GRAPHS CONTAINING FINITE INDUCED PATHS OF UNBOUNDED LENGTH

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ABSTRACT

The age $\mathcal{A}(G)$ of a graph G (undirected and without loops) is the collection of finite induced subgraphs of G, considered up to isomorphy and ordered by embeddability. It is well-quasi-ordered (wqo) for this order if it contains no infinite antichain. A graph is *path-minimal* if it contains finite induced paths of unbounded length and every induced subgraph G' with the same property admits an embedding of G. We construct $2^{\aleph_0}$ path-minimal graphs whose ages are pairwise incomparable with set inclusion and which are wqo. Our construction is based on uniformly recurrent sequences and lexicographical sums of labeled graphs.

Keywords (partially) ordered set · incomparability graph · graphical distance · isometric subgraph · paths · well quasi order · symbolic dynamic · sturmian words · uniformly recurrent sequences

1 Introduction and presentation of the results

We consider graphs that are undirected, simple and have no loops. Let $H = (X, F)$ be a graph and let $(G_x = (V_x, E_x))_{x\in X}$ be a family of graphs whose vertex sets are pairwise disjoint. The *lexicographical sum* of $(G_x)_{x\in X}$ (over the graph H) is the graph $H[G_x, x \in X]$ whose vertex set is $\cup_{x\in X} V_x$ and two vertices u_x and $v_{x'}$ are adjacent if $x = x'$ and $\{u_x, v_x\} \in E_x$ or $x \neq x'$ and $\{x, x'\} \in F$. If H is empty i.e. $F = \emptyset$, the lexicographical sum is called a *direct sum*. Else if H is a complete graph, then the lexicographical sum is called the *complete sum*.

Among those graphs with finite induced paths of unbounded length, we ask which one are unavoidable. If a graph is an infinite path, it can be avoided: it contains a direct sum of finite paths of unbounded length. This latter one, on the other hand, cannot be avoided. Indeed, two direct sums of finite paths of unbounded length embed in each other. Similarly, two complete sums of finite paths of unbounded length embed in each other. Hence, the direct sum, respectively the complete sum of finite paths of unbounded length are, in our sense, unavoidable. Are there other examples? This question is the motivation behind this article.

We recall that the *age* of a graph G is the collection $Age(G)$ of finite induced subgraphs of G, considered up to isomorphy and ordered by embeddability (cf. [7]). It is *well-quasi-ordered* (wqo) for this order if it contains no infinite antichain. A *path* is a graph P such that there exists a one-to-one map f from the set $V(P)$ of its vertices into an interval I of the chain $\mathbb{Z}$ of integers in such a way that $\{u, v\}$ belongs to $E(P)$, the set of edges of P, if and only if $|f(u) - f(v)| = 1$ for every $u, v \in V(P)$. If $I = \{1, \dots, n\}$, then we denote that path by P_n; its *length* is $n - 1$, so, if $n = 2$, P_2 is made of a single edge, whereas if $n = 1$, P_1 is a single vertex. We denote by P_∞ the path on $\mathbb{N}$. The *detour* of a graph G [4] is the supremum of the length of induced paths included in G. Our aim is to give a structural

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result on graphs with infinite detour (for the existence of infinite paths we refer to [14, 19, 27]. We say that a graph G is *path-minimal* if its detour is infinite end every induced subgraph G' with infinite detour embeds admits an embedding of G. Let $\oplus_n P_n$ respectively $\sum_n P_n$ be the direct sum, respectively the complete sum of paths P_n. These graphs are path-minimal graphs. There are others. Our main result is this.

Theorem 1 *There are $2^{\aleph_0}$ path-minimal graphs whose ages are pairwise incomparable and wqo.*

Our construction uses uniformly recurrent sequences, and in fact Sturmian sequences (or billiard sequences) [17, 6], and lexicographical sums of labelled graphs. The existence of $2^{\aleph_0}$ wqo ages is a non trivial fact. It was obtained for binary relations in [21] and for undirected graphs in [24] and in [25]. The proofs were based on uniformly recurrent sequences. These sequences were also used in [15].

We leave open the following:

Problems 1 *[(i)]*

If a graph admits an embedding of finite induced path of unbounded length, does it embed a path-minimal graph?

2. *If a graph is path-minimal, is its age wqo?*
3. *If a graph G is path-minimal, can G be equipped with an equivalence relation $\equiv$ whose blocks are paths in such a way that $(G, \equiv)$ is path-minimal?*

In some situations there are only two path-minimal graphs (up to equimorphy).

