Quasi-Classical Semantics for Expressive Description Logics*

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Abstract. Inconsistency handling in expressive description logics is an important problem because inconsistency may naturally occur in an open world. In this paper, we first present the quasi-classical semantics for description logic $\mathcal{SHIQ}$, which is based on quasi-classical logic. We show that this semantics can be used to deal with inconsistency and that it is reduced to the standard semantics when there is no inconsistency. Compared with the existing four-valued semantics for description logics, it strengthens the reasoning capability in the sense that it can satisfy some important inference rules, such as *Modus Ponens*. Finally, we analyze the computational complexity of consistency checking based on the quasi-classical semantics.

1 Introduction

In an open, constantly changing and collaborative environment like the forthcoming Semantic Web, a knowledge base (or an ontology) may well contain inconsistencies because of many reasons, such as modeling errors, migration from other formalisms, merging ontologies, and ontology evolution (see [1]). As the logical foundation of Ontology Web Language (OWL), expressive description logics (for short DLs) fail to tolerate inconsistent data. Therefore, handling inconsistency in expressive DLs, such as $\mathcal{SHIQ}$, is becoming an important topic in recent years.

There are two fundamentally different approaches to handling inconsistency. One is to repair inconsistency in ontologies to obtain consistent ontologies [2,3,4]. The other, called paraconsistent approach, does not simply repair inconsistencies but applies a non-standard reasoning to obtain meaningful answers from inconsistent ontologies [5,6]. For the latter, having inconsistencies is treated as a natural phenomenon in realistic data and is tolerated during reasoning, whilst inconsistencies are viewed as erroneous data for the first approach. So far, the main idea of paraconsistent methods for handling inconsistent ontologies [5] is based on Belnap's four-valued semantics [7]. However their reasoning capability is rather weak [6]. For instance, they fail in holding some important inference rules, such as *Modus Ponens*, *Modus Tollens* and *Disjunctive Syllogism*. In [6], a total negation is introduced to enable the resolution principle for paraconsistent reasoning

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in four-valued DLs. However, four-valued DLs with total negation still don't satisfy some intuitive equivalences mentioned above. The main problem of four-valued DLs is that they can not overcome those inherent shortcomings of four-valued semantics in reasoning [7].

To avoid shortcomings of four-valued DLs and strengthen paraconsistent reasoning capability, in this paper, following [8], we study a quasi-classical description logic $\mathcal{SHIQ}$ (QC $\mathcal{SHIQ}$ for short), which is based on quasi-classical logic [9,10] and description logic (for short DL) $\mathcal{SHIQ}$. We concentrate on $\mathcal{SHIQ}$ since it is the core of OWL DL [11,12]. Main contributions of this paper are summarized as follows:

- *QC semantics* is introduced to $\mathcal{SHIQ}$ to handle inconsistent ontologies. It composes of two kinds of semantics, namely, *QC weak semantics* $\models_w$ and *QC strong semantics* $\models_s$. QC weak semantics inherits the characteristics of four-valued semantics and QC strong semantics redefines the interpretation for disjunction and conjunction of concepts to make the three important inference rules hold.
- A new notion called *complement of concept* is introduced. We use $\overline{C}$ to denote the complement of concept C. The semantics of an *inclusion axiom* defined under the weak semantics and the strong semantics satisfy the *intuitive equivalences*. For instance, $\mathcal{I} \models_w C \sqsubseteq D$ iff $\mathcal{I} \models_w \overline{C} \sqcup D(a)$; and $\mathcal{I} \models_s C \sqsubseteq D$ iff $\mathcal{I} \models_s \neg C \sqcup D(a)$ for any individual a.
- A *QC model* is defined to express a possible world under QC strong semantics. A *QC entailment* based on QC model, written by "$\models_Q$", between an ontology and an axiom is presented which is shown to be able to handle inconsistency. Compared with the four-valued DLs, QC DLs satisfy three important inference rules: *Modus Ponens (MP)*: $\{C(a), C \sqsubseteq D\} \models_Q D(a)$, *Modus Tollens (MT)*: $\{\neg D(a), C \sqsubseteq D\} \models_Q \neg C(a)$, and *Disjunctive Syllogism (DS)*: $\{\neg C(a), C \sqcup D\} \models_Q D(a)$.
- Two basic query entailment problems, namely, instance checking and subsumption checking are defined and discussed. We show that the two basic inference problems can be reduced into the QC consistency problem.
- We prove that the complexity of deciding QC consistency for an ABox is EXPTIME-Complete.

