Temporal *DL-Lite* over Finite Traces (Preliminary Results)

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Abstract. We transfer results on the temporal *DL-Lite* family of logics, moving from infinite models of linear time to the case of *finite traces*. In particular, we investigate the complexity of the satisfiability problem in various fragments of $T_{\mathcal{U}}DL\text{-}Lite^{\mathcal{N}}_{bool}$, distinguishing the case of global axioms from the case where axioms are interpreted locally. We also consider satisfiability on traces bounded by a fixed number of time points.

1 Introduction

Temporal description logics based on linear temporal logic (*LTL*) are interpreted over a flow of time which is generally represented by the infinite linear order of the natural numbers [9, 10, 5]. A renewed interest in *LTL* interpreted over finite traces, i.e., over structures based on finite initial segments of the natural numbers [7], has recently motivated the study of temporal formalisms for knowledge representation interpreted on a finite, or even bounded, temporal dimension [3, 4]. Logics in the temporal *DL-Lite* family, suitable for temporal conceptual data modelling, have only been investigated over infinite temporal structures [1, 2].

We provide first results for the complexity of lightweight temporal description logics when interpreted over finite traces. In particular, we consider the problem of formula satisfiability (where formulas are intended to hold locally, at the first time point), and knowledge base satisfiability (in which concept inclusions hold globally, i.e., at all time points). Many of the complexity results presented in this work are obtained by adapting proofs from the infinite case to the finite traces case. However, straightforward adaptations are not always applicable. In particular, our lower bound in Theorem 8 is based on a new encoding of arithmetic progressions.

In Section 2, we introduce the syntax of a family of $TDL\text{-}Lite^{\mathcal{N}}$ logics, defining their semantics on finite temporal structures. Preliminary complexity results for the formula and the knowledge base satisfiability problems over finite traces are given in Section 3. Section 4 studies reasoning over traces with a fixed bound on the number of instants, given in binary as part of the input. In Section 5 we point directions for future work.

2 Temporal *DL-Lite*

We define the language of $T_{\mathcal{U}}$*DL-Lite*$^{\mathcal{N}}_{bool}$ as follows [1,2]. Let $\mathsf{N_C}, \mathsf{N_I}$ be countable sets of *concept* and *individual names*, respectively, and let $\mathsf{N_L}$ and $\mathsf{N_G}$ be countable and disjoint sets of *local* and *global role names*, respectively. The union $\mathsf{N_L} \cup \mathsf{N_G}$ is the set $\mathsf{N_R}$ of *role names.* $T_{\mathcal{U}}$*DL-Lite*$^{\mathcal{N}}_{bool}$ *roles* R, *basic concepts* B, *concepts* C, and *temporal concepts* D are given by the following grammar:

$$R ::= L \mid L^- \mid G \mid G^-, \qquad\qquad B ::= \bot \mid A \mid \geq qR,$$
$$C ::= B \mid D \mid \neg C \mid C_1 \sqcap C_2, \qquad\qquad D ::= C \mid C_1 \,\mathcal{U}\, C_2$$

where $L \in \mathsf{N_L}$, $G \in \mathsf{N_G}$, $A \in \mathsf{N_C}$, and $q \in \mathbb{N}, q > 0$, given in binary. A role R is said to be *local*, if it is of the form L or L^-, with $L \in \mathsf{N_L}$, and *global*, if it is of the form G or G^-, with $G \in \mathsf{N_G}$. A $T_{\mathcal{U}}$*DL-Lite*$^{\mathcal{N}}_{bool}$ *axiom* is either a *concept inclusion (CI)* of the form $C_1 \sqsubseteq C_2$, or an *assertion*, α, of the form $C(a)$ or $R(a, b)$, where C, C_1, C_2 are $T_{\mathcal{U}}$*DL-Lite*$^{\mathcal{N}}_{bool}$ concepts, R is a role, and $a, b \in \mathsf{N_I}$. $T_{\mathcal{U}}$*DL-Lite*$^{\mathcal{N}}_{bool}$ *formulas* have the form

$$\varphi, \psi ::= \alpha \mid C_1 \sqsubseteq C_2 \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \,\mathcal{U}\, \psi.$$

We will use the following standard equivalences for concepts: $\top \equiv \neg\bot$; $(C_1 \sqcup C_2) \equiv \neg(\neg C_1 \sqcap \neg C_2)$; $\bigcirc C \equiv \bot \,\mathcal{U}\, C$; $\bigcirc^{n+1} C \equiv \bigcirc \bigcirc^n C$, with $n \in \mathbb{N}$ (we set $\bigcirc^0 C \equiv C$); $\Diamond C \equiv \top \,\mathcal{U}\, C$; $\Box C \equiv \neg\Diamond\neg C$; $\Diamond^+ C \equiv C \sqcup \Diamond C$; and $\Box^+ C \equiv \neg\Diamond^+\neg C$ (similarly for formulas).

We consider also the restricted setting where formulas are limited to conjunctions of CIs (globally interpreted, cf. e.g. [2]) and assertions. In this case, a $T^f_{\mathcal{U}}$*DL-Lite*$^{\mathcal{N}}_{bool}$ *TBox* $\mathcal{T}$ is a finite set of CIs. A $T^f_{\mathcal{U}}$*DL-Lite*$^{\mathcal{N}}_{bool}$ *ABox* $\mathcal{A}$ is a finite set of assertions of the of the form

$$\bigcirc^n A(a), \quad \bigcirc^n \neg A(a), \quad \bigcirc^n S(a, b), \quad \bigcirc^n \neg S(a, b)$$

where $A \in \mathsf{N_C}$, $S \in \mathsf{N_R}$, and $a, b \in \mathsf{N_I}$. A $T^f_{\mathcal{U}}$*DL-Lite*$^{\mathcal{N}}_{bool}$ *knowledge base* (KB) $\mathcal{K}$ is a pair $(\mathcal{T}, \mathcal{A})$.

Semantics. A $T_{\mathcal{U}}$*DL-Lite*$^{\mathcal{N}}_{bool}$ *interpretation* is a structure $\mathfrak{M} = (\Delta^{\mathfrak{M}}, (I_n)_{n \in \mathfrak{T}})$, where $\mathfrak{T}$ is an interval of the form $[0, \infty)$ or $[0, l]$, with $l \in \mathbb{N}$, and each I_n is a classical DL interpretation with domain $\Delta^{\mathfrak{M}}$ (or simply Δ). We have that $A^{I_n} \subseteq \Delta^{\mathfrak{M}}$ and $S^{I_n} \subseteq \Delta^{\mathfrak{M}} \times \Delta^{\mathfrak{M}}$, for all $A \in \mathsf{N_C}$ and $S \in \mathsf{N_R}$: in particular, for all $G \in \mathsf{N_G}$ and $i, j \in \mathbb{N}$, $G^{I_i} = G^{I_j}$ (denoted simply by G^I). Moreover, $a^{I_i} = a^{I_j} \in \Delta^{\mathfrak{M}}$ for all $a \in \mathsf{N_I}$ and $i, j \in \mathbb{N}$, i.e., constants are *rigid designators* (with fixed interpretation, denoted simply by a^I). The stipulation that all time points share the same domain $\Delta^{\mathfrak{M}}$ is called the *constant domain assumption* (meaning that objects are not created nor destroyed over time). The interpretation of roles and concepts at instant n is defined as follows (where $S \in \mathsf{N_R}$):

$$(S^-)^{I_n} = \{(d', d) \in \Delta^{\mathfrak{M}} \times \Delta^{\mathfrak{M}} \mid (d, d') \in S^{I_n}\}, \qquad \bot^{I_n} = \emptyset,$$
$$(\geq qR)^{I_n} = \{d \in \Delta^{\mathfrak{M}} \mid |\{d' \in \Delta^{\mathfrak{M}} \mid (d, d') \in R^{I_n}\}| \geq q\},$$
$$(\neg C)^{I_n} = \Delta^{\mathfrak{M}} \setminus C^{I_n}, \qquad (C_1 \sqcap C_2)^{\mathcal{I}_n} = C_1^{I_n} \cap C_2^{I_n},$$
$$(C_1 \,\mathcal{U}\, C_2)^{I_n} = \{d \in \Delta^{\mathfrak{M}} \mid \exists m \in \mathfrak{T}, m > n \colon d \in C_2^{I_m} \wedge \forall i \in (n, m) \colon d \in C_1^{I_i}\}.$$

We say that a concept C is *satisfied in* $\mathfrak{M}$ if $C^{I_0} \neq \emptyset$. *Satisfaction of a formula* φ *in* $\mathfrak{M}$ *at time point* $n \in \mathfrak{T}$ (written $\mathfrak{M}, n \models \varphi$) is inductively defined as follows:

$\mathfrak{M}, n \models C \sqsubseteq D$	iff $C^{I_n} \subseteq D^{I_n}$,	$\mathfrak{M}, n \models \neg\psi$	iff not $\mathfrak{M}, n \models \phi$,
$\mathfrak{M}, n \models C(a)$	iff $a^I \in C^{I_n}$,	$\mathfrak{M}, n \models \psi \wedge \chi$	iff $\mathfrak{M}, n \models \psi$ and $\mathfrak{M}, n \models \chi$,
$\mathfrak{M}, n \models R(a, b)$	iff $(a^I, b^I) \in R^{I_n}$,	$\mathfrak{M}, n \models \psi \,\mathcal{U}\, \chi$	iff $\exists m \in \mathfrak{T}, m > n \colon \mathfrak{M}, m \models \chi$, and $\forall i \in (n, m) \colon \mathfrak{M}, i \models \psi$.

We say that φ is *satisfied in* $\mathfrak{M}$, writing $\mathfrak{M} \models \varphi$, if $\mathfrak{M}, 0 \models \varphi$, and that it is *satisfiable* if it is satisfied in some $\mathfrak{M}$. For a KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, we say that $\mathcal{K}$ is *satisfied in* $\mathfrak{M}$ if all CIs in $\mathcal{T}$ are satisfied in $\mathfrak{M}$ at *all* time points, i.e., $\mathfrak{M} \models C \sqsubseteq D$ iff $C^{I_n} \subseteq D^{I_n}$, for all $n \in \mathfrak{T}$ (they are *globally* satisfied), and all assertions in $\mathcal{A}$ are satisfied in $\mathfrak{M}$ at time point 0. $\mathcal{K}$ is *satisfiable* if it is satisfied in some $\mathfrak{M}$. In the following, we call *finite trace* an interpretation with $\mathfrak{T} = [0, l]$, often denoted by $\mathfrak{F} = (\Delta^{\mathfrak{F}}, (\mathcal{F}_n)_{n \in [0,l]})$, while *infinite traces*, based on $\mathfrak{T} = [0, \infty)$, will be denoted by $\mathfrak{I} = (\Delta^{\mathfrak{I}}, (\mathcal{I}_n)_{n \in [0,\infty)})$. We say that a $T_{\mathcal{U}}$*DL-Lite*$^{\mathcal{N}}_{bool}$ formula φ or KB $\mathcal{K}$ is *satisfiable on infinite*, *finite*, or k*-bounded traces*, if it is satisfied in a trace in the class of infinite, finite, or finite traces with at most $k \in \mathbb{N}, k > 0$ (given in binary) time points, respectively.

We consider the ($\Box\bigcirc$)-fragment, denoted $T_{\Box\bigcirc}$*DL-Lite*$^{\mathcal{N}}_{bool}$, with temporal concepts of the form

$$D ::= C \mid \Box C \mid \bigcirc C \qquad (\Box\bigcirc)$$

Furthermore, we define $T_{\mathcal{U}}$*DL-Lite*$^{\mathcal{N}}_{horn}$, $T_{\mathcal{U}}$*DL-Lite*$^{\mathcal{N}}_{krom}$, and $T_{\mathcal{U}}$*DL-Lite*$^{\mathcal{N}}_{core}$ as the fragments of $T_{\mathcal{U}}$*DL-Lite*$^{\mathcal{N}}_{bool}$ having, respectively, CIs of the form

$$D_1 \sqcap \ldots \sqcap D_k \sqsubseteq D \qquad (horn)$$
$$D_1 \sqsubseteq D_2, \qquad \neg D_1 \sqsubseteq D_2, \qquad D_1 \sqsubseteq \neg D_2 \qquad (krom)$$
$$D_1 \sqsubseteq D_2, \qquad D_1 \sqcap D_2 \sqsubseteq \bot \qquad (core)$$

and the respective ($\Box\bigcirc$)-fragments where temporal concepts D are defined from concepts C of the form

$$C ::= B \mid D.$$

Table 1 summarises the logics studied, and the corresponding complexity results.

3 Satisfiability on Finite Traces

We present complexity results for the finite satisfiability checking problem, distinguishing the case where formulas are allowed from the case with just knowledge bases (where axioms are interpreted globally).

		finite traces		k-bounded traces
		$T_{\mathcal{U}}$*DL-Lite*$^{\mathcal{N}}$	$T_{\Box\bigcirc}$*DL-Lite*$^{\mathcal{N}}$	$T_{\mathcal{U}}$*DL-Lite*$^{\mathcal{N}}$
bool	φ	ExpSpace Th. 2	ExpSpace Th. 2	NExpTime Th. 9
	$\mathcal{K}$	PSpace Th. 6	PSpace Th. 7	PSpace $\leq$ Th. 12
horn	φ	ExpSpace Th. 2	ExpSpace Th. 2	NExpTime Th. 10
	$\mathcal{K}$	PSpace	PSpace Th. 7	PSpace
krom	$\mathcal{K}$	PSpace	?	PSpace
core	$\mathcal{K}$	PSpace Th. 6	$\geq$ NP Th. 8	PSpace $\geq$ Th. 11

Table 1. Complexity results for *TDL-Lite*$^{\mathcal{N}}$ fragments on finite and bounded traces.

Formula satisfiability. Following [3], we define the *end of time formula* ψ_f as the conjunction of the following $T_{\bigcirc}$*DL-Lite*$^{\mathcal{N}}_{krom}$ formulas (i.e., with temporal concepts of the form $D ::= C \mid \bigcirc D$):

$$\psi_{f_1} = \top \sqsubseteq \neg E, \quad \psi_{f_2} = (\top \sqsubseteq \neg E)\,\mathcal{U}\,(\top \sqsubseteq E), \quad \psi_{f_3} = \Box^+(E \sqsubseteq \bigcirc E),$$

where E is a fresh concept name representing the end of time. The translation $\cdot^\dagger$ from $T_{\mathcal{U}}$*DL-Lite*$^{\mathcal{N}}_{bool}$ concepts and formulas to $T_{\mathcal{U}}$*DL-Lite*$^{\mathcal{N}}_{bool}$ concepts and formulas, respectively, is defined as follows:

$$\begin{array}{rlrl}
(\bot)^\dagger & \mapsto \bot & (C \sqsubseteq D)^\dagger & \mapsto C^\dagger \sqsubseteq D^\dagger \\
(A)^\dagger & \mapsto A & (C(a))^\dagger & \mapsto C^\dagger(a) \\
(\geq qR)^\dagger & \mapsto \geq qR & (R(a,b))^\dagger & \mapsto R(a,b) \\
(\neg C)^\dagger & \mapsto \neg C^\dagger & (\neg\varphi)^\dagger & \mapsto \neg\varphi^\dagger \\
(C \sqcap D)^\dagger & \mapsto C^\dagger \sqcap D^\dagger & (\varphi \wedge \psi)^\dagger & \mapsto \varphi^\dagger \wedge \psi^\dagger \\
(C\,\mathcal{U}\,D)^\dagger & \mapsto C^\dagger\,\mathcal{U}\,(D^\dagger \sqcap \neg E) & (\varphi\,\mathcal{U}\,\psi)^\dagger & \mapsto \varphi^\dagger\,\mathcal{U}\,(\psi^\dagger \wedge \top \sqsubseteq \neg E)
\end{array}$$

We obtain the reduction to the infinite traces case with the following lemma, that will be used to show the ExpSpace upper bound in Theorem 2.

Lemma 1. *A* $T_{\mathcal{U}}$*DL-Lite*$^{\mathcal{N}}_{bool}$ *formula* φ *is satisfiable on finite traces iff* $\psi_f \wedge \varphi^\dagger$ *is satisfiable on infinite traces.*

Theorem 2. $T_{\mathcal{U}}$*DL-Lite*$^{\mathcal{N}}_{bool}$ *and* $T_{\Box\bigcirc}$*DL-Lite*$^{\mathcal{N}}_{horn}$ *formula satisfiability on finite traces is* ExpSpace*-complete.*

Proof. The upper bound follows from the reduction in Lemma 1 to the same problem in $T_{\mathcal{U}}$*DL-Lite*$^{\mathcal{N}}_{bool}$ on infinite traces, which is known to be ExpSpace-complete [1]. The lower bound is obtained by a reduction of the $m \times 2^n$ corridor tiling problem to satisfiability on finite traces of formulas in $T_{\Box\bigcirc}$*DL-Lite*$^{\mathcal{N}}_{horn}$ (similar to [1, Theorem 10], modified for the finite traces case). □

Knowledge base satisfiability. In this section, $T_{\mathcal{U}}$*DL-Lite*$^{\mathcal{N}}_{bool}$ KB satisfiability on finite traces is reduced to *LTL* formula satisfiability on finite traces, which is known to be PSPACE-complete [7]. The proof is an adaptation of the reduction of $T_{\mathcal{U}}$*DL-Lite*$^{\mathcal{N}}_{bool}$ to *LTL* on infinite traces [2], and it proceeds in two steps. First, the $T_{\mathcal{U}}$*DL-Lite*$^{\mathcal{N}}_{bool}$ KB satisfiability problem is reduced to the formula satisfiability problem in the one-variable fragment of first-order temporal logic on finite traces, $T_{\mathcal{U}}\mathcal{QL}^1$ [8]. Then, the satisfiability of the resulting $T_{\mathcal{U}}\mathcal{QL}^1$ formulas is reduced to satisfiability on finite traces of $T_{\mathcal{U}}\mathcal{QL}^1$ formulas without positive occurrences of existential quantifiers, which are essentially *LTL* formulas.

First-order temporal logic on finite traces. The *alphabet* of $T_{\mathcal{U}}\mathcal{QL}$ consists of countably infinite and pairwise disjoint sets of *predicates* $\mathsf{N_P}$ (with $\mathsf{ar}(P) \in \mathbb{N}$ being the arity of $P \in \mathsf{N_P}$), *constants* (or *individual names*) $\mathsf{N_I}$, and *variables* Var; the *logical operators* $\neg, \wedge$; the *existential quantifier* $\exists$, and the *temporal operator* $\mathcal{U}$ (*until*). *Formulas* of $T_{\mathcal{U}}\mathcal{QL}$ have the form:

$$\varphi, \psi ::= P(\bar{\tau}) \mid \neg\varphi \mid \varphi \wedge \psi \mid \exists x \varphi \mid \varphi\,\mathcal{U}\,\psi,$$

where $P \in \mathsf{N_P}$, $\bar{\tau} = (\tau_1, \dots, \tau_{\mathsf{ar}(P)})$ is a tuple of *terms*, i.e., constants or variables, and $x \in \mathsf{Var}$. We write $\varphi(x_1, \dots, x_m)$ to indicate that the free variables of a formula φ are in $\{x_1, \dots, x_m\}$. For $p \in \mathbb{N}$, the *p-variable fragment* of $T_{\mathcal{U}}\mathcal{QL}$, denoted by $T_{\mathcal{U}}\mathcal{QL}^p$, consists of $T_{\mathcal{U}}\mathcal{QL}$ formulas with at most p variables ($T_{\mathcal{U}}\mathcal{QL}^0$ is simply propositional *LTL*).

A *first-order temporal model on a finite trace* (or simply a *first-order finite trace*) is a structure $\mathscr{M} = (\mathscr{D}, (\mathscr{I}_n)_{n\in\mathfrak{T}})$, where $\mathfrak{T}$ is an interval of the form $[0, l]$, with $l \in \mathbb{N}$, and each $\mathscr{I}_n$ is a classical first-order interpretation with domain $\mathscr{D}$. We have $P^{\mathscr{I}_n} \subseteq \mathscr{D}^{\mathsf{ar}(P)}$, for each $P \in \mathsf{N_P}$, and for all $a \in \mathsf{N_I}$ and $i, j \in \mathbb{N}$, $a^{\mathscr{I}_i} = a^{\mathscr{I}_j} \in \mathscr{D}$ (denoted simply by $a^{\mathscr{I}}$). An *assignment* in $\mathscr{M}$ is a function $\mathfrak{a}$ from terms to $\mathscr{D}$: $\mathfrak{a}(\tau) = \mathfrak{a}(x)$, if $\tau = x$, and $\mathfrak{a}(\tau) = a^{\mathscr{I}}$, if $\tau = a \in \mathsf{N_I}$ *Satisfaction* of a formula φ in $\mathscr{M}$ at time point $n \in \mathfrak{T}$ under assignment $\mathfrak{a}$ (written $\mathscr{M}, n \models^{\mathfrak{a}} \varphi$) is inductively defined as:

$\mathscr{M}, n \models^{\mathfrak{a}} P(\tau_1, \dots, \tau_{\mathsf{ar}(P)})$	iff	$\mathfrak{a}(\tau_1), \dots, \mathfrak{a}(\tau_{\mathsf{ar}(P)}) \in P^{\mathscr{I}_n}$,
$\mathscr{M}, n \models^{\mathfrak{a}} \neg\varphi$	iff	not $\mathscr{M}, n \models^{\mathfrak{a}} \varphi$,
$\mathscr{M}, n \models^{\mathfrak{a}} \varphi \wedge \psi$	iff	$\mathscr{M}, n \models^{\mathfrak{a}} \varphi$ and $\mathscr{M}, n \models^{\mathfrak{a}} \psi$,
$\mathscr{M}, n \models^{\mathfrak{a}} \exists x\varphi$	iff	$\mathscr{M}, n \models^{\mathfrak{a}'} \varphi$ for some assignment $\mathfrak{a}'$ that can differ from $\mathfrak{a}$ only on x,
$\mathscr{M}, n \models^{\mathfrak{a}} \varphi\,\mathcal{U}\,\psi$	iff	$\exists m \in \mathfrak{T}, m > n$: $\mathscr{M}, m \models^{\mathfrak{a}} \psi$ and $\forall i \in (n, m)$: $\mathscr{M}, i \models^{\mathfrak{a}} \varphi$.

The standard abbreviations are used for other connectives. We say that φ is *satisfied in* $\mathscr{M}$ (and $\mathscr{M}$ is a *model* of φ), writing $\mathscr{M} \models \varphi$, if $\mathscr{M}, 0 \models^{\mathfrak{a}} \varphi$, for some $\mathfrak{a}$. Moreover, φ is said to be *satisfiable* if it is satisfied in some $\mathscr{M}$. If a formula φ contains no free variables (i.e., φ is a *sentence*), then we omit the assignment $\mathfrak{a}$ in $\mathscr{M}, n \models^{\mathfrak{a}} \varphi$ and write $\mathscr{M}, n \models \varphi$. If φ has a single free variable x, then we write $\mathscr{M}, n \models \varphi[a]$ in place of $\mathscr{M}, n \models^{\mathfrak{a}} \varphi$ with $\mathfrak{a}(x) = a$.

