Proving a Conjecture on 8-Distance Coloring of the Infinite Hexagonal Grid*

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Abstract. Given $p \in \mathbb{N}$, a p distance coloring is a coloring $f : V \rightarrow \{1, 2, \cdots, n\}$ of the vertices of G such that $f(u) \neq f(v)$ for all pair of vertices u and v in G where $d(u, v)$, the distance between u and v, is at most p. Here $d(u, v)$ is defined as the minimum number of edges required to connect u and v in G. The p distance chromatic number $\lambda^p(G)$ of a graph G is the minimum n such that G admits a p distance coloring of G. Such type of distance coloring is relevant when frequency assignment problem is formulated as a graph coloring problem. In the context of frequency assignment problem, sometimes cellular network can be modelled as an infinite hexagonal grid T_H. Therefore $\lambda^p(T_H)$ has practical relevance. For even $p \geq 8$, it was conjectured by Jacko and Jendrol [Discussiones Mathematicae Graph Theory, 2005] that $\lambda^p(T_H) = \left[\frac{3}{8}\left(p + \frac{4}{3}\right)^2\right]$ where $[x]$ is an integer, $x \in \mathbb{R}$ and $x - \frac{1}{2} < [x] \leq x + \frac{1}{2}$. In this paper, we prove that $\lambda^8(T_H) = 33$ which coincides with the conjectured value.

Keywords: Distance coloring · hexagonal grid · lower bound · conjecture

1 Introduction

Assigning frequencies to the communication channels in a communication network is one of the fundamental challenges as frequencies must be allotted in such a way that interference cannot occur during communication. The frequency channel assignment problem (CAP) can be modelled as a graph coloring problem where vertices are represented as users and proximity between vertices can be measured in terms of minimum number of edges between them and color of a vertex represents frequency assigned to the corresponding user [1]. Sometimes CAP is formulated as a variant of graph coloring problem where two vertices at distance at most p cannot have same color. This type of graph coloring is called a p distance coloring of a graph $G(V, E)$. Wegner [2] introduced p distance coloring of G as a coloring $f : V \rightarrow \{1, 2, \cdots, n\}$ of the vertices of G such that $f(u) \neq f(v)$ for all pair of vertices u and v in G where $d(u, v) \leq p$. Here $d(u, v)$, the distance between u and v in G, is defined as the minimum number of edges required to connect u and v in G. The objective of such coloring problem is to find the p distance chromatic number $\lambda^p(G)$ where $\lambda^p(G)$ is the minimum n such that G admits a p distance coloring of G. Several authors studied the distance graph coloring problems [7–9]. Sometimes CAP in cellular network can be modelled as a graph coloring problem in infinite regular hexagonal grid or honeycomb grid T_H. In Fig. 1, honeycomb representation of T_H have been shown. The coordinates of the vertices of T_H have also been shown here.

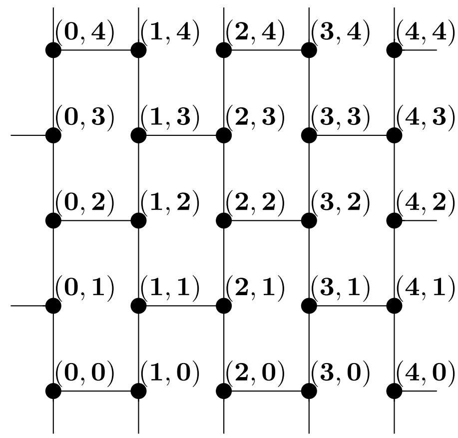


Fig. 1: Honeycomb representation of T_H and coordinates of its vertices.

Several authors studied p distance graph coloring for T_H [3–6]. When p is odd, the exact value of $\lambda^p(T_H)$ has been determined in [5]. But for even p, the exact value of $\lambda^p(T_H)$ has not been determined yet [5, 7, 10, 11]. In [5], it was conjectured that for every even $p \geq 8$,

$$\lambda^p(T_H) = \left[\frac{3}{8}\left(p + \frac{4}{3}\right)^2\right] \tag{1}$$

($[x]$ is an integer, $x \in \mathbb{R}$ and $x - \frac{1}{2} < [x] \leq x + \frac{1}{2}$) and is still an open problem.

It has been shown in [5] that

$$|T'_H(V', E')| = \left(\frac{3}{2}p^2 + \frac{3}{2}p + 1\right) \tag{2}$$

where $T'_H(V', E')$ is the maximum subgraph of T_H such that $d(u, v) \leq 2p$ for every pair of vertices $u, v \in V'$. Using equation (2), a lower bound of $\lambda^{2p}(T_H)$ was obtained in [5]. The result is as follows:

$$\lambda^{2p}(T_H) \;\geq \left(\frac{3}{2}p^2 + \frac{3}{2}p + 1\right) + 1. \tag{3}$$

equation (3) implies that $\lambda^8(T_H) \geq 32$ [5]. But this value is one less than that of the value obtained from equation (1).