Compared with QC $\mathcal{ALC}$ studied in [13], QC $\mathcal{SHIQ}$ studied in this paper has the following significant improvements: (1) we redefine the QC strong semantics of concept inclusion $C_1 \sqsubseteq C_2$ so that intuitive logical equivalences w.r.t. classical negation can be satisfied, which is not the case for QC $\mathcal{ALC}$; (2) we properly introduce the weak and strong satisfactions in QC $\mathcal{SHIQ}$, which avoids redundant definition of the semantics of disjunction and conjunction of concepts in QC $\mathcal{ALC}$; (3) we show that instance checking and subsumption checking can be reduced to the QC consistency problem, which has not been achieved in QC $\mathcal{ALC}$; and (4) we study the QC semantics for more expressive language constructors than those in $\mathcal{ALC}$.

The paper is organized as follows. Section 2 briefly reviews $\mathcal{SHIQ}$. Section 3 introduces QC semantics for $\mathcal{SHIQ}$. Section 4 discusses the reasoning problems in QC $\mathcal{SHIQ}$. Section 5 concludes this paper and discusses the future work. Due to the space limitation, proofs are omitted but are available in a technical report[1].

[1] http://www.is.pku.edu.cn/~zxw/publication/TRQCSHIQ.pdf

2 Preliminaries

In this section, we briefly review some basic notations in description logic (DL) $\mathcal{SHIQ}$. For comprehensive background reading, please refer to [11].

Let $\mathcal{L}$ be the language of $\mathcal{SHIQ}$, which contains a set of atomic concepts (or concept names), denoted by N_C; a set of individuals, denoted by N_I; and a set of role names, denoted by N_R. N_{R+} is a set of transitive role names in N_R and $\mathbf{R}$ is a transitive closure set of N_R. We denote $\mathbb{R} = \mathbf{R} \cup \{R^- \mid R \in \mathbf{R}\}$ where R^- is the inverse role of R. The inverse role of R can be also taken as the inverse transformation on R. In this way, the inverse role of R can be denoted by $Inv(R)$, i.e., $Inv(R) = R^-$ and $Inv(R^-) = R$. A *role inclusion axiom* is of the form $R \sqsubseteq S$ where roles R, S can be inverse. A *role hierarchy*, denoted $\mathcal{R}$, is a set of role inclusion axioms. A role R is a *sub-role* of S, denoted by $R \sqsubseteq^* S$, where "$\sqsubseteq^*$" is the transitive-reflexive closure of $\sqsubseteq$ over $\mathcal{R} \cup \{Inv(R) \sqsubseteq Inv(S) \mid R \sqsubseteq S \in \mathcal{R}\}$. A simple role S is a role which is neither transitive nor has any transitive sub-roles.

Concept descriptions in $\mathcal{SHIQ}$ are formed according to the following syntax rule:

$$C, D \rightarrow A \mid \top \mid \bot \mid \neg A \mid C \sqcup D \mid C \sqcap D \mid \exists R.C \mid \forall R.C \mid \geq nS.C \mid \leq nS.C$$

where $\top$ is the top concept, $\bot$ is the bottom concept, A is a concept name, C, D are concepts, R is a role, S is a simple role and n is nonnegative integer.

Let C, D be concepts, a, b individuals and R a role. In $\mathcal{SHIQ}$, *assertions* are of the form $C(a)$ or $R(a, b)$ or $a \not\doteq b$. A *general concept inclusion axiom (GCI)* is of the form $C \sqsubseteq D$. Informally, an axiom $C(a)$ means that the individual a is an instance of concept C, and an axiom $R(a, b)$ means that individual a is related with individual b via the property R. The inclusion axiom $C \sqsubseteq D$ means that each individual of C is an individual of D.