Reduction to $T_{\mathcal{U}}\mathcal{QL}^1$ formula satisfiability. Here we show how to adapt to the finite traces case the reduction of KB satisfiability to the one-variable fragment of first order temporal logic, given in [2]. For a $T_{\mathcal{U}}DL\text{-}Lite^{\mathcal{N}}_{bool}$ KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, let $\mathsf{ind}_{\mathcal{A}}$ be the set of all individual names occurring in $\mathcal{A}$, and $\mathsf{role}_{\mathcal{K}}$ the set of global and local role names occurring in $\mathcal{K}$ and their inverses. In this reduction, individual names $a \in \mathsf{ind}_{\mathcal{A}}$ are mapped to constants a, concept names A to unary predicates $A(x)$, and number restrictions $\geq q\, R$ to unary predicates $E_q R(x)$. Recall that, for $S \in \mathsf{N_R}$, the predicates $E_q S(x)$ and $E_q S^-(x)$ represent, at each moment of time, the sets of elements with *at least* q distinct S-successors and *at least* q distinct S-predecessors (in particular, $E_1 S(x)$ and $E_1 S^-(x)$ represent the domain and the range of S, respectively). By induction on the construction of a $T_{\mathcal{U}}DL\text{-}Lite^{\mathcal{N}}_{bool}$ concept C, we define the $T_{\mathcal{U}}\mathcal{QL}^1$ formula $C^*(x)$:

$$A^* = A(x), \qquad \bot^* = \bot, \qquad (\geq q\, R)^* = E_q R(x),$$
$$(C_1 \,\mathcal{U}\, C_2)^* = C_1^* \,\mathcal{U}\, C_2^*, \qquad (C_1 \sqcap C_2)^* = C_1^* \wedge C_2^*, \qquad (\neg C)^* = \neg C^*.$$

For a TBox $\mathcal{T}$, we consider the following sentence, saying that the CIs in $\mathcal{T}$ hold globally (the reflexive box $\Box^+$ plays here the role of the 'always' box operator $\boxast$ in [2], since we consider traces based on initial segments of the natural numbers):

$$\mathcal{T}^\dagger = \bigwedge_{C_1 \sqsubseteq C_2 \in \mathcal{T}} \Box^+ \forall x \big(C_1^*(x) \rightarrow C_2^*(x) \big).$$

Now we have to ensure that the predicates $E_q R(x)$ behave similarly to the number restrictions they replace. Denote by $Q_{\mathcal{T}}$ the set of numerical parameters in number restrictions of $\mathcal{T}$:

$$Q_{\mathcal{T}} \;=\; \{\, 1 \,\} \cup \{\, q \;\mid\; \geq q\, H \text{ occurs in } \mathcal{T} \,\}.$$

Then, the following three properties hold in $T_{\mathcal{U}}DL\text{-}Lite^{\mathcal{N}}_{bool}$ finite traces, for all roles R at all instants: (i) every individual with at least q' R-successors has at least q R-successors, for $q < q'$; (ii) if R is a global role, then every individual with at least q R-successors at some moment has at least q R-successors at all moments of time; (iii) if the domain of a role is not empty, then its range is not empty either. These conditions can be encoded by the following $T_{\mathcal{U}}\mathcal{QL}^1$ sentences:

$$\bigwedge_{R \in \mathsf{role}_{\mathcal{K}}} \; \bigwedge_{\substack{q, q' \in Q_{\mathcal{T}} \\ q < q' \\ \neg \exists q'' \in Q_{\mathcal{T}} :\, q < q'' < q'}} \Box^+ \forall x \big((\geq q'\, R)^*(x) \rightarrow (\geq q\, R)^*(x) \big), \tag{1}$$

$$\bigwedge_{\substack{R \in \mathsf{role}_{\mathcal{K}} \\ R \text{ global}}} \; \bigwedge_{q \in Q_{\mathcal{T}}} \Box^+ \; \forall x \Big[\big((\geq q\, R)^*(x) \rightarrow \Box (\geq q\, R)^*(x) \big) \wedge \big(\bigcirc (\geq q\, R)^*(x) \rightarrow (\geq q\, R)^*(x) \big) \Big], \tag{2}$$

$$\bigwedge_{R \in \mathsf{role}_{\mathcal{K}}} \Box^+ \big(\exists x\, (\exists R)^*(x) \;\rightarrow\; \exists x\, (\exists \mathsf{inv} R)^*(x) \big), \tag{3}$$

where $\mathsf{inv}R$ is the inverse of R, i.e., $\mathsf{inv}S = S^-$ and $\mathsf{inv}S^- = S$, for a role name S. Since we lack the past operators, to encode the condition on global roles, in (2) we use both the $\Box$ and the $\bigcirc$ operators, instead of the $\boxast$ as in [2].

The above reduction is extended to the ABox as in [2]. In particular, we assume that $\mathcal{A}$ contains $\bigcirc^n S^-(b,a)$ whenever it contains $\bigcirc^n S(a,b)$. For each $n \in [0,l]$ and each role R, the *temporal slice* $\mathcal{A}_n^R$ of $\mathcal{A}$ is defined by taking

$$\mathcal{A}_n^R = \begin{cases} \{R(a,b) \mid \bigcirc^m R(a,b) \in \mathcal{A} \text{ for some } m \in [0,l]\}, & R \text{ is a global role,} \\ \{R(a,b) \mid \bigcirc^n R(a,b) \in \mathcal{A}\}, & R \text{ is a local role.} \end{cases}$$

The translation $\mathcal{A}^\dagger$ of the $T_\mathcal{U}$*DL-Lite*$_{bool}^\mathcal{N}$ ABox $\mathcal{A}$ is now defined as follows:

$$\mathcal{A}^\dagger = \bigwedge_{\bigcirc^n A(a) \in \mathcal{A}} \bigcirc^n A(a) \wedge \bigwedge_{\bigcirc^n \neg A(a) \in \mathcal{A}} \bigcirc^n \neg A(a) \wedge \bigwedge_{\bigcirc^n R(a,b) \in \mathcal{A}} \bigcirc^n (\geq q_{\mathcal{A}(a)}^{R,n} R)^*(a) \wedge \bigwedge_{\substack{\bigcirc^n \neg S(a,b) \in \mathcal{A} \\ S(a,b) \in \mathcal{A}_n^S}} \bot,$$

where $q_{\mathcal{A}(a)}^{R,n}$ is the number of distinct R-successors of a in $\mathcal{A}$ at moment n:

$$q_{\mathcal{A}(a)}^{R,n} = \max\{q \in Q_\mathcal{T} \mid R(a,b_1), \dots, R(a,b_q) \in \mathcal{A}_n^R, \text{ for distinct } b_1, \dots, b_q\}.$$

Finally, we define the $T_\mathcal{U}\mathcal{QL}^1$ *translation* $\mathcal{K}^\dagger$ of $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ as the conjunction of $\mathcal{T}^\dagger$, $\mathcal{A}^\dagger$ and formulas (1)–(3). The size of $\mathcal{T}^\dagger$ and $\mathcal{A}^\dagger$ does not exceed the size of $\mathcal{T}$ and $\mathcal{A}$, respectively. Thus, the size of $\mathcal{K}^\dagger$ is linear in the size of $\mathcal{K}$. Moreover, we have that $\mathcal{K}$ and $\mathcal{K}^\dagger$ are equisatisfiable.

Lemma 3. *On finite traces, a $T_\mathcal{U}$DL-Lite$_{bool}^\mathcal{N}$ KB $\mathcal{K}$ is satisfiable iff the $T_\mathcal{U}\mathcal{QL}^1$ sentence $\mathcal{K}^\dagger$ is satisfiable.*

Reduction to LTL. As in [2], our next aim is to construct an *LTL* formula that is equisatisfiable, on finite traces, with $\mathcal{K}^\dagger$. First, we have that $\mathcal{K}^\dagger$ can be represented in the form $\mathcal{K}^{\dagger_0} \wedge \bigwedge_{R \in \mathsf{role}_\mathcal{K}} \vartheta_R$, where

$$\mathcal{K}^{\dagger_0} = \Box^+ \forall x \varphi(x) \wedge \psi, \qquad \vartheta_R = \Box^+ \forall x \big((\exists R)^*(x) \to \exists x (\exists \mathsf{inv}R)^*(x)\big),$$

for a quantifier-free first-order temporal formula $\varphi(x)$ with a single variable x and unary predicates only, and a variable-free formula ψ. In order to show that it is possible to replace ϑ_R by a formula without existential quantifiers, we require the following lemma.

Lemma 4. *For every $T_\mathcal{U}$DL-Lite$_{bool}^\mathcal{N}$ KB $\mathcal{K}$, if there is a first-order finite trace $\mathscr{M} = (\mathscr{D}, (\mathscr{I}_n)_{n \in [0,l]})$ satisfying $\mathcal{K}^{\dagger_0}$ such that $\mathscr{M}, n_0 \models (\exists R)^*[d]$, for some $n_0 \in [0,l]$ and $d \in \mathscr{D}$, then there is a first-order finite trace $\mathscr{M}'$ extending $\mathscr{M}$ with new elements and satisfying $\mathcal{K}^{\dagger_0}$ such that, for each $n \in [0,l]$, there is $d_n \in \mathscr{D}'$ with $\mathscr{M}', n \models (\exists R)^*[d_n]$.*

Next, for each $R \in \mathsf{role}_\mathcal{K}$, we take a fresh constant d_R and a fresh propositional variable p_R (recall that $\mathsf{inv}R$ is also in $\mathsf{role}_\mathcal{K}$), and consider the following $T_\mathcal{U}\mathcal{QL}^1$

formula:

$$\mathcal{K}^{\ddagger} = \mathcal{K}^{\dagger_0} \wedge \bigwedge_{R \in \mathsf{role}_{\mathcal{K}}} \vartheta'_R, \text{ with}$$

$$\vartheta'_R = \Box^+ \forall x \big[\big((\exists R)^*(x) \to \Box^+ p_R \big) \wedge \big(\bigcirc p_R \to p_R \big) \big] \wedge \big(p_{\mathsf{inv}R} \to (\exists R)^*(d_R) \big)$$

($p_{\mathsf{inv}R}$ and p_R indicate that $\mathsf{inv}R$ and R are non-empty whereas d_R and $d_{\mathsf{inv}R}$ witness that at 0). Notice that in ϑ'_R we again use both $\Box$ and $\bigcirc$ operators, instead of the $\circledast$ used in [2].

Lemma 5. *On finite traces, a $T_{\mathcal{U}}$DL-Lite$^{\mathcal{N}}_{bool}$ KB $\mathcal{K}$ is satisfiable iff the $T_{\mathcal{U}}\mathcal{QL}^1$ sentence $\mathcal{K}^{\ddagger}$ is satisfiable.*

We can thus state the following result.

Theorem 6. *$T_{\mathcal{U}}$DL-Lite$^{\mathcal{N}}_{bool}$ and $T_{\mathcal{U}}$DL-Lite$^{\mathcal{N}}_{core}$ KB satisfiability on finite traces is* PSpace*-complete.*

Proof. The PSpace-hardness for $T_{\mathcal{U}}$DL-Lite$^{\mathcal{N}}_{core}$ is obtained by observing that the PSpace-hardness proof in [2, Theorem 4.5] works also in the case of finite traces. For PSpace-membership, we have that $\mathcal{K}^{\ddagger}$ can be considered as a *LTL* formula (as it does not contain existential quantifiers, and because all the universally quantified variables can be instantiated by all the constants in the formula, which only results in a polynomial blow-up). Moreover, the translation $\cdot^{\ddagger}$ can be done in logarithmic space in the size of $\mathcal{K}$ [2]. Thus, from Lemma 5 and PSpace-membership of *LTL* on finite traces [7], we get the matching upper bounds. $\Box$

Similar complexity results can be obtained restricting the temporal operators to just $\Box$ and $\bigcirc$ combined with the horn fragment.

Theorem 7. *The $T_{\Box\bigcirc}$DL-Lite$^{\mathcal{N}}_{bool}$ and $T_{\Box\bigcirc}$DL-Lite$^{\mathcal{N}}_{horn}$ KB satisfiability on finite traces is* PSpace*-complete.*

Proof. The upper bound follows from Theorem 6. The lower bound can be obtained from the PSpace result in [2, Theorem 4.5] noting that the only axiom requiring the $\mathcal{U}$ operator has the form $S_{ia} \sqsubseteq S_{ia} \,\mathcal{U}\, D_i$ which can be replaced by the horn axiom $H_{iq} \sqcap S_{ja} \sqsubseteq \bigcirc S_{ja}$, for $i \neq j$. The proof then proceeds similarly to Theorem 6. $\Box$

When we further reduce to the core fragment we can obtain the following hardness result. The proof is by reduction of 3SAT as in [2, Theorem 5.4], however, we cannot directly adapt such proof as it relies on an encoding of arithmetic progressions. In this encoding, each time point may represent an assignment satisfying a propositional formula in 3CNF. A symbol d is used to mark time points which do *not* represent a satisfying assignment. The 3CNF formula is satisfiable iff d does not hold at some time point. On finite traces, we cannot explictly encode all (infinitely many) values of the arithmetic progressions. We solve this by encoding a finite portion of the arithmetic progressions *backwards*. Essentially, we mark the last time point using a concept expression of the form $\Box\bot$, which is only true at the last time point. Then, we check whether there is an arbitrarily large but finite trace where d does not hold at time point zero.

Theorem 8. *$T_{\Box\bigcirc}$DL-Lite$^{\mathcal{N}}_{core}$ (and $T_{\Box\bigcirc}$DL-Lite$^{\mathcal{N}}_{krom}$) KB satisfiability on finite traces is* NP*-hard.*

Proof. The proof is by reduction of 3SAT. Let $f = \bigwedge_{i=1}^{n} C_i$ be a 3CNF with m variables $p_1, \ldots, p_m$ and n clauses $C_1, \ldots, C_n$. By a propositional assignment for f we understand a function $\sigma\colon \{p_1, \ldots, p_m\} \to \{0, 1\}$. We will represent such assignments by sets of positive natural numbers. More precisely, let $P_1, \ldots, P_m$ be the first m prime numbers; it is known that P_m does not exceed $O(m^2)$. We say that a natural number k *represents* an assignment σ if k modulo P_i is equivalent to $\sigma(p_i)$, for all i, $1 \leq i \leq m$.

Not every natural number represents an assignment. Consider the following arithmetic progressions:

$$j + P_i \cdot \mathbb{N}, \qquad \text{for } 1 \leq i \leq m \text{ and } 2 \leq j < P_i. \tag{4}$$

Every element of $j + P_i \cdot \mathbb{N}$ is equivalent to j modulo P_i, and so, since $j \geq 2$, cannot represent an assignment. Moreover, every natural number that cannot represent an assignment belongs to one of these arithmetic progressions (see Fig. 1 from [2]).

Let C_i be a clause in f, for example, $C_i = p_{i_1} \vee \neg p_{i_2} \vee p_{i_3}$. Consider the following progression:

$$P^0_{i_1} P^1_{i_2} P^0_{i_3} + P_{i_1} P_{i_2} P_{i_3} \cdot \mathbb{N}. \tag{5}$$

where by $P^0_{i_1} P^1_{i_2} P^0_{i_3}$ we denote the least natural number, say P, such that P modulo P_{i_1} is equivalent to 0, P modulo P_{i_2} is equivalent to 1, and P modulo P_{i_3} is equivalent to 0. Then a natural number represents an assignment making C_i true iff it *does not* belong to the progressions (4) and (5). Thus, a natural number represents a satisfying assignment for f iff it does not belong to any of the progressions of the form (4) and (5), for clauses in f.

We now show how to encode arithmetic progressions as $T_{\Box\bigcirc}$*DL-Lite*$^{\mathcal{N}}_{core}$ KBs. We use a concept name D signalling a truth assignment for f whenever D does not hold. To take advantage of the finite traces for KBs in $T_{\Box\bigcirc}$*DL-Lite*$^{\mathcal{N}}_{core}$ we encode arithmetic progressions starting from the last point of the finite trace and then going backwards, halting whenever D becomes false. To express that D cannot be true in all instants we use the following axioms, with the concept $\Box\bot$ being true just at the last point of the finite trace and F a newly introduced concept name:

$$\Box\bot \sqsubseteq D, \qquad D \sqcap F \sqsubseteq \bot$$

together with the ABox assertion $F(a)$. Each of the arithmetic progressions (4) and (5) have the form $a + b\mathbb{N}$ (with $a > 0$ and $b > 1$) and can be encoded with the following axioms:

$$\begin{aligned} \Box\bot &\sqsubseteq U_0, & \bigcirc U_{j-1} &\sqsubseteq U_j, \text{ for } j = 1, \ldots, a,\\ U_a &\sqsubseteq V_0, & \bigcirc V_{j-1} &\sqsubseteq V_j, \text{ for } j = 1, \ldots, b,\\ V_b &\sqsubseteq V_0, & V_0 &\sqsubseteq D, \end{aligned}$$

where $U_0, \ldots, U_a$ and $V_0, \ldots, V_b$ are fresh atomic concepts. Note that the size of the resulting $T_{\Box\bigcirc}$*DL-Lite*$^{\mathcal{N}}_{core}$ KB is $O(n \cdot m^6)$. One can check that the above KB is satisfiable on finite traces iff f is satisfiable.

The NP upper bound presented in [2, Theorem 4.7], using a translation to the krom fragment of *LTL* on infinite traces, cannot be immediately applied in the finite case, since the complexity of this fragment on finite traces is unknown.

4 Satisfiability on Bounded Traces

In this section we consider satisfiability of $T_{\mathcal{U}}$*DL-Lite*$^{\mathcal{N}}$ formulas and KBs on traces with at most k time points, with k given in *binary*. We start by considering the formula satisfiability problem. We establish that the complexity of the satisfiability problem for $T_{\mathcal{U}}$*DL-Lite*$^{\mathcal{N}}_{bool}$ (and consequently for $T_{\mathcal{U}}$*DL-Lite*$^{\mathcal{N}}_{horn}$) is in NExpTime for traces bounded by k (in binary, given as part of the input). We formalise this result with the following theorem.

Theorem 9. *$T_{\mathcal{U}}$DL-Lite$^{\mathcal{N}}_{bool}$ formula satisfiability on k-bounded traces is in* NExpTime.

Proof. (Sketch) Our result follows from the fact that one can translate any $T_{\mathcal{U}}$*DL-Lite*$^{\mathcal{N}}_{bool}$ formula into an equisatisfiable $T_{\mathcal{U}}\mathcal{QL}^1$ formula [1]. Satisfiability of a $T_{\mathcal{U}}\mathcal{QL}^1$ formula can be solved using *quasimodels* [8, Theorem 11.30], a classical technique used to abstract models. For finite traces, the same notions can be adopted. In particular, one can show that there is a model for a $T_{\mathcal{U}}\mathcal{QL}^1$ formula with k time points if and only if there is a quasimodel for it where the sequence of quasistates has lengh k [3]. If the number of time points is bouned by k (in binary), then satisfiability of this translation can be decided in NExpTime by guessing an exponential size sequence of sets of types and then checking in exponential time that it forms a quasimodel. $\Box$

The next theorem establishes a matching lower bound for $T_{\mathcal{U}}$*DL-Lite*$^{\mathcal{N}}_{horn}$ (and consequently for $T_{\mathcal{U}}$*DL-Lite*$^{\mathcal{N}}_{bool}$) on k-bounded traces.

Theorem 10. *$T_{\mathcal{U}}$DL-Lite$^{\mathcal{N}}_{horn}$ formula satisfiability on k-bounded traces is* NExpTime*-hard.*

Proof. (Sketch) Suppose we are given a finite set T of tile types, a $t_0 \in T$ and a natural number k in binary. We can assume w.l.o.g. that $k = 2^n$. The problem is to decide whether T tiles the grid $2^n \times 2^n$ in such a way that t_0 is placed at $(0, 0)$. We construct essentially the same $T_{\mathcal{U}}$*DL-Lite*$^{\mathcal{N}}_{horn}$ formula $\varphi_{T,t_0,n}$ as in Theorem 10 in [1]. An exponential counter can be used to mark with a concept name M the $2^n - 1$ time points of the trace with 2^n time points. One can then use M on the left side of inclusions to ensure that the $\bigcirc$ operator on the right side is only 'applied' when there is a next time point. We exclude axioms used to encode that the top and bottom sides of the corridor are white, which are not needed for the bounded tiling problem. The main difference is that in the mentioned proof, the formula on infinite traces is used to prove ExpSpace-hardness by reduction from the corridor problem. Here, the number of time points is bounded by k, and so, we can only encode the bounded tiling problem, which gives us NExpTime-hardness [6]. $\square$

We now consider the KB satisfiability problem. The same PSpace-hardness proof for $T_{\mathcal{U}}DL\text{-}Lite^{\mathcal{N}}_{core}$ can applied for $T_{\mathcal{U}}DL\text{-}Lite^{\mathcal{N}}_{core}$ (see Theorems 6 and 7). The main point here is to show that the bound on the number of time points does not affect this hardness proof. Indeed, the proof is by reduction from a polynomial space bounded Turing machine, where each configuration can be encoded in a time point. One can assume w.l.o.g. that the length of a computation is exponential in the size of the input (by removing repetitions in a sequence of configurations). Since k is given in binary, if the length of k is polynomial in the size of the KB given as input, then traces may still have an exponential number of time points (w.r.t the size of the formula). So, the same encoding of the problem holds in this setting.

Theorem 11. *$T_{\mathcal{U}}DL\text{-}Lite^{\mathcal{N}}_{core}$ KB satisfiability on k-bounded traces is* PSpace-*hard.*

The upper bound for $T_{\mathcal{U}}DL\text{-}Lite^{\mathcal{N}}_{bool}$ (and consequently for its fragments) is obtained in the same way as for $T_{\mathcal{U}}DL\text{-}Lite^{\mathcal{N}}_{bool}$ (Theorems 6 and 7), with a translation to *LTL*. The important point here is that the procedure is adapted to ensure that the number of time points is bounded by k. The exact number of time points $t \leq k$ can be guessed and stored in binary using polynomial space w.r.t. the size of k (as a string). Then the procedure is as for *LTL*, with the difference that when we reach t we have to check whether all the 'until's have been realised, that is, whether the finite trace can finish at this time point.

Theorem 12. *$T_{\mathcal{U}}DL\text{-}Lite^{\mathcal{N}}_{bool}$ KB satisfiability on k-bounded traces is in* PSpace.

5 Conclusion

We presented preliminary results on the complexity of reasoning in the $TDL\text{-}Lite^{\mathcal{N}}$ family of languages interpreted on finite traces. Our results show that in terms of complexity, there is not much change between reasoning on finite and infinite traces (except when there is a bound on the time points). However, on the semantical side, there are several expressions that, on finite traces, become satisfiable ($\Box\bot$) and unsatisfiable ($\Box^+ \bigcirc \top$).

We plan to investigate syntactical and semantical ways of characterising the distinction between reasoning on finite and infinite traces. Also, we plan to improve the landscape of complexity results, in particular, to study satisfiability on finite traces in sub-boolean fragments of $T_{\mathcal{U}}DL\text{-}Lite^{\mathcal{N}}_{bool}$, such as the krom and core fragments.

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