Again in [5], it was shown that when $p = 4m$ where m is a positive integer,

$$\lambda^p(T_H) \leq \frac{3}{8}\left(p + \frac{4}{3}\right)^2 + \frac{1}{3}. \tag{4}$$

equation (4) implies that $\lambda^8(T_H) \leq 33$. Moreover, using a computer routine that explores all possible colorings of a subgraph of T_H with 109 vertices, authors in [5] found that 33 colors were required for the subgraph for 8 distance coloring and this value exactly coincides with the value obtained from the conjecture stated in equation (1) as well as with the upper bound stated in equation (4). In this paper, we prove that $\lambda^8(T_H) \geq 33$. Since $\lambda^8(T_H) \leq 33$ [5], we get $\lambda^8(T_H) = 33$, thus answering the conjecture stated in equation (1) positively.

2 Problem statement and key ideas

Definition 1. *A vertex with coordinates (i, j) in T_H is said to be a right vertex, or x_r if it is connected to the vertex with coordinates $(i + 1, j)$ by an edge.*

A right vertex x_r with coordinates (i, j) is adjacent to the vertices having coordinates $(i+1, j)$, $(i, j+1)$ and $(i, j-1)$ but not adjacent to the vertex with coordinates $(i-1, j)$.

Definition 2. *A vertex with coordinates (i, j) in T_H is said to be a left vertex, or x_l if it is connected to the vertex with coordinates $(i - 1, j)$ by an edge.*

A left vertex x_l with coordinates (i, j) is adjacent to the vertices having coordinates $(i - 1, j)$, $(i, j + 1)$ and $(i, j - 1)$ but not adjacent to the vertex with coordinates $(i + 1, j)$. In Fig. 2, a right and a left vertex in T_H are shown.

Definition 3. *A subgraph D_x^{2p} ($p \in \mathbb{N}$) of T_H centered at vertex $x \in V(T_H)$ is the maximum ordered vertex induced subgraph of T_H such that for each pair of vertices $u, v \in V(D_x^{2p})$, $d(u, v) \leq 2p$ and for every vertex $w \in V(D_x^{2p})$, $d(w, x) \leq p$.*

In Fig. 3 different D_x^{2p} in T_H are shown.

Note that for any right vertex x_r and left vertex x_l, $D_{x_r}^{2p}$ and $D_{x_l}^{2p}$ are isomorphic. So any property that holds for $D_{x_r}^{2p}$ also holds for $D_{x_l}^{2p}$. Therefore, we will state and prove our results for $D_{x_r}^{2p}$ and these also hold for $D_{x_l}^{2p}$.

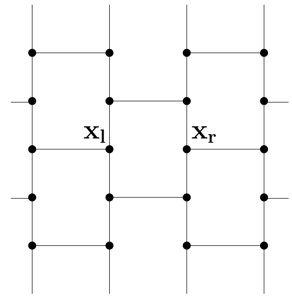
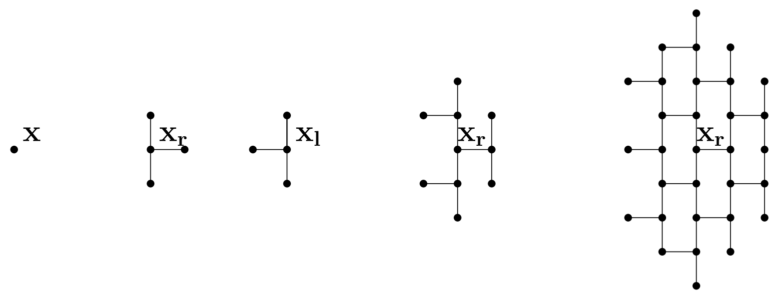


Fig. 2: x_r and x_l are right and left vertices in T_H.



$a.\ D_x^0$ $\quad b.\ D_{x_r}^2$ $\quad c.\ D_{x_l}^2$ $\quad d.\ D_{x_r}^4$ $\quad e.\ D_{x_r}^8$

Fig. 3: Different D_x^{2p}.

Consider a right vertex x_r and we assume the coordinates of x_r are $(0, 0)$. The subgraph $D_{x_r}^{16}$ is shown in Fig. 4. Note that there are $3j$ vertices which are at distance j from x_r, where $1 \leq j \leq 8$. In Fig. 4, the $3j$ vertices are denoted as $v_1^j, v_2^j, \cdots, v_{3j}^j$ where $1 \leq j \leq 8$. We now define $\mathcal{F}_j = \bigcup\limits_{1 \leq i \leq 3j} \{v_i^j\}$ where $1 \leq j \leq 8$. For $5 \leq q \leq 8$, we define the following sets of vertices.

$$V_{i-j}^q = \begin{cases} \{v_l^q : i \leq l \leq j\} & \text{when } i \leq j \\ V_{i-3q}^q \cup V_{1-j}^q & \text{when } i > j \end{cases}$$

3 Results

Consider $V' = V(D_{x_r}^{16}) \setminus V(D_{x_r}^8) = \mathcal{F}_5 \cup \mathcal{F}_6 \cup \mathcal{F}_7 \cup \mathcal{F}_8$. In the following discussion, we investigate how many times at maximum a color used in $D_{x_r}^8$ can be reused in V' and where it can be reused in V'.