A knowledge base (or ontology) comprises two components, a *TBox* and an *ABox*. In $\mathcal{SHIQ}$, a TBox includes two parts: a set of concept inclusion axioms $\mathcal{T}$ and a role *hierarchy* $\mathcal{R}$; an ABox consists of concept assertions, role assertions, and individual inequalities.

The formal definition of the classical (model-theoretic) semantics of $\mathcal{SHIQ}$ is given by means of an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ which consists of a non-empty domain $\Delta^{\mathcal{I}}$ and a mapping $\cdot^{\mathcal{I}}$ satisfying the conditions in Table 1, where the mapping $\cdot^{\mathcal{I}}$ interprets concepts as subsets of the domain and roles as binary relations on the domain $\Delta^{\mathcal{I}}$.

An interpretation $\mathcal{I}$ *satisfies* a terminology $\mathcal{T}$ (role hierarchy $\mathcal{R}$) iff for any concept inclusion $C \sqsubseteq D$ (role inclusion $R \sqsubseteq S$) in $\mathcal{T}$ ($\mathcal{R}$), $\mathcal{I}$ is an interpretation of $C \sqsubseteq D$ ($R \sqsubseteq S$). In this case, $\mathcal{I}$ is named a *model* of $\mathcal{T}$ ($\mathcal{R}$), denoted $\mathcal{I} \models \mathcal{T}$ ($\mathcal{I} \models \mathcal{R}$). A concept C is called *satisfiable* w.r.t. a terminology $\mathcal{T}$ and a hierarchy $\mathcal{R}$ iff there is a model $\mathcal{I}$ of $\mathcal{T}$ and $\mathcal{R}$ with $C^{\mathcal{I}} \neq \emptyset$. A concept D *subsumes* a concept C w.r.t $\mathcal{T}$ and $\mathcal{R}$ iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds for each model $\mathcal{I}$ of $\mathcal{T}$ and $\mathcal{R}$. For an interpretation $\mathcal{I}$, an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$ is called an *instance* of a concept C iff $a^{\mathcal{I}} \in C^{\mathcal{I}}$. An interpretation $\mathcal{I}$ *satisfies* an ABox $\mathcal{A}$ (written $\mathcal{I} \models \mathcal{A}$) if it satisfies every assertion in $\mathcal{A}$. An ABox $\mathcal{A}$ is consistent w.r.t. $\mathcal{T}$ and $\mathcal{R}$ iff there is a model $\mathcal{I}$ of $\mathcal{T}$ and $\mathcal{R}$ that satisfies each assertion in $\mathcal{A}$.

Table 1. Syntax and semantics of DL $\mathcal{SHIQ}$

Constructor Name	Syntax	Semantics
atomic concept A	A	$A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$
abstract role R	R	$R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$
individuals I	o	$o^{\mathcal{I}} \in \Delta^{\mathcal{I}}$
trans.role	$R \in N_{\mathbf{R}_+}$	$R^{\mathcal{I}} = (R^{\mathcal{I}})^+$
inverse role	R^-	$\{(x,y) \mid (y,x) \in R^{\mathcal{I}}\}$
top concept	$\top$	$\Delta^{\mathcal{I}}$
bottom concept	$\bot$	$\emptyset$
conjunction	$C_1 \sqcap C_2$	$C_1^{\mathcal{I}} \cap C_2^{\mathcal{I}}$
disjunction	$C_1 \sqcup C_2$	$C_1^{\mathcal{I}} \cup C_2^{\mathcal{I}}$
negation	$\neg C$	$\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
exists restriction	$\exists R.C$	$\{x \mid \exists y, (x,y) \in R^{\mathcal{I}}$ and $y \in C^{\mathcal{I}}\}$
value restriction	$\forall R.C$	$\{x \mid \forall y, (x,y) \in R^{\mathcal{I}}$ implies $y \in C^{\mathcal{I}}\}$
qualifying atleast restriction	$\geq nR.C$	$\{x \mid \sharp(\{y.(x,y) \in R^{\mathcal{I}}\}$ and $y \in C^{\mathcal{I}}) \geq n\}$
qualifying atmost restriction	$\leq nR.C$	$\{x \mid \sharp(\{y.(x,y) \in R^{\mathcal{I}}\}$ and $y \in C^{\mathcal{I}}) \leq n\}$
Axiom Name	**Syntax**	**Semantics**
concept assertion	$C(a)$	$a^{\mathcal{I}} \in C^{\mathcal{I}}$
role assertion	$R(a,b)$	$(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$
concept inclusion	$C_1 \sqsubseteq C_2$	$C_1^{\mathcal{I}} \subseteq C_2^{\mathcal{I}}$
role inclusion	$R \sqsubseteq S$	$R^{\mathcal{I}} \subseteq S^{\mathcal{I}}$
individual inequality	$a \not\doteq b$	$a^{\mathcal{I}} \not\doteq b^{\mathcal{I}}$

3 Quasi-Classical Description Logic $\mathcal{SHIQ}$

Quasi-classical DL $\mathcal{SHIQ}$ is based on quasi-classical logic defined in [9] and $\mathcal{SHIQ}$. In this section, we present the syntax and the semantics of QC $\mathcal{SHIQ}$.

3.1 Syntax of QC $\mathcal{SHIQ}$

The syntax of QC $\mathcal{SHIQ}$ is almost the same as that of $\mathcal{SHIQ}$. One difference is that a new concept constructor called *complement of a concept* is introduced, which is the syntax form of "*total negation*" in [14]. The complement of concept C is denoted by $\overline{C}$. The intuition behind a complement of a concept is that we want to use it to characterize QC inconsistency, i.e., to reverse both the information of being true and of being false. Its formal interpretation will be given in Section 3.2.

The definition of a *QC ABox (resp. QC terminology, QC role hierarchy)* is similar to that of an ABox (resp. terminology, role hierarchy). The only difference is that we allow a complement of a concept in a complex concept. A *QC ontology* comprises a QC terminology, a QC role hierarchy and a QC ABox.

Given the language $\mathcal{L}$ of $\mathcal{SHIQ}$, the language of QC $\mathcal{SHIQ}$ is denoted $\mathcal{L}^*$ and is defined by $\mathcal{L}^* = \mathcal{L} \cup \{\overline{C} \mid C \text{ is a concept in } \mathcal{L}\}$. In the following, we discuss QC $\mathcal{SHIQ}$ based on the language $\mathcal{L}^*$.

For an atomic concept A, we call A, $\neg A$, $\overline{A}$ and $\overline{\neg A}$ as *concept literals*. A concept C is in *Negation Normal Form (NNF)* if negation ($\neg$) occurs only in front of concept names. A concept C is in *QC NNF*, if concept C is in NNF and complement only occurs over a concept name or negation of a concept name. A *role-involved literal* has the form $\forall R.C$ or $\exists R.C$ with C a concept in NNF. A *role-involved literal* has the form $\forall R.C$, $\exists R.C$, $\geq nR.C$ or $\leq nR.C$ with C a concept in QC NNF. A *literal* is either a concept literal or a role-involved literal, written by L. A *clause* is the disjunction of finite literals. Let $L_1 \sqcup \cdots \sqcup L_n$ be a clause, then $Lit(L_1 \sqcup \cdots \sqcup L_n)$ is the set of literals $\{L_1, \ldots, L_n\}$ that are in the clause. A clause is the *empty clause*, denoted by $\Diamond$, if it has no literals. We define $\sim$ be a *complementation operation* such that $\sim A$ is $\neg A$ and $\sim (\neg A)$ is A.

Let $L_1 \sqcup \cdots \sqcup L_n$ be a clause that includes a literal disjunct L_i. The *focus* of $L_1 \sqcup \cdots \sqcup L_n$ by L_i, denoted $\otimes(L_1 \sqcup \cdots \sqcup L_n, L_i)$, is defined as the clause obtained by removing L_i from $Lit(L_1 \sqcup \cdots \sqcup L_n)$. In the case of a clause with just one disjunct, we assume $\otimes(L, L) = \bot$. For instance, given a clause $L_1 \sqcup L_2 \sqcup L_3$, $\otimes(L_1 \sqcup L_2 \sqcup L_3, L_2) = L_1 \sqcup L_3$.

3.2 Semantics of QC $\mathcal{SHIQ}$

In this subsection, we define the QC semantics for QC $\mathcal{SHIQ}$ by introducing two semantics, namely, QC weak semantics and QC strong semantics.

Firstly, we introduce the notion of a weak interpretation and the notion of a strong interpretation over domain $\Delta^{\mathcal{I}}$ by assigning to each concept C a pair $\langle +C, -C \rangle$ of subsets of $\Delta^{\mathcal{I}}$. Intuitively, $+C$ is the set of elements known to belong to the extension of C, while $-C$ is the set of elements known to be not contained in the extension of C. $+C$ and $-C$ are not necessarily disjoint or mutually complemental with respect to the domain. The *complemental set* of a set S w.r.t. an interpretation $\mathcal{I}$, denoted by $\overline{S}$, is $\overline{S} = \Delta^{\mathcal{I}} \setminus S$.

In QC $\mathcal{SHIQ}$, a weak interpretation is a reformulation of a four-valued interpretation in four-valued DLs (see [6]).

Definition 1 *Let $\mathcal{I}$ be a pair $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ with $\Delta^{\mathcal{I}}$ as domain, where $\cdot^{\mathcal{I}}$ is a function assigning elements of $\Delta^{\mathcal{I}}$ to individuals, subsets of $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ to concepts and subsets of $(\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}})^2$ to roles. $\mathcal{I}$ is a* weak interpretation *in QC $\mathcal{SHIQ}$ if the conditions in Table 2 are satisfied, where $C_i^{\mathcal{I}} = \langle +C_i, -C_i \rangle$ for $i = 1, 2$, $C^{\mathcal{I}} = \langle +C, -C \rangle$ and $R^{\mathcal{I}} = \langle +R, -R \rangle$.*

Definition 2 *Let $\models_w$ be a satisfiability relation between a set of weak interpretation and a set of axioms, called* weak satisfaction*. For a weak interpretation $\mathcal{I}$, we define $\models_w$ as follows, where C, C_1, C_2 are concepts, R, R_1, R_2 are roles and a is an individual:*

(1) $\mathcal{I} \models_w R_1 \sqsubseteq R_2$ *iff* $+R_1 \subseteq +R_2$, *if* $R_i^{\mathcal{I}} = \langle +R_i, -R_i \rangle$, $i = 1, 2$*;*

(2) $\mathcal{I} \models_w$ *Trans*(R) *iff* $+R = (+R)^+$, *if* $R^{\mathcal{I}} = \langle +R, -R \rangle$ *and* $(R^+)^{\mathcal{I}} = \langle +R^+, -R^+ \rangle$*;*

(3) $\mathcal{I} \models_w C(a)$ *iff* $a^{\mathcal{I}} \in +C$, $C^{\mathcal{I}} = \langle +C, -C \rangle$*;*

(4) $\mathcal{I} \models_w R(a, b)$ *iff* $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in +R$, $R^{\mathcal{I}} = \langle +R, -R \rangle$*;*

(5) $\mathcal{I} \models_w C_1 \sqsubseteq C_2$ *iff* $+C_1 \subseteq +C_2$, *for* $i = 1, 2$, $C_i^{\mathcal{I}} = \langle +C_i, -C_i \rangle$*;*

(6) $\mathcal{I} \models_w a \neq b$ *iff* $a^{\mathcal{I}} \neq b^{\mathcal{I}}$.

Table 2. