Synthesis of Weighted Marked Graphs from Circular Labelled Transition Systems

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Abstract. Several works have proposed methods for the analysis and synthesis of Petri net subclasses from labelled transition systems (LTS). In this paper, we focus on Choice-Free (CF) Petri nets, in which each place has at most one output, and their subclass of Weighted Marked Graphs (WMGs). We provide new conditions for the WMG-synthesis from a circular LTS, i.e. forming a single circuit, and discuss the difficulties in extending these new results to the CF case.

Keywords: Weighted Petri net, choice-free net, synthesis, labelled transition system, cycles, cyclic words, circular solvability.

1 Introduction

Petri nets form a highly expressive and intuitive operational model of discrete event systems, capturing the mechanisms of synchronisation, conflict and concurrency. Many of their fundamental behavioural properties are decidable, allowing to model and analyse numerous artificial and natural systems. However, most interesting model checking problems are worst-case intractable, and the efficiency of synthesis algorithms varies widely depending on the constraints imposed on the desired solution. In this study, we focus on the Petri net synthesis problem from a labelled transition system (LTS), which consists in determining the existence of a Petri net whose reachability graph is isomorphic to the given LTS, and building such a Petri net solution when it exists.

In previous studies on analysis or synthesis, structural restrictions on nets encompassed *plain* nets (each weight equals 1; also called ordinary nets) [25], *homogeneous* nets (i.e. for each place p, all the output weights of p are equal) [28, 23], *free-choice* nets (the net is plain, hence also homogeneous, and any two

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transitions sharing an input place have the same set of input places) [12, 28], join-free nets (each transition has at most one input place) [28, 11, 22, 23], etc.

More recently, another kind of restriction has been considered, limiting the number of different transition labels of the LTS [2, 3, 18, 19].

In this paper, we study the problem of solvability of LTS with weighted marked graphs (each place has at most one output transition and one input transition) and choice-free nets (each place has at most one output transition). Both these classes are important for real-world applications, and they are widely studied in the literature [27, 21, 15, 9, 26, 8, 16, 7]. In this work, we focus mainly on finite *circular LTS*, meaning strongly connected LTS that contain a unique *cycle*[4]. In this context, we investigate the *cyclic solvability* of a word w, meaning the existence of a Petri net solution to the finite circular LTS induced by the infinite *cyclic word* w^∞.

An important purpose of studying such constrained LTS is to better understand the relationship between LTS decompositions and their solvability by Petri nets. Indeed, the unsolvability of simple subgraphs of the given LTS, typically elementary paths (i.e. not containing any node twice) and cycles (i.e. closed paths, whose start and end states are equal), often induces simple conditions of unsolvability for the entire LTS, as highlighted in other works [2, 18, 4]. Moreover, cycles appear systematically in the reachability graph of live and/or reversible Petri nets [27], which are used to model various real-world applications, such as embedded systems [20].

Contributions. In this work, we study further the links between simple LTS structures and the reachability graph of WMGs and CF nets, as follows. First, we show that a binary LTS is CF-solvable if and only if it is WMG-solvable. Then, we provide new general conditions for the WMG-solvability of a cyclic word over an arbitrary alphabet, together with an algorithm synthesizing a cyclical WMG-solution for a given word. We also discuss the difficulties of extending these results to the CF class.

Organisation of the paper. After recalling classical definitions, notations and properties in Section 2, we present the equivalence of CF- and WMG-solvability for 2-letter words in Section 3. Then, in Section 4, we focus on circular LTS: we develop a new characterisation of WMG-solvability and a dedicated synthesis algorithm. We also provide a number of examples, which demonstrate that some of the presented results cannot be applied to the class of CF-nets. Finally, Section 5 presents our conclusions and perspectives.

[4] A set A of k arcs in a LTS G defines a cycle of G if the elements of A can be ordered as a sequence $a_1 \dots a_k$ such that, for each $i \in \{1, \dots, k\}$, $a_i = (n_i, \ell_i, n_{i+1})$ and $n_{k+1} = n_1$, i.e. the i-th arc a_i goes from node n_i to node n_{i+1} until the first node n_1 is reached, closing the path.

2 Classical Definitions, Notations and Properties

LTS, sequences and reachability. A *labelled transition system with initial state*, *LTS* for short, is a quadruple $TS = (S, \to, T, \iota)$ where S is the set of *states*, T is the set of *labels*, $\to \subseteq (S \times T \times S)$ is the *transition relation*, and $\iota \in S$ is the *initial state*. A label t is *enabled* at $s \in S$, written $s[t\rangle$, if $\exists s' \in S \colon (s, t, s') \in \to$, in which case s' is said to be *reachable* from s by the firing of t, and we write $s[t\rangle s'$. Generalising to any (firing) sequences $\sigma \in T^*$, $s[\varepsilon\rangle$ and $s[\varepsilon\rangle s$ are always true, with ε being an empty sequence; and $s[\sigma t\rangle s'$, i.e., σt is *enabled* from state s and leads to s' if there is some s'' with $s[\sigma\rangle s''$ and $s''[t\rangle s'$. For clarity, in case of long formulas we write $|_r \sigma |_s \tau |_q$ instead of $r[\sigma\rangle s[\tau\rangle q$, thus fixing some intermediate states along a firing sequence. A state s' is *reachable* from state s if $\exists \sigma \in T^* \colon s[\sigma\rangle s'$. The set of states reachable from s is noted $[s\rangle$. $TS = (S, \to, T, \iota)$ is *fully reachable* if $S = [\iota\rangle$.

Petri nets and reachability graphs. A (finite, place-transition) *weighted Petri net*, or *weighted net*, is a tuple $N = (P, T, W)$ where P is a finite set of *places*, T is a finite set of *transitions*, with $P \cap T = \emptyset$ and W is a *weight* function $W \colon ((P \times T) \cup (T \times P)) \to \mathbb{N}$ giving the weight of each arc. A *Petri net system*, or *system*, is a tuple $\mathcal{S} = (N, M_0)$ where N is a net and M_0 is the *initial marking*, which is a mapping $M_0 \colon P \to \mathbb{N}$ (hence a member of $\mathbb{N}^P$) indicating the initial number of *tokens* in each place. The *incidence matrix* C of the net is the integer $P \times T$-matrix with components $C(p, t) = W(t, p) - W(p, t)$.
A place $p \in P$ is *enabled by* a marking M if $M(p) \geq W(p, t)$ for every output transition t of p. A transition $t \in T$ is *enabled by* a marking M, denoted by $M[t\rangle$, if for all places $p \in P$, $M(p) \geq W(p, t)$. If t is enabled at M, then t can *occur* (or *fire*) in M, leading to the marking M' defined by $M'(p) = M(p) - W(p, t) + W(t, p)$; we note $M[t\rangle M'$. A marking M' is *reachable* from M if there is a sequence of firings leading from M to M'. The set of markings reachable from M is denoted by $[M\rangle$. The *reachability graph of* $\mathcal{S}$ is the labelled transition system $RG(\mathcal{S})$ with the set of vertices $[M_0\rangle$, the set of labels T, initial state M_0 and transitions $\{(M, t, M') \mid M, M' \in [M_0\rangle \land M[t\rangle M'\}$. A system $\mathcal{S}$ is *bounded* if $RG(\mathcal{S})$ is finite.

Vectors. The *support* of a vector is the set of the indices of its non-null components. Consider any net $N = (P, T, W)$ with its incidence matrix C. A *T-vector* is an element of $\mathbb{N}^T$; it is called *prime* if the greatest common divisor of its components is one (i.e., its components do not have a common non-unit factor). A *T-semiflow* ν of the net is a non-null T-vector such that $C \cdot \nu = \mathbb{0}$. A T-semiflow is called *minimal* when it is prime and its support is not a proper superset of the support of any other T-semiflow [27].
The *Parikh vector* $\mathbf{P}(\sigma)$ of a finite sequence σ of transitions is a T-vector counting the number of occurrences of each transition in σ, and the *support* of σ is the support of its Parikh vector, i.e., $supp(\sigma) = supp(\mathbf{P}(\sigma)) = \{t \in T \mid \mathbf{P}(\sigma)(t) > 0\}$.

Strong connectedness and cycles in LTS. The LTS is said *reversible* if, $\forall s \in [\iota\rangle$, we have $\iota \in [s\rangle$, i.e., it is always possible to go back to the initial state;

reversibility implies the strong connectedness of the LTS.
A sequence $s[\sigma\rangle s'$ is called a *cycle*, or more precisely a *cycle at (or around) state* s, if $s = s'$. A non-empty cycle $s[\sigma\rangle s$ is called *small* if there is no non-empty cycle $s'[\sigma'\rangle s'$ in TS with $\mathbf{P}(\sigma') \lneqq \mathbf{P}(\sigma)$ (the definition of Parikh vectors extending readily to sequences over the set of labels T of the LTS). A cycle $s[\sigma\rangle s$ is *prime* if $\mathbf{P}(\sigma)$ is prime. TS has the *prime cycle property* if every small cycle has a prime Parikh vector.
A *circular LTS* is a finite, strongly connected LTS that contains a unique cycle; hence, it has the shape of an oriented circle. The circular LTS *induced by* a word $w{=}w_1 \dots w_k$ is the LTS with initial state s_0 defined as $s_0[w_1\rangle s_1[w_2\rangle s_2 \dots [w_k\rangle s_0$.

All notions defined for labelled transition systems apply to Petri nets through their reachability graphs.

Petri net subclasses. A net N is *plain* if no arc weight exceeds 1; *pure* if $\forall p \in P\colon (p^\bullet \cap {}^\bullet p) = \emptyset$, where $p^\bullet = \{t \in T \mid W(p,t){>}0\}$ and ${}^\bullet p = \{t \in T \mid W(t,p){>}0\}$; CF (*choice-free* [10, 27]) or ON (place-output-nonbranching [4]) if $\forall p \in P\colon |p^\bullet| \leq 1$; a WMG (*weighted marked graph* [26]) if $|p^\bullet| \leq 1$ and $|{}^\bullet p| \leq 1$ for all places $p \in P$. The latter form a subclass of the choice-free nets; other subclasses are *marked graphs* [9], which are plain with $|p^\bullet| = 1$ and $|{}^\bullet p| = 1$ for each place $p \in P$, and *T-systems* [12], which are plain with $|p^\bullet| \leq 1$ and $|{}^\bullet p| \leq 1$ for each place $p \in P$.

Isomorphism and solvability. Two LTS $TS_1 = (S_1, \to_1, T, s_{01})$ and $TS_2 = (S_2, \to_2, T, s_{02})$ are isomorphic if there is a bijection $\zeta\colon S_1 \to S_2$ with $\zeta(s_{01}) = s_{02}$ and $(s,t,s') \in \to_1 \Leftrightarrow (\zeta(s), t, \zeta(s')) \in \to_2$, for all $s, s' \in S_1$.
If an LTS TS is isomorphic to $RG(\mathcal{S})$, where $\mathcal{S}$ is a system, we say that $\mathcal{S}$ *solves* TS. Solving a word $w = \ell_1 \dots \ell_k$ amounts to solve the acyclic LTS defined by the single path $\iota[\ell_1\rangle s_1 \dots [\ell_k\rangle s_k$. A finite word w is *cyclically solvable* if the circular LTS induced by w is solvable. An LTS is WMG-solvable if a WMG solves it.

Separation problems. Let $TS = (S, \to, T, s_0)$ be a given labelled transition system. The theory of regions [1] characterises the solvability of an LTS through the solvability of a set of *separation problems*. In case the LTS is finite, we have to solve $\frac{1}{2}{\cdot}|S|{\cdot}(|S|{-}1)$ states separation problems and up to $|S|{\cdot}|T|$ event/state separation problems, as follows:

- A *region* of $(S, \to, T, s_0)$ is a triple $(\mathbb{R}, \mathbb{B}, \mathbb{F}) \in (S \to \mathbb{N}, T \to \mathbb{N}, T \to \mathbb{N})$ such that for all $s[t\rangle s'$, $\mathbb{R}(s) \geq \mathbb{B}(t)$ and $\mathbb{R}(s') = \mathbb{R}(s) - \mathbb{B}(t) + \mathbb{F}(t)$. A region models a place p, in the sense that $\mathbb{B}(t)$ models $W(p,t)$, $\mathbb{F}(t)$ models $W(t,p)$, and $\mathbb{R}(s)$ models the token count of p at the marking corresponding to s.
- A *states separation problem* (SSP for short) consists of a set of states $\{s, s'\}$ with $s \neq s'$, and it can be solved by a place distinguishing them, i.e., has a different number of tokens in the markings corresponding to the two states.
- An *event/state separation problem* (ESSP for short) consists of a pair $(s,t) \in S{\times}T$ with $\neg s[t\rangle$. For every such problem, one needs a place p such that $M(p) < W(p,t)$ for the marking M corresponding to state s, where W

refers to the arcs of the hoped-for net. On the other hand, for every edge $(s', t, s'') \in\to$ we must guarantee $M'(p) \geq W(p,t)$, M' being the marking corresponding to state s'.

If the LTS is infinite, also the number of separation problems (of each kind) becomes infinite.

A synthesis procedure does not necessarily lead to a connected solution. However, the technique of decomposition into prime factors described in [13, 14] can always be applied first, so as to handle connected partial solutions and recombine them afterwards. Hence, in the sequel, we focus on connected nets, without loss of generality. In the next section, we consider the synthesis problem of CF nets with exactly two different labels.

3 Reversible Binary CF Synthesis

In this section, we link the CF-solvability of a reversible LTS with 2 letters to the WMG-solvability.

Lemma 1 (Pure CF-solvability).
If a reversible LTS has a CF-solution, it has a pure CF-solution.

Proof. Let $TS = (S, \to, T, \iota)$ be a reversible LTS. First, we observe that, if $t \in T$ does not occur in $\to$, TS is solvable iff $TS' = (S, \to, T \setminus \{t\}, \iota)$ is solvable and a possible solution of TS is obtained by adding to any solution of TS' a transition t and a fresh place p, initially empty, with an arc from p to t (e.g. with weight 1), so that t is pure. We may thus assume that each label of T occurs in $\to$.

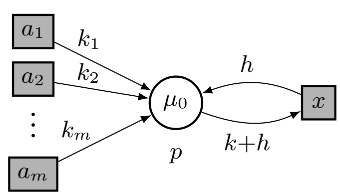

Fig. 1. A general pure ($h = 0$) or non-pure ($h > 0$) choice-free place p with initial marking μ_0. Place p has at most one outgoing transition named x. The set $\{a_1, \dots, a_m\}$ comprises all other transitions, i.e., $T = \{x, a_1, \dots, a_m\}$, and k_j denotes the weight of the arc from a_j to p (which could be zero).

The general form of a place in a CF-solution is exhibited in Figure 1. If $h = 0$, we are done, so that we shall assume $h > 0$. If $-h \leq k < 0$, the marking of p cannot decrease, and since x occurs in $\to$, the system cannot be reversible. If $k = 0$, for the same reason all the k_i's must be null too, $\mu_0 \geq h$, and we may drop p. Hence we assume that $k > 0$ and $\exists i : k_i > 0$.

Once x occurs, the marking of p is at least h, remains so, and since the system is reversible, all the reachable markings have at least h tokens in p. But then, if we replace p by a place p' with initially $\mu_0 - h$ tokens, the same k_i's and $h = 0$, we shall get exactly the same reachability graph, but with h tokens less in p' than in p. This will wipe out the side condition[5] for p, and repeating this for each side condition, we shall get an equivalent pure and choice-free solution. □

Theorem 1 (Reversible binary CF-solvability).
A binary reversible LTS is CF-solvable iff it is WMG-solvable.

Proof. If we have two labels, from Lemma 1, if there is a CF-solution, there will be one with places of the form exhibited in Figure 2, hence a WMG-solution. □

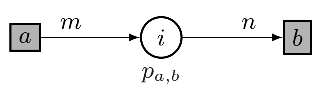

Fig. 2. A generic pure CF-place with two labels.

In the next section, the number of letters is no more restricted.

4 Cyclic WMG- and CF-solvablity

In this section, we recall and extend the conditions for WMG-solvability of some restricted classes of LTS formed by a single circuit, which were suggested in [15]. We gradually study the separation problems (SSPs in Subsection 4.1 and ESSPs in Subsection 4.2) for cyclical solvability with WMGs, and develop a language-theoretical characterisation of WMG-cyclically solvable sequences. The characterisation gives rise to a synthesis algorithm which is presented later. Unfortunately, most of these results cannot be directly extended to the more general class of CF-nets, which is demonstrated by examples in Subsection 4.3.

In the following, two distinct labels a and b are called *(circularly) adjacent* in a word w if $w = (w_1 a b w_2)$ or $w = (b w_3 a)$ for some $w_1, w_2, w_3 \in T^*$. We denote by $p_{a,*}$ any place $p_{a,b}$ where b is adjacent to a. Also, since $|T| > 1$, at least one label is adjacent to t_0, and at least one is adjacent to the ones we exhibited, etc., until we get the whole set T, and we may start from any label t_i instead of t_0.

Theorem 2 (Sufficient condition for cyclic WMG-solvability [15]).
Consider any word w over any finite alphabet T such that $\mathbf{P}(w)$ is prime. Suppose the following: $\forall u = w_{|t_1 t_2}$ (i.e., the projection[6] of w on $\{t_1, t_2\}$) for some

[5] A place p is a *side-condition* if ${}^\bullet p \cap p^\bullet \neq \emptyset$.

[6] The projection of a word $w \in A^*$ on a set $A' \subseteq A$ of labels is the maximum subword of w whose labels belong to A', noted $w_{|A'}$. For example, the projection of the word $w = \ell_1\, \ell_2\, \ell_3\, \ell_2$ on the set $\{\ell_1,\, \ell_2\}$ is the word $\ell_1\, \ell_2\, \ell_2$.

circularly adjacent labels t_1, t_2 in w, $u = v^\ell$ for some positive integer ℓ, $\mathbf{P}(v)$ is prime, and v is cyclically solvable by a circuit (i.e. a circular net as in Fig. 3). Then, w is cyclically solvable with a WMG.

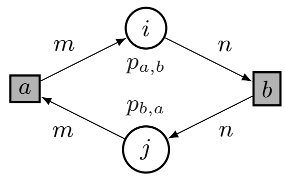

Fig. 3. A generic WMG solving a finite circular LTS induced by a word w over the alphabet $\{a, b\}$.

Theorem 3 (Cyclic WMG-solvability of ternary words [15]).
Consider a ternary word w over the alphabet T with Parikh vector (x, x, y) such that $gcd(x, y) = 1$. Then, w is cyclically solvable with a WMG if and only if $\forall u = w_{|t_1 t_2}$ such that $t_1 \neq t_2 \in T$, and $w = (w_1 t_1 t_2 w_2)$ or $w = (t_2 w_3 t_1)$, $u = v^\ell$ for some positive integer ℓ, $\mathbf{P}(v)$ is prime, and v is cyclically solvable by a circuit.

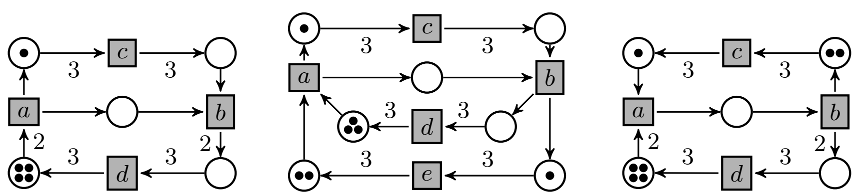

Fig. 4. The WMG on the left solves *aacbbdabd* cyclically, and the WMG in the middle solves *aacbbeabd* cyclically. On the right, the WMG solves *abcabdabd* cyclically.

For a circular LTS, the solvability of its binary projections by circuits is a sufficient condition, as specified by Theorem 2, but it turns out not to be a necessary one. Indeed, for the cyclically solvable sequence $w_1 = aacbbdabd$ (cf. left of Fig. 4), its binary projection on $\{a, b\}$ is $w_{1\,|a,b} = aabbab$ which is not cyclically solvable with a WMG (neither generally solvable). Looking only at the Parikh vector of the sequence is also not enough to establish its cyclical (un)solvability. For instance, sequences $w_2 = abcabdabd$ and $w_3 = abcbadabd$ are Parikh-equivalent: $\mathbf{P}(w_2) = \mathbf{P}(w_3) = (3, 3, 1, 2)$ (and also Parikh-equivalent to w_1), but w_2 is cyclically solvable with a WMG (e.g. with the WMG on the right of Fig. 4) and w_3 is not WMG-cyclically solvable.

All the binary projections of w_1 and w_3 are cyclically WMG-solvable, except $w_{i\,|a,b}$. But only the unsolvability of $w_{3\,|a,b}$ implies the unsolvability of w_3. Since all the w_i are Parikh-equivalent, then so are their binary projections. So, to find the difference we have to look at the sequences themselves, without abstracting to Parikh-vectors. Since the projections $w_{1\,|a,b}$ and $w_{3\,|a,b}$ are equivalent (up to cyclical rotation and swapping a and b), it is also not enough to look only at the 'problematic' binary projections. We then look at the conditions for solvability of separation problems.

4.1 SSPs for Prime Cycles

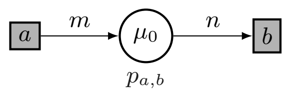

Fig. 5. A place of general form in a WMG.

Lemma 2 (SSPs are solvable for prime cycles). *If for the cyclical transition system $TS = (S, \to, T, s_0)$ defined by some word $w = t_0 \dots t_k$, where $S = \{s_0, \dots, s_k\}$, $\to = \{(s_{i-1}, t_{i-1}, s_i) \mid 1 \leq i \leq k\} \cup \{(s_k, t_k, s_0)\}$ with $t_i \in T$, $\mathbf{P}(w)$ is prime, then all the SSPs are solvable.*

Proof. If $|T| = 1$, then $k = 0$ (otherwise $\mathbf{P}(t_0 \dots t_k)$ is not prime) and $|S| = 1$, so that there is no SSP to solve. We may thus assume $|T| > 1$.

For $0 \leq i, j \leq k$ such that $s_i \neq s_j$ (so that $i \neq j$), we note $\mathbf{P}_{ij} = \mathbf{P}(t_i t_{i+1} \dots t_{j-1})$ if $i < j$ and $\mathbf{P}_{ij} = \mathbf{P}(t_i t_{i+1} \dots t_{k-1} t_k t_0 t_1 \dots t_{j-1})$ if $i > j$. For each pair of distinct labels $a, b \in T$ that are adjacent in TS, construct places $p_{a,b}$ (and $p_{b,a}$ since adjacency is commutative) as in Fig. 5 with

$$m = \frac{\mathbf{P}(w)(b)}{\gcd(\mathbf{P}(w)(a), \mathbf{P}(w)(b))},\ n = \frac{\mathbf{P}(w)(a)}{\gcd(\mathbf{P}(w)(a), \mathbf{P}(w)(b))}, \tag{1}$$

and $\mu_0 = n \cdot \mathbf{P}(w)(b)$. Clearly, the markings of $p_{a,b}$ reachable by repeatedly firing $u = w_{|ab}$ are always non-negative, and the initial marking is reproduced after each repetition of the sequence u. Consider two distinct states $s_i, s_j \in S$ (w.l.o.g. $i < j$). We now demonstrate that there is at least one place of the form $p_{a,b}$ such that $M_i(p_{a,b}) \neq M_j(p_{a,b})$, where M_l denotes the marking corresponding to state s_l for $0 \leq l \leq k$. If $j - i = 1$, then any place of the form $p_{t_i,*}$ distinguishes states s_i and s_j. The same is true if $j - i > 1$ but $\forall l \in [i, j-1] : t_l = t_i$. Otherwise, choose some letter a from $t_i \dots t_{j-1}$ and an adjacent letter b. Then $M_j(p_{a,b}) = M_i(p_{a,b}) + m \cdot \mathbf{P}_{ij}(a) - n \cdot \mathbf{P}_{ij}(b)$. If $M_i(p_{a,b}) \neq M_j(p_{a,b})$, place $p_{a,b}$ distinguishes s_i and s_j. Otherwise we have $m \cdot \mathbf{P}_{ij}(a) = n \cdot \mathbf{P}_{ij}(b)$, hence, due to the choice of m and n:

$$\frac{\mathbf{P}_{ij}(a)}{\mathbf{P}(w)(a)} = \frac{\mathbf{P}_{ij}(b)}{\mathbf{P}(w)(b)}$$

(so that b also belongs to $t_i \dots t_{j-1}$). Consider some other letter c which is adjacent to a or b. If place $p_{a,c}$ distinguishes s_i and s_j, we are done. Otherwise, due to the choice of the arc weights for these places, we have

$$\frac{\mathbf{P}_{ij}(a)}{\mathbf{P}(w)(a)} = \frac{\mathbf{P}_{ij}(c)}{\mathbf{P}(w)(c)} = \frac{\mathbf{P}_{ij}(b)}{\mathbf{P}(w)(b)}.$$

Since $t_i \dots t_{j-1}$ is finite, by progressing along the adjacency relation, either we find a place which has different markings at s_i and s_j, or for all $a, b \in \mathit{supp}(t_i \dots t_{j-1})$ we have

$$\frac{\mathbf{P}_{ij}(a)}{\mathbf{P}(w)(a)} = \frac{\mathbf{P}_{ij}(b)}{\mathbf{P}(w)(b)}.$$

If $\mathit{supp}(t_i \dots t_{j-1}) = \mathit{supp}(w)$, $\mathbf{P}(w)$ is proportional to $\mathbf{P}(t_i \dots t_{j-1})$, but since $t_i \dots t_{j-1}$ is smaller than w (otherwise $s_i = s_j$) this contradicts the primality of $\mathbf{P}(w)$. Hence, there exist adjacent c and d such that $c \in \mathit{supp}(w) \setminus \mathit{supp}(t_i \dots t_{j-1})$ and $d \in \mathit{supp}(t_i \dots t_{j-1})$. For the place $p_{c,d}$ we have $M_j(p_{c,d}) \neq M_i(p_{c,d})$. $\square$

This property has some similarities with Theorem 4.1 in [17], but the preconditions are different.

The reachability graph of any CF net, hence of any WMG, satisfies the prime cycle property [5, 6]. Thus, primeness of a sequence allows us skip checking of SSPs when looking at solvability within these two classes of Petri nets.

4.2 ESSPs in Cyclical Solvability with WMGs

Now, consider further conditions for the cyclical WMG-solvability of a sequence $w = t_0 \dots t_k$, where $\mathbf{P}(w)$ is prime. Let us assume that the system $((P, T, W), M_0)$ is a WMG solving w cyclically. Due to the definition of WMGs, all the places that we have to consider are of the form schematised in Fig. 6. The arc weights may differ due to the integer parameter $l \geq 1$, but the ratio $\frac{W(a,p_{a,b})}{W(p_{a,b},b)} = \frac{m}{n}$ is determined by the Parikh vector of w and its cyclical solvability (let us notice that the initial marking is to be defined). Moreover, we have to consider only these places, which are connected to the pairs of adjacent transitions in w. Indeed, if $w = u_1 \mid_{s_i} a \mid_{s_{i+1}} b \, u_2$, where s_i is the state reached after performing u_1 and s_{i+1} is the state reached after performing $u_1 a$, then any place that solves the ESSP[7] $\neg M_i[b\rangle$ is an input place for b. On the other hand, any place whose marking at M_{s_i} differs from its marking at $M_{s_{i+1}}$ is connected to a. Hence, a place $p \in P$ solving $\neg M_i[b\rangle$ is of the form $p_{a,b}$. Since p is only affected by a and b, it also disables b at all the states between s_l and s_i in w when it is of the form $w = u_3 \mid_{s_j} t_j \mid_{s_{j+1}} b^+ \mid_{s_l} u_4 \mid_{s_i} abu_2$ with $\mathbf{P}(u_4)(b) = 0$ (in the case there is no b in the prefix between s_0 and abu_2, $s_l = s_0$). Analogously, if $t_j \neq b$, there must be a place $q \in P$ of the form $p_{t_j,b}$ that solves $\neg M_{s_j}[b\rangle$. Doing so, we ascertain that the places of the form schematised in Fig. 6 for the adjacent pairs of transitions are sufficient to handle all the ESSPs.

[7] Assuming $b \neq a$.

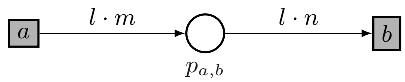

Fig. 6. A general place from a to b of a WMG solution (l may be any multiple of $1/\gcd(m,n)$).

In fact, for every pair of adjacent transitions a and b in w, a single place of the form $p_{a,b}$ is sufficient. Indeed, assume there are $p_1, p_2 \in P$ of the form $p_{a,b}$. If $\frac{M_0(p_1)}{\gcd(W(a,p_1),W(p_1,b))} \geq \frac{M_0(p_2)}{\gcd(W(a,p_2),W(p_2,b))}$ then for any $M \in [M_0\rangle$, $M(p_1) < W(p_1,b)$ implies $M(p_2) < W(p_2,b)$. Hence, p_1 is redundant in the system, and the following is true.

Lemma 3. *If $w \in T^*$ is cyclically solvable by a WMG, there exists a WMG $S = ((P,T,W), M_0)$, where P consists of places $p_{a,b}$, for each pair of distinct circularly adjacent a and b (i.e., either $w = u_1abu_2$ or $w = bu_3a$).*

Let w be cyclically solvable with a WMG $S = ((P,T,W), M_0)$ as in Lemma 3, and place $p \in P$ be of the form $p_{a,b}$ (as in Fig. 6, with $l = 1$) for an adjacent pair ab. Choose two successive ab's in $w = u_1 \,|_{s_r} a \,|_{s_{r+1}} b \,|_{s_{r+2}} \ldots |_{s_q} a \,|_{s_{q+1}} b \, u_2$ with possibly other letters between s_{r+2} and s_q (if there is only one ab, apply the argumentation while wrapping around w circularly, i.e., with $s_r = s_q$). Since p solves ESSPs $\neg s_r[b\rangle$ and $\neg s_q[b\rangle$, the next inequalities hold true, where μ_r denotes the marking of $p_{a,b}$ at state s_r:

$$\begin{array}{rll} \neg s_r[b\rangle : & \mu_r & < n \\ s_{r+1}[b\rangle : & \mu_r + m & \geq n \\ \forall j : \; r \leq j \leq q : & \mu_r + \mathbf{P}_{rj}(a) \cdot m - \mathbf{P}_{rj}(b) \cdot n & \geq 0 \\ \neg s_q[b\rangle : & \mu_r + \mathbf{P}_{rq}(a) \cdot m - \mathbf{P}_{rq}(b) \cdot n & < n \end{array} \tag{2}$$

From the first and the third line of (2) we get $\mathbf{P}_{rj}(a) \cdot m - \mathbf{P}_{rj}(b) \cdot n > -n$. This implies:

$$\frac{\mathbf{P}_{rj}(b) - 1}{\mathbf{P}_{rj}(a)} < \frac{m}{n}, \; r < j \leq q. \tag{3}$$

From the third and the fourth line of (2) we obtain

$$(\mathbf{P}_{rq}(a) - \mathbf{P}_{rj}(a)) \cdot m - (\mathbf{P}_{rq}(b) - \mathbf{P}_{rj}(b)) \cdot n < n.$$

For $\mathbf{P}_{jq}(a) \neq 0$, since $\mathbf{P}_{rq} = \mathbf{P}_{rj} + \mathbf{P}_{jq}$ this inequality can be written as

$$\frac{m}{n} < \frac{\mathbf{P}_{jq}(b) + 1}{\mathbf{P}_{jq}(a)}. \tag{4}$$

Thus, from (3) and (4) we have a necessary condition for solvability in the following sense.

Lemma 4 (A necessary condition for cyclical solvability with a WMG). *If $w \in T^*$ is cyclically solvable by a WMG, then for any adjacent transitions a and b in w, and any two successive-up-to-rotation occurrences of ab in $w = u_1 \,|_{s_r} a\, b\, \ldots\, |_{s_q} a\, b\, u_2$, the inequality*

$$\frac{\mathbf{P}_{rj}(b) - 1}{\mathbf{P}_{rj}(a)} < \frac{m}{n} < \frac{\mathbf{P}_{jq}(b) + 1}{\mathbf{P}_{jq}(a)} \tag{5}$$

holds true, where m, n are as in (1), $r < j \le q$, and the right inequality is omitted when $\mathbf{P}_{jq}(a) = 0$.

In particular, Lemma 4 explains the cyclical unsolvability of the word $w_3 = |_{s_r} ab\, c\, b\,|_{s_j}\, a\, d\,|_{s_q}\, ab\, d$. Indeed, $\mathbf{P}(w_3)(b) = 3 = \mathbf{P}(w_3)(a)$, so that $m/n = 1$ and $1 \not< \frac{0+1}{1} = \frac{\mathbf{P}_{jq}(b)+1}{\mathbf{P}_{jq}(a)}$.

Lemma 5 (A sufficient condition for cyclical solvability by a WMG). *If $w \in T^*$ has a prime Parikh vector, and for each pair of circularly adjacent ab in $w = \ldots\, |_q\, a\, b\, \ldots$, the inequality*

$$\frac{m}{n} < \frac{\mathbf{P}_{jq}(b) + 1}{\mathbf{P}_{jq}(a)},\ j \ne q \tag{6}$$

holds true, then w is cyclically solvable by a WMG.

Proof: We have earlier proved (Lemma 2) that all SSPs are solvable for prime cycles. Let us now consider the ESSPs at states s as in $w = \ldots |_s ab \ldots$, i.e. $\neg s[b\rangle$. Since we are looking for a WMG-solution, all the sought places are of the form $p_{a,b}$ (see Lemma 3 and Fig. 6) with m, n as in (1). To define the initial marking of $p_{a,b}$, let us put $n \cdot \mathbf{P}(w)(b)$ tokens on it and fire the sequence w once completely. Choose some state s' in $w = \ldots |_{s'}\, a \ldots$ such that the number k of tokens on $p_{a,b}$ at state s' is minimal (it may be the case that such an s' is not unique; we can choose any such state). Define $M_0(p_{a,b}) = n \cdot \mathbf{P}(w)(b) - k$ as the initial marking of $p_{a,b}$. By construction, the firing of w reproduces the markings of $p_{a,b}$ and M_0 guarantees their non-negativity. Let us now demonstrate that the constructed place $p_{a,b}$ solves all the ESSPs $\neg s[b\rangle$, where $w = \ldots |_s ab \ldots$. Consider such a state s in w (w.l.o.g. we assume $s \ne s'$, since s' certainly disables b). From $w = u_1 \,|_{s'} a\, \ldots\, |_s\, a\, b\, u_2$, and from inequality (6) for $s_j = s'$ we get $\mathbf{P}_{s's}(a) \cdot m - \mathbf{P}_{s's}(b) \cdot n < n$. Since $M_{s'}(p_{a,b}) = 0$, $M_s(p_{a,b}) = M_{s'}(p_{a,b}) + \mathbf{P}_{s's}(a) \cdot m - \mathbf{P}_{s's}(b) \cdot n < n$, i.e., $p_{a,b}$ disables b at state s.

Now, we show that places of the form $p_{a,b}$ also solve the other ESSPs against b, i.e., at the states where b is not the subsequent transition. Sequence w (up to rotation) can be written as $w = u_1\, b^{x_1}\, u_2\, b^{x_2}\, \ldots\, u_l\, b^{x_l}$, $1 \le l \le \mathbf{P}(w)(b)$, and for $1 \le i \le l$: $x_i > 0$, $u_i \in (T \setminus \{b\})^+$. Transition b has to be deactivated at all the states between neighbouring b-blocks. Consider an arbitrary pair of such blocks b^{x_j} and $b^{x_{j+1}}$ in $w = \ldots\, b^{x_j}\, u_j\, b^{x_{j+1}}\, \ldots = \ldots\, b^{x_j}\, |_s\, u'_j\, |_r\, t\, b^{x_{j+1}}\, \ldots$, with $u_j = u'_j t$. Place $p_{t,b}$ does not allow b to fire at state r. We have to check that b is not activated at any state between s and r, i.e., it is not activated 'inside' u'_j. If u'_j is empty, then $s = r$, and we are done. Let $u'_j \neq \varepsilon$. Due to $\mathbf{P}(u'_j)(b) = \mathbf{P}(u_j)(b) = 0$, the marking of place $p_{t,b}$ cannot decrease from s to r, i.e., $M_s(p_{t,b}) \leq M_{s''}(p_{t,b}) \leq M_r(p_{t,b})$ for any s'' 'inside' u'_j. Since $p_{t,b}$ deactivates b at r, it then deactivates b at all states between s and r, inclusively. $\square$

From Lemma 4 and Lemma 5 we can deduce the following characterisation.

Theorem 4 (A characterisation of cyclical solvability with a WMG). *A sequence $w \in T^*$ is cyclically solvable with a WMG iff $\mathbf{P}(w)$ is prime and for any pair of circularly adjacent labels in w, for instance $w = \ldots\,|_q\, ab\, \ldots$,*

$$\frac{m}{n} < \frac{\mathbf{P}_{jq}(b) + 1}{\mathbf{P}_{jq}(a)},\ j \neq q$$

holds true with m, n as in (1). A WMG-solution can be found with the places of the form $p_{a,b}$ for every such pair of a and b.

Based on the characterisation from Theorem 4 and the considerations above, Algorithm 1 below synthesizes a cyclical WMG-solution for a given sequence $w \in T^*$, if one exists, with a runtime in $\mathcal{O}(|w|^2)$. For a comparison, the general region-based synthesis typically uses ILP-solvers, and for Karmarkar's algorithm [24] (which is known to be efficient) we may expect a running time of $\mathcal{O}(|w|^3 \cdot L(|w|))$, with a logarithmic factor $L(|w|) = \log(|w|) \cdot \log(\log(|w|))$. Note that, with this general approach, some redundant places may be constructed, but they can be reduced in a post-processing phase.

4.3 CF-solvability vs WMG-solvability of Cycles

The class of WMGs is clearly a proper subset of the class of CF nets. If we are only looking at cyclical solvability of sequences, this inclusion remains strict, i.e., there exist sequences which are cyclically solvable by a CF net but have no cyclical solution in the form of a WMG. E.g., the sequence $w = abcbad$ has a cyclical CF-solution (cf. Fig. 7). On the other hand, for $a\,|_r\, b\, c\,|_q\, b\, a\, d$ we have $\frac{\mathbf{P}(w)(a)}{\mathbf{P}(w)(b)} = \frac{2}{2} \not< \frac{0+1}{1} = \frac{\mathbf{P}_{rq}(a)+1}{\mathbf{P}_{rq}(b)}$ which, by Theorem 3, implies the cyclical unsolvability of w by a WMG.

From Lemma 3, for the cyclical solvability by a WMG it is enough to use only places between adjacent transitions. For the sequence $abcbad$ in Fig. 7, transition b follows a and c, and the input place of b in the CF-solution is an output place for both a and c. The situation is similar for transition a, which follows b and d. However, this is not always the case when we are looking for a solution in the class of CF nets. For instance, the sequence $cabdaaabeab$ is cyclically solvable by a CF net (see Fig. 8). In this sequence, b always follows a. But in order to solve ESSPs against b, we need a place which is an output place for c and e (in addition to a).

Algorithm 1: WMG-cycles

input : $w \in T^*,\ T = \{t_0, \ldots, t_{n-1}\}$
output: A WMG N cyclically solving w if it exists
var: $T[0\,..\,|T|-1] = (t_0, \ldots, t_{n-1})$, $v[0\,..\,|w|-1]$, a, b, na, nb, ia, ib, M, $Mmin$;
compute the Parikh-vector $\mathbf{P}[0\,..\,|T|-1]$ of w;
if $\mathbf{P}$ *is not prime* **then** **return** *unsolvable* ; `// Parikh-primeness`
$b \leftarrow w[0]$;
for $j = 0$ **to** $|T|-1$ **do** `// index of` b
  if $b = T[j]$ **then** $ib \leftarrow j$;
for $i = 0$ **to** $|w|-1$ **do**
  $v \leftarrow w[i] \ldots w[|w|-1]w[0] \ldots w[i-1]$; `// rotation of` w
  $a \leftarrow b$, $b \leftarrow v[1]$, $ia \leftarrow ib$; `// fix first adjacent pair`
  for $j = 0$ **to** $|T|-1$ **do**
    if $b = T[j]$ **then** $ib \leftarrow j$;
  $na \leftarrow 1$, $nb \leftarrow 1$;
  if $a \neq b$ **then**
    for $k = 2$ **to** $|w|-1$ **do**
      if $\frac{\mathbf{P}[ib]}{\mathbf{P}[ia]} \geq \frac{\mathbf{P}[ib]-nb+1}{\mathbf{P}[ia]-na}$ **then**
        return *unsolvable* ; `// check solvability condition`
      if $v[k] = T[ia]$ **then** $na \leftarrow na + 1$;
      if $v[k] = T[ib]$ **then** $nb \leftarrow nb + 1$;
    $M \leftarrow \mathbf{P}[ia] \cdot \mathbf{P}[ib]$, $Mmin \leftarrow M$;
    for $k = 0$ **to** $|w|-1$ **do** `// find initial marking`
      if $w[k] = a$ **then** $M \leftarrow M + \mathbf{P}[ib]$;
      if $w[k] = b$ **then** $M \leftarrow M - \mathbf{P}[ia]$;
      if $M < Mmin$ **then** $Mmin \leftarrow M$;
    add new place $p_{T[ia],T[ib]}$ to N with
    $W(T[ia], p) = \mathbf{P}[ib]$, $W(p, T[ib]) = \mathbf{P}[ia]$, $M_0 = \mathbf{P}[ia] \cdot \mathbf{P}[ib] - Mmin$;
return N

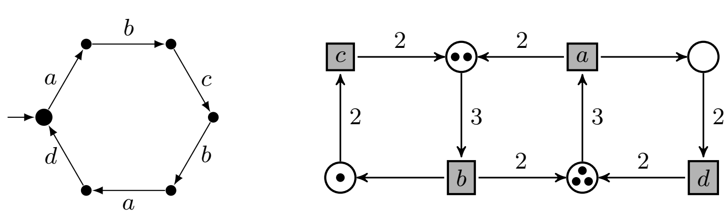

Fig. 7. Sequence *abcbad* is cyclically solved by the CF net on the right.

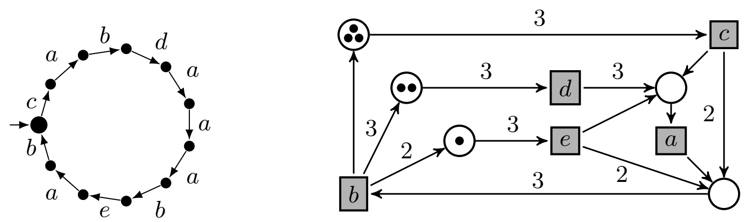

Fig. 8. Sequence *cabdaaabeab* is cyclically solved by the CF-net on the right.

Indeed, if there is a place $p_{a,b}$ as on the left of Fig. 9 which solves ESSPs against b, then for $ca\,|_s\, bdaa\,|_q\, abeab$

$$\begin{array}{lll} s[b\rangle & : \mu_0 + 3 & \geq 5 \\ \neg q[b\rangle & : \mu_0 + 3 \cdot 3 - 5 & < 5 \end{array}$$

Subtracting the first inequality from the second one, we get $6 - 5 < 0$, which is a contradiction. Hence, a place of form $p_{a,b}$ cannot solve all ESSPs against b in *cabdaaabeab*.

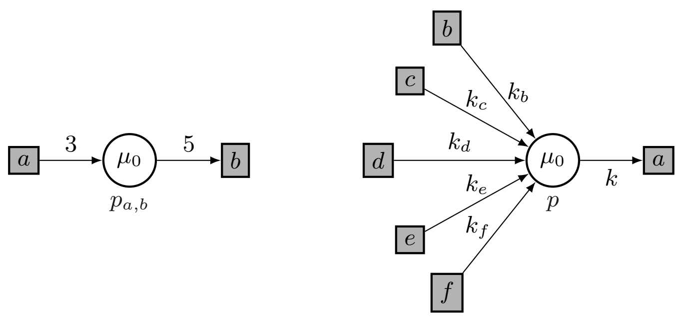

Fig. 9. A place from a to b (left); a place of a CF net with 6 transitions (right).

In a WMG, a place can have at most one input transition. This restriction is relaxed for choice-free nets and multiple inputs are allowed. Nevertheless, this does not imply that a single input place for each transition will always be sufficient. As an instance, consider the sequence *bcafdeaaabcdaafdcaaa* (see Fig. 10) which is cyclically solvable with a CF net.

Assume that we can solve all ESSPs against transition a with a single place p as on the right of Fig. 9 (we know that we do not need any side-conditions). Then, for p and $w = |_{s_0}\, b\ c\ a\ f\ d\,|_{s_5}\, e\,|_{s_6}\, a\ a\ a\ b\ c\,|_{s_{11}}\, d\,|_{s_{12}}\, a\ a\ f\ d\,|_{s_{16}}\, c\,|_{s_{17}}\, a\ a\ a$, the

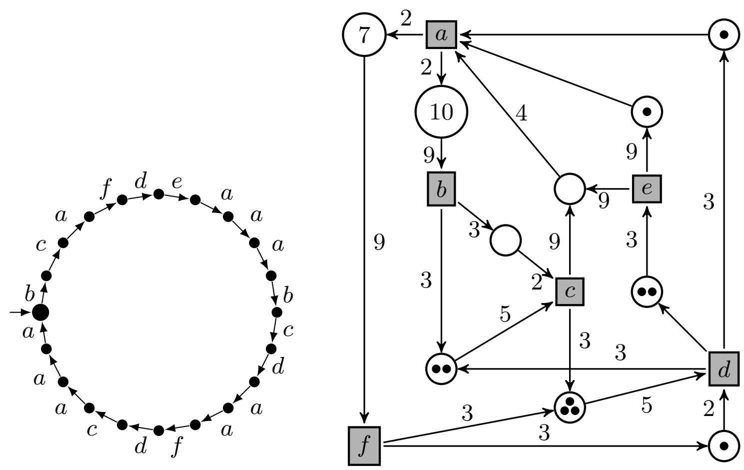

Fig. 10. $w = bcafdeaaabcdaafdcaaa$ is cyclically solved by the CF net on the right.

following system of inequalities must hold true:

$$
\begin{array}{lllll}
\text{cycle} & : 2 \cdot k_b + 3 \cdot k_c + 3 \cdot k_d + k_e + 2 \cdot k_f & = & 9 \cdot k & (0) \\
\neg s_5[a\rangle & : \mu_0 + k_b + k_c + k_d + k_f - k & < & k & (1) \\
s_6[aaa\rangle & : \mu_0 + k_b + k_c + k_d + k_e + k_f - k & \geq & 3 \cdot k & (2) \\
\neg s_{11}[a\rangle & : \mu_0 + 2 \cdot k_b + 2 \cdot k_c + k_d + k_e + k_f - 4 \cdot k & < & k & (3) \\
s_{12}[aa\rangle & : \mu_0 + 2 \cdot k_b + 2 \cdot k_c + 2 \cdot k_d + k_e + k_f - 4 \cdot k & \geq & 2 \cdot k & (4) \\
\neg s_{16}[a\rangle & : \mu_0 + 2 \cdot k_b + 2 \cdot k_c + 3 \cdot k_d + k_e + 2 \cdot k_f - 6 \cdot k & < & k & (5) \\
s_{17}[aaa\rangle & : \mu_0 + 2 \cdot k_b + 3 \cdot k_c + 3 \cdot k_d + k_e + 2 \cdot k_f - 6 \cdot k & \geq & 3 \cdot k & (6)
\end{array}
$$

From the system above we obtain:

$$
\begin{array}{llll}
(2) - (1) & : k_e & > & 2 \cdot k \\
(4) - (3) & : k_d & > & k \\
(6) - (5) & : k_c & > & 2 \cdot k
\end{array}
$$

which implies $3 \cdot k_c + 3 \cdot k_d + k_e > 13 \cdot k$, contradicting the equality (0). Hence, the ESSPs against a cannot be solved by a single place.

5 Conclusions and Perspectives

In this work, we specialised previous methods of analysis and synthesis to the CF nets and their WMG subclass, two useful subclasses of weighted Petri nets allowing to model various real-world applications.

We highlighted the correspondance between CF-solvability and WMG-solvability for binary alphabets. We also tackled the case of an LTS formed of a single circuit with an arbitrary number of letters, for which we developed a characterisation of WMG-solvability together with a dedicated and efficient synthesis algorithm. Finally, we discussed the applicability of our conditions to CF synthesis.

As a natural continuation of the work, we expect extensions of our results in two directions: generalising the class of goal-nets (e.g. to choice-free or fork-attribution nets), and relaxing the restrictions for the LTS under consideration.

Acknowledgements

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