As $|V(D_{x_r}^8)| = 1 + |\mathcal{F}_1| + |\mathcal{F}_2| + |\mathcal{F}_3| + |\mathcal{F}_4| = 31$, we need 31 distinct colors to color the vertices of $D_{x_r}^8$ and hence $\lambda^8(D_{x_r}^8) \geq 31$. In Fig. 4, we denote the $3j$ colors of the vertices in $\mathcal{F}_j$ as $c_1^j, c_2^j, \cdots, c_{3j}^j$, where $1 \leq j \leq 4$. It is evident that color c assigned

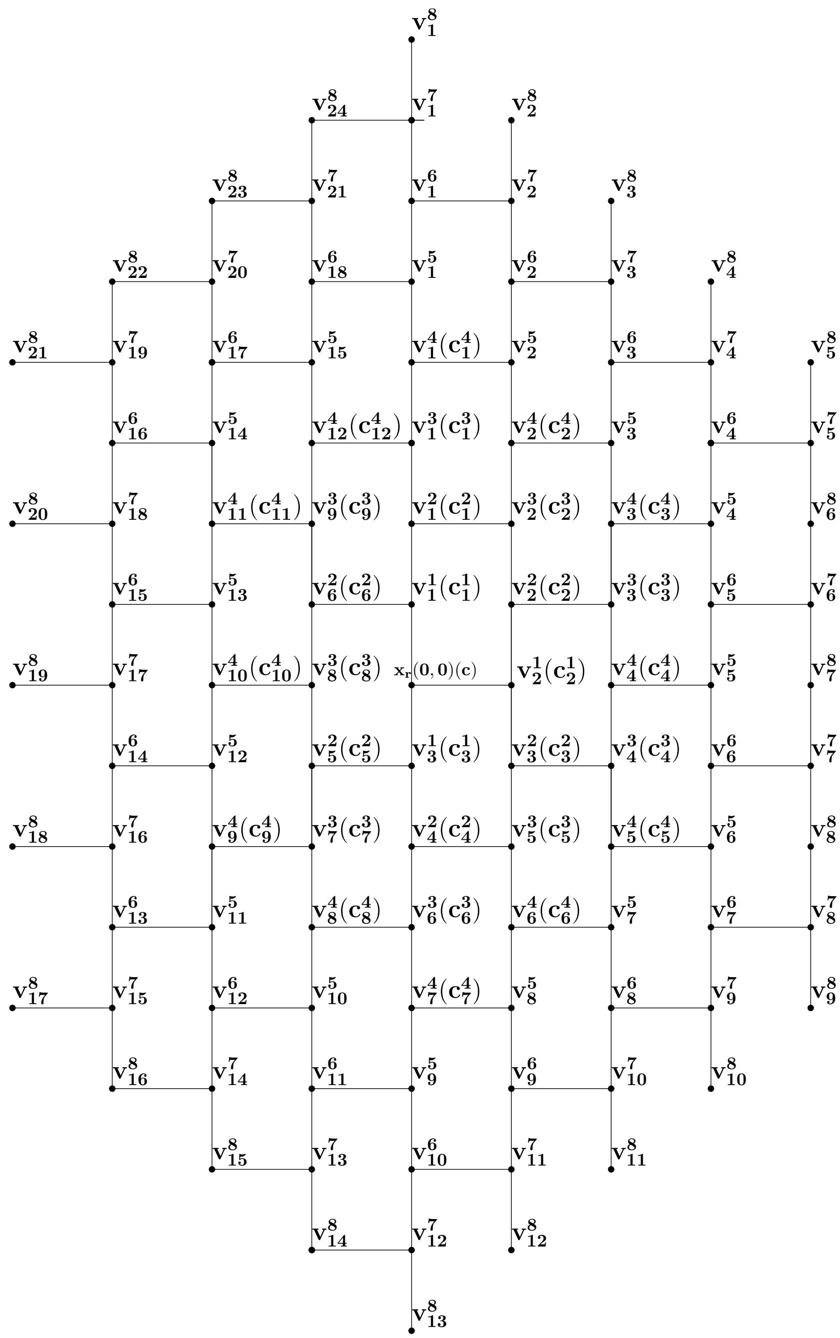


Fig. 4: All the vertices v with $1 \leq d(v, x_r) \leq 8$ and color of all the vertices u with $d(u, x_r) \leq 4$ (Color is mentioned within brackets next to the corresponding vertex).

to x_r cannot be reused at all in V' due to the reuse distance. In subsequent Observations we will state and prove how many times at most the colors $c_1^j, c_2^j, \cdots, c_{3j}^j$ can be reused in V' and where they can be be reused to attain their maximum reusability.

Observation 1 *Each color c_i^1 with $i \in \{1,2,3\}$ can be reused at most three times in V'. For maximum reusability, each color c_i^1 must be reused three times in $\mathcal{F}_8$.*

Proof. There are three vertices in $\mathcal{F}_1$ where the colors c_i^1 with $i \in \{1, \cdots, 3\}$ are used. We first consider the color c_1^1 used in vertex v_1^1. Observe that c_1^1 can be reused in $R = V_{5-17}^8 = R_1 \cup R_2 \cup R_3$ where $R_1 = V_{5-8}^8$, $R_2 = V_{9-12}^8$ and $R_3 = V_{13-17}^8$ are three disjoint subsets of vertices. Note that every pair of vertices in R_1 are at distance at most 8, and the same is true for R_2 and R_3. Hence c_1^1 can be reused at most three times in V', once in R_1, once in R_2 and once in R_3. Note that v_1^1 and v_3^1 are symmetric with respect to x_r where c_1^1 and c_3^1 are used respectively. Hence result obtained regarding how many times the color c_1^1 can be reused in V' also holds for c_3^1. So we are remaining to consider the color c_2^1 used in vertex v_2^1. For c_2^1, the corresponding sets R, R_1, R_2 and R_3 can easily be obtained as $R = V_{13-1}^8$, $R_1 = V_{13-16}^8$, $R_2 = V_{17-21}^8$ and $R_3 = V_{22-1}^8$ respectively. Hence c_2^1 can also be reused at most three times in V'.

It is evident that each c_i^1 with $i \in \{1,2,3\}$ can only be reused in $\mathcal{F}_8$. Hence for maximum reusability, each color c_i^1 must be reused three times in $\mathcal{F}_8$.

Observation 2 *Each color c_i^2 with $i \in \{1, \cdots, 6\}$ can be reused at most two times in V'. For maximum reusability, each color c_i^2 must be reused two times in $\mathcal{F}_7 \cup \mathcal{F}_8$.*

Proof. There are six vertices in $\mathcal{F}_2$ where the colors c_i^2 with $i \in \{1, \cdots, 6\}$ are used. Observe that the vertices v_1^2 and v_4^2 are symmetric with respect to x_r where c_1^2 and c_4^2 are used respectively. Similar fact holds for c_2^2 and c_3^2; c_5^2 and c_6^2. Hence results obtained regarding how many times the colors c_1^2, c_2^2 and c_5^2 can be reused in V' also hold for c_4^2, c_3^2 and c_6^2 respectively. Therefore we need to consider the colors c_1^2, c_2^2 and c_5^2 only.

- We will first consider the color c_1^2. It can be reused in $R = V_{8-15}^7 \cup V_{9-17}^8$. Observe that $R = R_1 \cup R_2$ where $R_1 = V_{8-11}^7 \cup V_{9-12}^8$ and $R_2 = V_{12-15}^7 \cup V_{13-17}^8$ are two vertex disjoint subsets. Note that there does not exist any pair of vertices in R_1 at distance 9 or more. Same fact holds for R_2. Hence c_1^2 can be reused at most two times in V', once in R_1 and once in R_2. It is evident that each c_i^2 with $i \in \{1, \cdots, 6\}$ can only be reused in $\mathcal{F}_7 \cup \mathcal{F}_8$. Hence c_1^2 must be reused two times in $\mathcal{F}_7 \cup \mathcal{F}_8$ to attain its maximum reusability. Similar result holds for c_4^2 also.
  For the colors c_2^2 and c_5^2 the sets R, R_1 and R_2 are stated below.
- c_2^2: $R = V_{12-19}^7 \cup V_{13-21}^8 = R_1 \cup R_2$; $R_1 = V_{12-15}^7 \cup V_{13-17}^8$; $R_2 = V_{16-19}^7 \cup V_{18-21}^8$.
- c_5^2: $R = V_{1-8}^7 \cup V_{1-9}^8 = R_1 \cup R_2$; $R_1 = V_{1-4}^7 \cup V_{1-4}^8$; $R_2 = V_{5-8}^7 \cup V_{5-9}^8$.

Observation 3 *Each color c_i^3 with $i \in \{1, \cdots, 9\}$ can be reused at most three times in V'. For maximum reusability, each c_i^3 with $i \in \{2,5,8\}$ must be reused at least two times in $\mathcal{F}_7 \cup \mathcal{F}_8$ and each c_i^3 with $i \in \{1, \cdots, 9\} \setminus \{2,5,8\}$ must be reused at least once in $\mathcal{F}_7 \cup \mathcal{F}_8$.*

Proof. Note that the pair of colors c_1^3 and c_6^3 are assigned to the vertices v_1^3 and v_6^3 which are symmetric with respect to x_r. Similar fact holds for c_2^3 and c_5^3; c_3^3 and c_4^3; c_9^3 and c_7^3. Hence results obtained regarding how many times the colors c_1^3, c_2^3, c_3^3 and c_9^3 can be reused in V' also hold for c_6^3, c_5^3, c_4^3 and c_7^3 respectively. Therefore, we need to consider the colors c_1^3, c_2^3, c_3^3, c_9^3 only. We will consider the case for the color c_8^3 separately.

- First consider the color c_1^3. It can be reused in $R = V_{7-13}^6 \cup V_{8-15}^7 \cup V_{5-18}^8 = R_1 \cup R_2 \cup R_3$ where $R_1 = V_{8-8}^7 \cup V_{5-9}^8$, $R_2 = V_{7-10}^6 \cup V_{9-12}^7 \cup V_{10-13}^8$ and $R_3 = V_{11-13}^6 \cup V_{13-15}^7 \cup V_{14-18}^8$ are three disjoint subsets of vertices. Observe that every pair of vertices belonging to the same subset are at distance at most 8. Hence c_1^3 can be reused at most three times, once in $R_1 \subset \mathcal{F}_7 \cup \mathcal{F}_8$, once in R_2 and once in R_3. To reuse c_1^3 three times in V', it must be reused at least once at $\mathcal{F}_7 \cup \mathcal{F}_8$. Similar result holds for c_6^3 also.
  For each of the colors c_2^3, c_3^3, c_9^3 and c_8^3 the set R and its corresponding partitions R_1, R_2, R_3 are as stated below.
- c_2^3: $R = V_{10-13}^6 \cup V_{12-15}^7 \cup V_{9-21}^8 = R_1 \cup R_2 \cup R_3$; $R_1 = V_{9-12}^8$; $R_2 = V_{10-13}^6 \cup V_{12-15}^7 \cup V_{13-16}^8$; $R_3 = V_{17-21}^8$; To reuse c_2^3 three times, it must be used once in $R_1 \subset \mathcal{F}_8 \subset \mathcal{F}_7 \cup \mathcal{F}_8$, once in R_2 and once in $R_3 \subset \mathcal{F}_8 \subset \mathcal{F}_7 \cup \mathcal{F}_8$. Hence to reuse c_2^3 three times, it must be used at least two times in $\mathcal{F}_7 \cup \mathcal{F}_8$. Similar result holds for c_5^3 also.
- c_3^3: $R = V_{10-16}^6 \cup V_{12-19}^7 \cup V_{12-1}^8 = R_1 \cup R_2 \cup R_3$; $R_1 = V_{10-12}^6 \cup V_{12-14}^7 \cup V_{12-16}^8$; $R_2 = V_{13-16}^6 \cup V_{15-18}^7 \cup V_{17-20}^8$; $R_3 = V_{19-19}^7 \cup V_{21-1}^8$; To reuse c_3^3 three times, it must be used once in R_1, once in R_2 and once in $R_3 \subset \mathcal{F}_7 \cup \mathcal{F}_8$. Hence to reuse c_3^3 three times, it must be used at least once in $\mathcal{F}_7 \cup \mathcal{F}_8$. Similar result holds for c_4^3 also.
- c_9^3: $R = V_{4-10}^6 \cup V_{5-12}^7 \cup V_{4-17}^8 = R_1 \cup R_2 \cup R_3$; $R_1 = V_{4-6}^6 \cup V_{5-7}^7 \cup V_{4-8}^8$; $R_2 = V_{7-10}^6 \cup V_{8-11}^7 \cup V_{9-12}^8$; $R_3 = V_{12-12}^7 \cup V_{13-17}^8$; To reuse c_9^3 three times, it must be used once in R_1, once in R_2 and once in $R_3 \subset \mathcal{F}_7 \cup \mathcal{F}_8$. Hence to reuse c_9^3 three times, it must be used at least once in $\mathcal{F}_7 \cup \mathcal{F}_8$. Similar result holds for c_7^3 also.
- c_8^3: at $R = V_{4-7}^6 \cup V_{5-8}^7 \cup V_{1-13}^8 = R_1 \cup R_2 \cup R_3$; $R_1 = V_{1-4}^8$; $R_2 = V_{4-7}^6 \cup V_{5-8}^7 \cup V_{5-9}^8$; $R_3 = V_{10-13}^8$; To reuse c_8^3 three times, it must be used once in $R_1 \subset \mathcal{F}_8 \subset \mathcal{F}_7 \cup \mathcal{F}_8$, once in R_2 and once in $R_3 \subset \mathcal{F}_8 \subset \mathcal{F}_7 \cup \mathcal{F}_8$. Hence to reuse c_8^3 three times, it must be used at least two times in $\mathcal{F}_7 \cup \mathcal{F}_8$.

Observation 4 *Each color c_i^4 with $i \in \{1, \cdots, 12\}$ can be reused at most three times in V'. For maximum reusability, each c_i^4 with $i \in \{1, \cdots, 12\}$ must be reused at least two times in $\mathcal{F}_7 \cup \mathcal{F}_8$.*

Proof. Note that the colors c_1^4 and c_7^4 are used at v_1^4 and v_7^4 respectively which are symmetric with respect to x_r. Similar fact also hold for c_2^4 and c_6^4; c_3^4 and c_5^4; c_8^4 and c_{12}^4; c_9^4 and c_{11}^4. Hence result obtained regarding how many times the colors $c_1^4, c_2^4, c_3^4, c_8^4$ and c_9^4 can be reused in V' are same for the colors $c_7^4, c_6^4, c_5^4, c_{12}^4$ and c_{11}^4 respectively. Therefore we need to consider the colors $c_1^4, c_2^4, c_3^4, c_8^4$ and c_9^4 only. We will consider the remaining two colors c_4^4 and c_{10}^4 separately.

- We first consider the color c_1^4. It can be reused at $R = V_{6-11}^5 \cup V_{7-13}^6 \cup V_{7-16}^7 \cup V_{8-18}^8 = R_1 \cup R_2 \cup R_3$ where $R_1 = V_{6-8}^5 \cup V_{7-9}^6 \cup V_{7-10}^7 \cup V_{8-11}^8$, $R_2 =$

$V^5_{9-11} \cup V^6_{10-13} \cup V^7_{11-14} \cup V^8_{12-16}$ and $R_3 = V^7_{15-16} \cup V^8_{17-18}$ are three disjoint subsets. Observe that every pair of vertices belonging to the same subset are at distance at most 8. Hence c^4_1 can be reused at most three times, once in R_1, once in R_2 and once in R_3.

Observe that c^4_1 can be reused two times in $\mathcal{F}_5 \cup \mathcal{F}_6$ only when c^4_1 is used once in $u \in V^5_{6-8} \cup V^6_{7-9} \subset R_1$ and once in $v \in V^5_{9-11} \cup V^6_{10-13} \subset R_2$ such that $d(u,v) \geq 9$. Note that for any such (u,v) pair, there does not exist any $w \in R_3$ such that $d(u,w) \geq 9$ and $d(v,w) \geq 9$. That is, in that case, c^4_1 cannot be reused once more in R_3. This implies that for maximum reusability, c^4_1 can be reused at most once in $\mathcal{F}_5 \cup \mathcal{F}_6$. Hence c^4_1 must be used at least two times in $\mathcal{F}_7 \cup \mathcal{F}_8$ to attain its maximum reusability. Similar result holds for c^4_7 also.

For each of the colors c^4_2, c^4_3, c^4_4, c^4_8, c^4_9 and c^4_{10} the set R and its corresponding partitions R_1, R_2, R_3 are stated below.

- c^4_2: $R = V^5_{9-11} \cup V^6_{10-13} \cup V^7_{8-19} \cup V^8_{9-21} = R_1 \cup R_2 \cup R_3$;
  $R_1 = V^7_{8-11} \cup V^8_{9-12}$; $R_2 = V^5_{9-11} \cup V^6_{10-13} \cup V^7_{12-15} \cup V^8_{13-17}$; $R_3 = V^7_{16-19} \cup V^8_{18-21}$;
  To reuse c^4_2 three times, it must be reused once in $R_1 \subset \mathcal{F}_7 \cup \mathcal{F}_8$, once in R_2 and once in $R_3 \subset \mathcal{F}_7 \cup \mathcal{F}_8$ and hence it must be reused at least two times in $\mathcal{F}_7 \cup \mathcal{F}_8$. Similar result holds for c^4_6 also.
- c^4_3: $R = V^5_{9-14} \cup V^6_{10-16} \cup V^7_{11-20} \cup V^8_{12-22} = R_1 \cup R_2 \cup R_3$;
  $R_1 = V^5_{9-11} \cup V^6_{10-13} \cup V^7_{11-14} \cup V^8_{12-16}$; $R_2 = V^7_{15-16} \cup V^8_{17-18}$; $R_3 = V^5_{12-14} \cup V^6_{14-16} \cup V^7_{17-20} \cup V^8_{19-22}$;
  Observe that c^4_3 can be reused two times in $\mathcal{F}_5 \cup \mathcal{F}_6$ only when c^4_3 is used once in $u \in V^5_{9-11} \cup V^6_{10-13} \subset R_1$ and once in $v \in V^5_{12-14} \cup V^6_{14-16} \subset R_3$ such that $d(u,v) \geq 9$. Note that for any such (u,v) pair, there does not exist any $w \in R_2$ such that $d(u,w) \geq 9$ and $d(v,w) \geq 9$. That is, in that case, c^4_3 cannot be reused once more in R_2. This implies that for maximum reusability, c^4_1 can be reused at most once in $\mathcal{F}_5 \cup \mathcal{F}_6$. Hence c^4_3 must be used at least two times in $\mathcal{F}_7 \cup \mathcal{F}_8$ to attain its maximum reusability. Similar result holds for c^4_5 also.
- c^4_4: $R = V^5_{11-14} \cup V^6_{13-16} \cup V^7_{12-1} \cup V^8_{13-1} = R_1 \cup R_2 \cup R_3$.
  $R_1 = V^7_{12-15} \cup V^8_{13-17}$; $R_2 = V^5_{11-14} \cup V^6_{13-16} \cup V^7_{16-18} \cup V^8_{18-20}$; $R_3 = V^7_{19-1} \cup V^8_{21-1}$;
  To reuse c^4_4 three times, it must be reused once in $R_1 \subset \mathcal{F}_7 \cup \mathcal{F}_8$, once in R_2 and once in $R_3 \subset \mathcal{F}_7 \cup \mathcal{F}_8$ and hence it must be reused at least two times in $\mathcal{F}_7 \cup \mathcal{F}_8$ to attain its maximum reusability.
- c^4_8: $R = V^5_{1-4} \cup V^6_{1-4} \cup V^7_{19-8} \cup V^8_{21-9} = R_1 \cup R_2 \cup R_3$;
  $R_1 = V^7_{19-1} \cup V^8_{21-1}$; $R_2 = V^5_{1-4} \cup V^6_{1-4} \cup V^7_{2-4} \cup V^8_{2-4}$; $R_3 = V^7_{5-8} \cup V^8_{5-9}$;
  To reuse c^4_8 three times, it must be reused once in $R_1 \subset \mathcal{F}_7 \cup \mathcal{F}_8$, once in $R_3 \subset \mathcal{F}_7 \cup \mathcal{F}_8$ and once in R_2 and hence it must be reused at least two times in $\mathcal{F}_7 \cup \mathcal{F}_8$ to attain its maximum reusability. Similar result holds for c^4_{12} also.
- c^4_9: $R = V^5_{1-6} \cup V^6_{1-7} \cup V^7_{21-9} \cup V^8_{24-10} = R_1 \cup R_2 \cup R_3$;
  $R_1 = V^5_{1-3} \cup V^6_{1-3} \cup V^7_{21-4} \cup V^8_{24-3}$; $R_2 = V^7_{5-5} \cup V^8_{4-5}$; $R_3 = V^5_{4-6} \cup V^6_{4-7} \cup V^7_{6-9} \cup V^8_{6-10}$;
  Observe that c^4_9 can be reused two times in $\mathcal{F}_5 \cup \mathcal{F}_6$ only when c^4_9 is used once in $u \in V^5_{1-3} \cup V^6_{1-3} \subset R_1$ and once in $v \in V^5_{4-6} \cup V^6_{4-7} \subset R_3$ such that $d(u,v) \geq 9$. Note that for any such (u,v) pair, there does not exist any $w \in R_2$ such that