Theorem 2 *If the incomparability graph of a poset admits an embedding of finite induced paths of unbounded length, then it admits an embedding of the direct sum or the complete sum of finite induced paths of unbounded length.*

If $G := (V, E)$ is a graph, and x, y are two vertices of G, we denote by $d_G(x, y)$ the length of the shortest path joining x and y if any, and $d_G(x, y) := +\infty$ otherwise. This defines a distance on V with values in the completion $\overline{\mathbb{N}}^+ := \mathbb{N}^+ \cup \{+\infty\}$ of non-negative integers. This distance is the *graphic distance*. If A is a subset of V, the graph G' induced by G on A is an *isometric subgraph* of G if $d_{G'}(x, y) = d_G(x, y)$ for all $x, y \in A$.

If instead of induced path we consider isometric paths, then

Theorem 3 *If a graph admits an embedding of isometric finite paths of unbounded length, then it admits an embedding of a direct sum of such paths.*

We examine the primality of the graphs we obtain. Prime (or indecomposable) graphs are the building blocks of the construction of graphs ([2, 8, 9, 11, 12, 23, 26]). Direct and complete sums of finite paths of unbounded length are not prime and not equimorphic to prime graphs. We construct $2^{\aleph_0}$ examples, none of them being equimorphic to a prime one. We construct also $2^{\aleph_0}$ which are prime. These examples are minimal in the sense of [22], but not in the sense of [23] nor in the sense of [18] p. 92.

We conclude this introduction with:

1.0.1 An outline of the proof of Theorem 1.

It uses two main ingredients. One is the so called uniformly recurrent sequences (or words).

A *uniformly recurrent* word with domain $\mathbb{N}$ is a sequence $u := (u(n))_{n \in \mathbb{N}}$ of letters such that for any given integer n there is some integer $m(u, n)$ such that every factor v of u of length at most n appears as a factor of every factor of u of length at least $m(u, n)$. [1, 3, 16, 6]. To a uniformly recurrent word u on the alphabet $\{0, 1\}$ we associate P_u, the path on $\mathbb{N}$ with a loop at every vertex n for which $u(n) = 1$ and no loop at vertices for which $u(n) = 0$. Next comes the second ingredient.

Fix a binary operation $\star$ on $\{0, 1\}$. Define the lexicographical sum of copies of P_u over the chain ω, denoted by $P_u \cdot_\star \omega$, and made of pairs $(i, v) \in \omega \times P_u$, with an edge between two vertices (i, n) and (j, m) of $P_{u_\alpha} \cdot_\star \omega$, such that $i < j$, if $u(n) \star u(m) = 1$. Since u is uniformly recurrent, the set $Fac(u)$ of finite factors of u is wqo w.r.t the factor ordering, hence by a theorem of Higman [10] (see also [20]), the ages of P_u and of $G_{(u,\star)} := P_u \cdot_\star \omega$ are wqo. Deleting the loops, we get a graph that we denote $\widehat{G}_{(u,\star)}$ and whose age is also wqo. Let $\widehat{Q}_{(u,\star)}$ be the restriction of $\widehat{G}_{(u,\star)}$ to the set $\{(m, n) : n < m + 4\}$ of $V := \mathbb{N} \times \mathbb{N}$. This restriction has the same age as $\widehat{G}_{(u,\star)}$ and it is path-minimal. If the operation $\star$ is constant and equal to 0, respectively equal to 1, $\widehat{Q}_{(u,\star)}$ is a direct sum, respectively a complete sum of paths. To conclude the proof of the theorem, we need to prove that there is some operation $\star$ and $2^{\aleph_0}$ words u such that the ages of $\widehat{G}_{(u,\star)}$ are incomparable. This is the substantial part of the proof. For that, we prove that if $\star$ is the Boolean sum or a projection and u is uniformly recurrent then every long enough path in $\widehat{G}_{(u,\star)}$ is contained in some projection (subset of the form $\{i\} \times \mathbb{N}$). This is a rather technical fact. We think that it holds for any operation. We deduce that if $Fac(u)$ and $Fac(u')$ are not equal up to reversal or to addition (mod 2) of the constant word 1 the ages of $\widehat{G}_{(u,\star)}$ and $\widehat{G}_{(u',\star)}$ are incomparable w.r.t. set inclusion. To complete the proof of Theorem 1, we then use the fact that there are $2^{\aleph_0}$ uniformly recurrent words u_α on the two-letter alphabet $\{0, 1\}$ such that for $\alpha \neq \beta$ the collections $Fac(u_\alpha)$ and $Fac(u_\beta)$ of their finite factors are distinct, and in fact incomparable with respect to set inclusion (this is a well-known fact of symbolic dynamic, e.g. Sturmian words with different slopes will do [6], Chapter 6 page 143).

The proofs will appear in the full version of the paper.

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