf{\overline{A}}$ are in between $\mathbf{P}^{\mathbf{NP}[O(\log n)]}$ and $\mathbf{P}^{\mathbf{NP}[O(\log^2 n)]}$ [16];
– $\mathsf{B}$, $\mathsf{E}$, $\mathsf{Prop}$ (the propositional fragment of HS) are **co-NP**-complete [2, 14].

In the next sections, we shall reconsider the MC problem for the fragment $\mathsf{A\overline{A}B\overline{B}\overline{E}}$ (and the symmetric fragment $\mathsf{A\overline{A}E\overline{B}\overline{E}}$), proving in a much simpler way (compared to [15]) its membership to **EXPSPACE**. We shall show that, given a track $\rho$ and $h \geq 0$, there is a track $\rho'$, whose length is at most $(|W|+2)^{h+2}$, such that for every $\mathsf{A\overline{A}B\overline{B}\overline{E}}$ formula $\psi$, with $\mathrm{d_B}(\psi) \leq h$, $\mathcal{K}, \rho \models \psi$ iff $\mathcal{K}, \rho' \models \psi$.

## 3 Track Bisimilarity

In this short section, we introduce the notions of *prefix-bisimilarity* and *suffix-bisimilarity* between a pair of tracks $\rho$ and $\rho'$ of a Kripke structure. As proved by Proposition 2 below, prefix-bisimilarity (resp., suffix-bisimilarity) is a sufficient condition for two tracks $\rho$ and $\rho'$ to be indistinguishable with respect to satisfiability of (some classes of) $\mathsf{A\overline{A}B\overline{B}\overline{E}}$ (resp., $\mathsf{A\overline{A}E\overline{B}\overline{E}}$) formulas, respectively.

**Definition 6 (Prefix-bisimilarity and Suffix-bisimilarity).** *Let $h \geq 0$ and $\rho$ and $\rho'$ be two tracks of a Kripke structure $\mathcal{K}$. We say that $\rho$ and $\rho'$ are $h$-*prefix bisimilar *if the following conditions inductively hold:*

- *for $h = 0$: $\mathrm{fst}(\rho) = \mathrm{fst}(\rho')$, $\mathrm{lst}(\rho) = \mathrm{lst}(\rho')$, and $\mathrm{states}(\rho) = \mathrm{states}(\rho')$.*
- *for $h > 0$: $\rho$ and $\rho'$ are 0-prefix bisimilar and for each proper prefix $\nu$ of $\rho$ (resp., proper prefix $\nu'$ of $\rho'$), there exists a proper prefix $\nu'$ of $\rho'$ (resp., proper prefix $\nu$ of $\rho$) such that $\nu$ and $\nu'$ are $(h-1)$-prefix bisimilar.*

*The notion of $h$-*suffix bisimilarity *is defined in a symmetric way by considering suffixes of tracks instead of prefixes.*

*Property 1.* Given a Kripke structure $\mathcal{K}$, for all $h \geq 0$, $h$-prefix (resp., $h$-suffix) bisimilarity is an equivalence relation over $\mathrm{Trk}_{\mathcal{K}}$.

Moreover, $h$-suffix bisimilarity and $h$-prefix bisimilarity propagate downwards.

*Property 2.* Given a Kripke structure $\mathcal{K}$ and two tracks $\rho, \rho' \in \mathrm{Trk}_{\mathcal{K}}$, for all $h > 0$, if $\rho$ and $\rho'$ are $h$-prefix (resp., $h$-suffix) bisimilar, then they are also $(h-1)$-prefix (resp., $(h-1)$-suffix) bisimilar.

The following result can easily be proved by induction on $h \geq 0$.

**Proposition 1.** *Let $h \geq 0$, and $\rho$ and $\rho'$ be two $h$-prefix (resp., $h$-suffix) bisimilar tracks of a Kripke structure $\mathcal{K}$. Then, for each track $\rho''$ of $\mathcal{K}$,*

1. *$\rho'' \star \rho$ and $\rho'' \star \rho'$ are $h$-prefix (resp., $h$-suffix) bisimilar;*
2. *$\rho \star \rho''$ and $\rho' \star \rho''$ are $h$-prefix (resp., $h$-suffix) bisimilar.*

By Proposition 1 and a straightforward induction on the structural complexity of formulas, we obtain that $h$-prefix (resp., $h$-suffix) bisimilarity preserves the satisfiability of $\mathsf{A\overline{A}B\overline{B}\overline{E}}$ (resp., $\mathsf{A\overline{A}E\overline{B}\overline{E}}$) formulas having nesting depth of modality $B$ (resp., $E$) at most $h$.

**Proposition 2.** *Let $h \geq 0$, and $\rho$ and $\rho'$ be two $h$-prefix (resp., $h$-suffix) bisimilar tracks of a Kripke structure $\mathcal{K}$. For each $\mathsf{A\overline{A}B\overline{B}\overline{E}}$ (resp., $\mathsf{A\overline{A}E\overline{B}\overline{E}}$) formula $\psi$ with $\mathrm{d_B}(\psi) \leq h$ (resp., $\mathrm{d_E}(\psi) \leq h$), it holds that $\mathcal{K}, \rho \models \psi$ iff $\mathcal{K}, \rho' \models \psi$.*

## 4 The Fragments $\mathsf{A\overline{A}B\overline{B}\overline{E}}$ and $\mathsf{A\overline{A}E\overline{B}\overline{E}}$: Exponential-Size Model-Track Property

In this section, we focus on the fragment $\mathsf{A\overline{A}B\overline{B}\overline{E}}$ (the case of $\mathsf{A\overline{A}E\overline{B}\overline{E}}$ is completely symmetric). We shall show how to determine a subset of positions of a track $\rho$

(a *prefix sampling* of $\rho$), starting from which it is possible to build another track $\rho'$, of bounded exponential size, which is indistinguishable from $\rho$ with respect to the fulfilment of $\mathsf{A\overline{A}B\overline{B}\overline{E}}$ formulas up to a given nesting depth of modality $B$ (*exponential-size model-track property*). We start by introducing the notions of *induced track*, *prefix-skeleton sampling*, and *h-prefix sampling*, and prove some related properties.

**Definition 7 (Induced track).** *Let $\rho$ be a track of length $n$ of a Kripke structure $\mathcal{K}$. A* track induced by $\rho$ *is a track $\pi$ of $\mathcal{K}$ such that there exists an increasing sequence of $\rho$-positions $i_1 < \ldots < i_k$, with $i_1 = 1$, $i_k = n$, and $\pi = \rho(i_1) \cdots \rho(i_k)$.*

Note that if $\pi$ is induced by $\rho$, then $\mathrm{fst}(\pi) = \mathrm{fst}(\rho)$, $\mathrm{lst}(\pi) = \mathrm{lst}(\rho)$, and $|\pi| \leq |\rho|$ (in particular, $|\pi| = |\rho|$ iff $\pi = \rho$). Intuitively, a track induced by $\rho$ is obtained by contracting $\rho$, namely, by concatenating some subtracks of $\rho$, provided that the resulting sequence is a track of $\mathcal{K}$ as well.

In the following, given a set $I$ of natural numbers, by "two consecutive elements of $I$" we refer to a pair of elements $i, j \in I$ s.t. $i < j$ and $I \cap [i, j] = \{i, j\}$.

**Definition 8 (Prefix-skeleton sampling).** *Let $\rho$ be a track of a Kripke structure $\mathcal{K} = (\mathcal{AP}, W, \delta, \mu, w_0)$. Given two $\rho$-positions $i$ and $j$, with $i \leq j$, the* prefix-skeleton sampling *of $\rho(i, j)$ is the* minimal *set $P$ of $\rho$-positions in the interval $[i, j]$ satisfying: $(i)$ $i, j \in P$; $(ii)$ for each state $w \in W$ occurring along $\rho(i + 1, j - 1)$, the minimal position $k \in [i + 1, j - 1]$ such that $\rho(k) = w$ is in $P$.*

From Definition 8, it immediately follows that the prefix-skeleton sampling $P$ of (any) track $\rho(i, j)$ is such that $|P| \leq |W| + 2$ and $i + 1 \in P$ whenever $i < j$.

**Definition 9 ($h$-prefix sampling).** *Let $\rho$ be a track of a Kripke structure $\mathcal{K}$. For each $h \geq 1$, the* $h$-prefix sampling of $\rho$ *is the* minimal *set $P_h$ of $\rho$-positions inductively satisfying the following conditions:*

- *Base case: $h = 1$. $P_1$ is the prefix-skeleton sampling of $\rho$;*
- *Inductive step: $h > 1$. $(i)$ $P_h \supseteq P_{h-1}$ and $(ii)$ for all pairs of consecutive positions $i, j$ in $P_{h-1}$, the prefix-skeleton sampling of $\rho(i, j)$ is in $P_h$.*

The following upper bound to the cardinality of prefix samplings holds.

*Property 3.* Let $h \geq 1$ and $\rho$ be a track of a Kripke structure $\mathcal{K}$. The $h$-prefix sampling $P_h$ of $\rho$ is such that $|P_h| \leq (|W| + 2)^h$.

We now prove a technical lemma that will be used in the proof of Lemma 2.

**Lemma 1.** *Let $h \geq 1$, $\rho$ be a track of $\mathcal{K}$, and $i, j$ be two consecutive $\rho$-positions in the $h$-prefix sampling of $\rho$. Then, for all $\rho$-positions $n, n' \in [i + 1, j]$ such that $\rho(n) = \rho(n')$, it holds that $\rho(1, n)$ and $\rho(1, n')$ are $(h - 1)$-prefix bisimilar.*

*Proof.* The proof is by induction on $h \geq 1$.

- Base case: $h = 1$. The 1-prefix sampling of $\rho$ is the prefix-skeleton sampling of $\rho$. Hence, being $i$ and $j$ consecutive positions in this sampling, for each position $k \in [i, j - 1]$, there is $\ell \leq i$ such that $\rho(\ell) = \rho(k)$. Since $\rho(n) = \rho(n')$, $\mathrm{states}(\rho(1, n)) = \mathrm{states}(\rho(1, n'))$, so $\rho(1, n)$ and $\rho(1, n')$ are 0-prefix bisimilar.

– Inductive step: $h > 1$. By definition of $h$-prefix sampling, there are two consecutive positions $i', j'$ in the $(h-1)$-prefix sampling of $\rho$ such that $i, j$ are consecutive positions of the prefix-skeleton sampling of $\rho(i', j')$.
  If $i = i'$, then $j = i + 1$, hence, being $n, n' \in [i+1, j]$, we get that $n = n'$, and the result trivially holds.
  Now, assume that $i \neq i'$, thus $i > i'$. As in the base case, we easily deduce that $\rho(1, n)$ and $\rho(1, n')$ are 0-prefix bisimilar. It remains to show that for each proper prefix $\nu$ of $\rho(1, n)$ (resp., proper prefix $\nu'$ of $\rho(1, n')$), there is a proper prefix $\nu'$ of $\rho(1, n')$ (resp., proper prefix $\nu$ of $\rho(1, n)$) such that $\nu$ and $\nu'$ are $(h-2)$-prefix bisimilar. Let us consider a proper prefix $\nu$ of $\rho(1, n)$ (the proof for the other direction is symmetric). Hence, $\nu = \rho(1, m)$ for some $m < n$. We distinguish two cases:
  - $m \leq i$. Hence $\rho(1, m)$ is a proper prefix of $\rho(1, n')$ and the result follows.
  - $m > i$: since $i$ and $j$ are consecutive positions of the prefix-skeleton sampling of $\rho(i', j')$, $i > i'$, and $m \in [i+1, j-1]$ (hence $m < j'$), there exists $m' \in [i'+1, i]$ such that $\rho(m') = \rho(m)$ and $m'$ is in the prefix-skeleton sampling of $\rho(i', j')$. Let $\nu' = \rho(1, m')$. Evidently $\nu'$ is a proper prefix of $\rho(1, n')$ (as $n' \geq i + 1$). Moreover, since $m, m' \in [i'+1, j']$ and $i', j'$ are consecutive positions in the $(h-1)$-prefix sampling of $\rho$, by the inductive hypothesis $\nu = \rho(1, m)$ and $\nu' = \rho(1, m')$ are $(h-2)$-prefix bisimilar. □

The next lemma and the following theorem show how to derive, from any track $\rho$ of a Kripke structure, another track $\rho'$, induced by $\rho$ and $h$-prefix bisimilar to $\rho$, such that $|\rho'| \leq (|W| + 2)^{h+2}$. By Proposition 2, $\rho'$ is indistinguishable from $\rho$ w.r.t. the fulfilment of any $\mathsf{A\overline{A}B\overline{B}\overline{E}}$ formula $\psi$ with $\mathrm{d_B}(\psi) \leq h$.

In order to build $\rho'$, we first compute the $(h+1)$-prefix sampling $P_{h+1}$ of $\rho$. Next, for all the pairs of consecutive $\rho$-positions $i, j \in P_{h+1}$, we consider a track induced by $\rho(i, j)$, with no repeated occurrences of any state, except at most the first and last ones (hence, it is no longer than $(|W| + 2)$). The track $\rho'$ is just the ordered concatenation (by means of the $\star$ operator) of all these tracks. The aforementioned bound on $|\rho'|$ holds as, by Property 3, $|P_{h+1}| \leq (|W|+2)^{h+1}$. The following preparatory lemma states that $\rho$ and $\rho'$ are indeed $h$-prefix bisimilar.