$d(u,w) \geq 9$ and $d(v,w) \geq 9$. That is, in that case, c_9^4 cannot be reused once more in R_2. This implies that for maximum reusability, c_9^4 can be reused at most once in $\mathcal{F}_5 \cup \mathcal{F}_6$. Hence c_9^4 must be used at least two times in $\mathcal{F}_7 \cup \mathcal{F}_8$ to attain its maximum reusability. Similar result holds for c_{11}^4 also.

– c_{10}^4: $R = V_{4-6}^5 \cup V_{4-7}^6 \cup V_{1-12}^7 \cup V_{1-13}^8 = R_1 \cup R_2 \cup R_3$;
$R_1 = V_{1-4}^7 \cup V_{1-4}^8$; $R_2 = V_{4-6}^5 \cup V_{4-7}^6 \cup V_{5-8}^7 \cup V_{5-9}^8$; $R_3 = V_{9-12}^7 \cup V_{10-13}^8$;
To reuse c_{10}^4 three times, it must be reused once in $R_1 \subset \mathcal{F}_7 \cup \mathcal{F}_8$, once in R_2 and once in $R_3 \subset \mathcal{F}_7 \cup \mathcal{F}_8$ and hence it must be reused at least two times at $\mathcal{F}_7 \cup \mathcal{F}_8$.

Now we investigate whether the colors used in $D_{x_r}^8$ are sufficient to color the vertices of V' or new color/s is/are to be introduced to color them. Note that if new color/s is/are necessary then the required number of new color/s will depend on the number of vertices of V' which cannot be colored with the colors used in $D_{x_r}^8$ and how many times at maximum a new color can be used in V'. In following Observation, we first state that how many times at maximum a color not appearing in $D_{x_r}^8$ can be used in V' and then based on this result, in Theorem 1, we finally state the minimum number of colors required to color the vertices of $D_{x_r}^{16}$.

Observation 5 *A color c_n not appearing in $D_{x_r}^8$ can be used at most five times in V'.*

Proof. The color c_n can be used in $V' = \mathcal{F}_5 \cup \mathcal{F}_6 \cup \mathcal{F}_7 \cup \mathcal{F}_8 = V_{1-24}^8 \cup V_{1-21}^7 \cup V_{1-18}^6 \cup V_{1-15}^5$. Observe that V' can be partitioned into six disjoint subsets $R_1 = V_{21-1}^8 \cup V_{19-1}^7 \cup V_{16-1}^6 \cup V_{14-1}^5$, $R_2 = V_{17-20}^8 \cup V_{15-18}^7 \cup V_{13-15}^6 \cup V_{11-13}^5$, $R_3 = V_{12-16}^8 \cup V_{11-14}^7 \cup V_{9-12}^6 \cup V_{8-10}^5$, $R_4 = V_{7-11}^8 \cup V_{7-10}^7 \cup V_{5-8}^6 \cup V_{5-7}^5$, $R_5 = V_{2-6}^8 \cup V_{2-5}^7 \cup V_{2-4}^6 \cup V_{2-4}^5$ and $R_6 = V_{6-6}^7$ where every pair of vertices in a subset is at most distance 8 apart. If c_n is not used in v_6^7, the only vertex in R_6, then c_n can be used at most five times in V'. In other words, to use c_n six times in V', c_n must be used in v_6^7. If c_n is used in v_6^7 then the set of vertices where c_n can be reused is $R' = V_{11-2}^8 \cup V_{11-1}^7 \cup V_{9-1}^6 \cup V_{9-15}^5$. Now observe that R' can be partitioned into four disjoint subsets $R_1' = V_{11-15}^8 \cup V_{11-13}^7 \cup V_{9-12}^6 \cup V_{9-10}^5$, $R_2' = V_{16-19}^8 \cup V_{14-18}^7 \cup V_{13-15}^6 \cup V_{11-13}^5$, $R_3' = V_{20-24}^8 \cup V_{19-21}^7 \cup V_{16-1}^6 \cup V_{14-15}^5$ and $R_4' = V_{1-2}^8 \cup V_{1-1}^7$ where every pair of vertices in a subset is at most distance 8 apart. This implies that c_n can be used at most five times in V' regardless of whether c_n is used or not used in v_6^7.

Now we state and prove our main theorem.

Theorem 1. $\lambda^8(T_H) \geq 33$.

Proof. As $|V(D_{x_r}^8)| = 31$ and $|V(D_{x_r}^{16})| = 109$, $(109-31) = 78$ vertices are to be colored with the colors from $c_1^j, c_2^j, \cdots, c_{3j}^j$, where $1 \leq j \leq 4$ (Note that color c of x_r cannot be reused in $D_{x_r}^{16}$ as any vertex in $D_{x_r}^{16}$ is at distance at most 8 from x_r). From Observation 1, Observation 2, Observation 3 and Observation 4, using these colors we can color at most $(\underbrace{(3\times 3)}_{c_i^1, i\in\{1,2,3\}} + \underbrace{(6\times 2)}_{c_i^2, i\in\{1,2,\cdots,6\}} + \underbrace{(9\times 3)}_{c_i^3, i\in\{1,2,\cdots,9\}} + \underbrace{(12\times 3)}_{c_i^4, i\in\{1,2,\cdots,12\}}) = 84$ vertices in V' if each of them are reused with their maximum potential of reusability.

As discussed in Observation 1, all $c_i^1, i \in \{1,2,3\}$ must be reused three times in $\mathcal{F}_8$ to attain their maximum reusability in V'; Hence all c_i^1s, $i \in \{1,2,3\}$ together must occupy 9 vertices in $\mathcal{F}_8$ here. As discussed in Observations 2, all $c_i^2, i \in \{1,2,\cdots,6\}$ must be reused two times in $\mathcal{F}_7 \cup \mathcal{F}_8$ to attain their maximum reusability; Hence all c_i^2s, $i \in \{1,2,\cdots,6\}$ together must occupy 12 vertices in $\mathcal{F}_8 \cup \mathcal{F}_7$ here. As discussed in Observations 3, each of c_2^3, c_5^3 and c_8^3 must be used at least two times in $\mathcal{F}_7 \cup \mathcal{F}_8$ to attain their maximum reusability and each of $c_i^3, i \in \{1,2,\cdots,9\} \setminus \{2,5,8\}$ must be reused at least once in $\mathcal{F}_7 \cup \mathcal{F}_8$ to attain their maximum reusability. So, all c_i^3s, $i \in \{1,2,\cdots,9\}$ together must occupy at least 12 vertices in $\mathcal{F}_7 \cup \mathcal{F}_8$ here. As discussed in Observations 4, all $c_i^4, i \in \{1,2,\cdots,12\}$ must be reused at least two times in $\mathcal{F}_7 \cup \mathcal{F}_8$ to attain their maximum reusability. So, all c_i^4s, $i \in \{1,2,\cdots,12\}$ together must occupy at least 24 vertices in $\mathcal{F}_7 \cup \mathcal{F}_8$ here. Therefore, to satisfy the maximum reusability of each color used in $D_{x_r}^8$, total positions required at $\mathcal{F}_7 \cup \mathcal{F}_8$ is at least $9+12+12+24=57$. However, total positions available at $\mathcal{F}_7 \cup \mathcal{F}_8$ is $21+24=45$ only. Since two or more colors cannot be given at the same vertex, these colors together must loose the potential of maximum reusability by at least $57-45=12$ in V'. Hence they together can color maximum $(84-12)=72$ vertices in V'. Since V' has 78 vertices, new color/s must be needed to color at least $(78-72)=6$ vertices of V'. From observation 5, a new color can color at most five vertices in V'. So at least two new colors are required to color these six vertices. Since 31 distinct colors are required for $D_{x_r}^8$, at least $(31+2)=33$ colors are required for $D_{x_r}^{16}$. Hence $\lambda^8(D_{x_r}^{16}) \geq 33$.

As for any left vertex x_l, $D_{x_r}^{16}$ and $D_{x_l}^{16}$ are isomorphic, so $\lambda^8(D_{x_l}^{16}) \geq 33$. As $D_{x_r}^{16}$ and $D_{x_l}^{16}$ are subgraphs of T_H, we conclude that $\lambda^8(T_H) \geq 33$. Hence the proof.

4 Conclusion

In our work we show that $\lambda^p(T_H) \geq 33$ and this exactly coincides with the value obtained from the conjecture and upper bound of $\lambda^p(T_H)$ as obtained in [5] when $p=8$. Exact value of $\lambda^p(T_H)$ for even $p>8$ is still unknown and determining it is an interesting problem for future work.

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