**Lemma 2.** *Let $h \geq 1$, $\rho$ be a track of $\mathcal{K}$, and $\rho' = \rho(i_1)\rho(i_2)\cdots\rho(i_k)$ be a track induced by $\rho$, where $1 = i_1 < i_2 < \ldots < i_k = |\rho|$ and $P_{h+1} \subseteq \{i_1, \ldots, i_k\}$, with $P_{h+1}$ the $(h+1)$-prefix sampling of $\rho$. Then, for all $j \in [1, k]$, $\rho'(1, j)$ and $\rho(1, i_j)$ are $h$-prefix bisimilar.*

Notice that, in particular, $\rho$ and $\rho'$ are $h$-prefix bisimilar.

*Proof.* Let $Q = \{i_1, \ldots, i_k\}$ (hence $P_{h+1} \subseteq Q$) and let $j \in [1, k]$. We prove by induction on $j$ that $\rho'(1, j)$ and $\rho(1, i_j)$ are $h$-prefix bisimilar. As for the base case $(j = 1)$, the result holds, since $i_1 = 1$.

Now assume that $j > 1$. We first show that $\rho(1, i_j)$ and $\rho'(1, j)$ are 0-prefix bisimilar. Clearly, $\rho(1) = \rho(i_1) = \rho'(1)$, $\rho(i_j) = \rho'(j)$, and $\mathrm{states}(\rho'(1, j)) \subseteq \mathrm{states}(\rho(1, i_j))$. Now, if, by contradiction, there was a state $w$ such that $w \in \mathrm{states}(\rho(1, i_j)) \setminus \mathrm{states}(\rho'(1, j))$, then for all $l \in Q$, with $l \leq i_j$, $\rho(l) \neq w$. However, the prefix-skeleton sampling $P_1$ of $\rho$ is contained in $Q$, and the minimal $\rho$-position $l'$ such that $\rho(l') = w$ belongs to $P_1$. Since $w \in \text{states}(\rho(1, i_j))$, $l' \leq i_j$. Thus, we get a contradiction, implying that $\text{states}(\rho'(1, j)) = \text{states}(\rho(1, i_j))$.

It remains to prove that: (1) for each proper prefix $\nu'$ of $\rho'(1, j)$, there exists a proper prefix $\nu$ of $\rho(1, i_j)$ such that $\nu$ and $\nu'$ are $(h-1)$-prefix bisimilar, and (2) for each proper prefix $\nu$ of $\rho(1, i_j)$, there exists a proper prefix $\nu'$ of $\rho'(1, j)$ such that $\nu$ and $\nu'$ are $(h-1)$-prefix bisimilar.

As for (1), let $\nu'$ be a proper prefix of $\rho'(1, j)$. Hence, there exists $m \in [1, j-1]$ such that $\nu' = \rho'(1, m)$. By the inductive hypothesis, $\rho'(1, m)$ and $\rho(1, i_m)$ are $h$-prefix bisimilar, and thus $(h-1)$-prefix bisimilar as well (Property 2). Since $\rho(1, i_m)$ is a proper prefix of $\rho(1, i_j)$, by choosing $\nu' = \rho(1, i_m)$ (1) follows.

As for (2), assume that $\nu$ is a proper prefix of $\rho(1, i_j)$. Therefore, there exists $n \in [1, i_j - 1]$ such that $\nu = \rho(1, n)$. We distinguish two cases:

- $n \in P_{h+1}$. Since $n < i_j$, there exists $m \in [1, j-1]$ such that $n = i_m$. By the inductive hypothesis, $\rho(1, n)$ and $\rho'(1, m)$ are $h$-prefix bisimilar, and thus $(h-1)$-prefix bisimilar as well (Property 2). Since $\rho'(1, m)$ is a proper prefix of $\rho'(1, j)$, by choosing $\nu' = \rho'(1, m)$ (2) follows.
- $n \notin P_{h+1}$. It follows that there exist two consecutive positions $i'$ and $j'$ in $P_{h+1}$, with $i' < j'$, such that $n \in [i'+1, j'-1]$. By definition of $(h+1)$-prefix sampling, there exist two consecutive positions $i''$ and $j''$ in the $h$-prefix sampling of $\rho$, with $i'' < j''$, such that $i'$ and $j'$ are two consecutive positions in the prefix-skeleton sampling of $\rho(i'', j'')$.
  First, we observe that $i' \neq i''$ (otherwise, $j' = i' + 1$, which contradicts the fact that $[i'+1, j'-1] \neq \emptyset$, as $n \in [i'+1, j'-1]$). Thus, by definition of prefix-skeleton sampling applied to $\rho(i'', j'')$, and since $n \in [i'+1, j'-1]$, there must be $\ell \in [i''+1, i']$ such that $\rho(\ell) = \rho(n)$ and $\ell$ is in the prefix-skeleton sampling of $\rho(i'', j'')$. Hence $\ell \in P_{h+1}$ by definition of $(h+1)$-prefix sampling. As a consequence, since $\ell < n < i_j$, there exists $m \in [1, j-1]$ such that $\ell = i_m$. By applying Lemma 1, we deduce that $\rho(1, n)$ and $\rho(1, i_m)$ are $(h-1)$-prefix bisimilar. Moreover, by the inductive hypothesis, $\rho(1, i_m)$ and $\rho'(1, m)$ are $(h-1)$-prefix bisimilar. Thus, by choosing $\nu' = \rho'(1, m)$, $\nu'$ is a proper prefix of $\rho'(1, j)$ which is $(h-1)$-prefix bisimilar to $\nu = \rho(1, n)$. □

**Theorem 1 (Exponential-size model-track property for $\mathsf{A\overline{A}B\overline{B}\overline{E}}$).** *Let $\rho$ be a track of a Kripke structure $\mathcal{K}$ and $h \geq 0$. Then, there exists a track $\rho'$ induced by $\rho$, whose length is at most $(|W|+2)^{h+2}$, such that for every $\mathsf{A\overline{A}B\overline{B}\overline{E}}$ formula $\psi$ with $\mathrm{d_B}(\psi) \leq h$, it holds that $\mathcal{K}, \rho \models \psi$ iff $\mathcal{K}, \rho' \models \psi$.*

*Proof.* Let $P_{h+1}$ be the $(h+1)$-prefix sampling of $\rho$. For all pairs of consecutive $\rho$-positions $i$ and $j$ in $P_{h+1}$, there exists a track induced by $\rho(i, j)$ having length at most $|W|+2$, featuring no repeated occurrences of any internal state. We now define $\rho'$ as the track of $\mathcal{K}$ obtained by concatenating in order all these induced tracks by means of the $\star$ operator. It is immediate to see that $\rho' = \rho(i_1)\rho(i_2)\cdots\rho(i_k)$, for some indexes $1 = i_1 < i_2 < \cdots < i_k = |\rho|$, where $\{i_1, \ldots, i_k\}$ contains the $(h+1)$-prefix sampling $P_{h+1}$ of $\rho$. It holds that $|\rho'| \leq |P_{h+1}| \cdot (|W|+2)$ and since, by Property 3, $|P_{h+1}| \leq (|W|+2)^{h+1}$, we obtain that $|\rho'| \leq (|W|+2)^{h+2}$.

Moreover, by Lemma 2, $\rho$ and $\rho'$ are $h$-prefix bisimilar. By Proposition 2, the result follows. □

Theorem 1 allows us to easily devise an **EXPSPACE** MC algorithm for $\mathsf{A\overline{A}B\overline{B}\overline{E}}$ formulas (and symmetrically for $\mathsf{A\overline{A}E\overline{B}\overline{E}}$ formulas) which is basically the same as that presented in [15]. However, in that paper, the authors prove—in a much more involved way—the existence of a bound on the length of equivalent induced tracks which is greater than the present one, that is, $O(|W|^{2h+4})$.

## 5 Conclusions and Future Work

In this paper, we dealt with the problem of finding bounded representatives of tracks of a Kripke structure to solve the MC problem for the HS fragments $\mathsf{A\overline{A}B\overline{B}\overline{E}}$ and $\mathsf{A\overline{A}E\overline{B}\overline{E}}$. The proposed solution slightly reduces the bounds for track representatives given for the same problem in [15]; moreover, it substantially simplifies the constructions and the complexity of the proofs. As for future work, we would like to precisely characterize the complexity of MC for $\mathsf{A\overline{A}B\overline{B}\overline{E}}$ and $\mathsf{A\overline{A}E\overline{B}\overline{E}}$. At the moment, we only know that it belongs to **EXPSPACE** and it is **PSPACE**-hard [14]. More generally, we are looking for possible improvements to known complexity results for MC of (full) HS. We know that it is **EXPSPACE**-hard (we proved **EXPSPACE**-hardness of its fragment $\mathsf{BE}$ [2]), while the only available decision procedure is nonelementary [13].

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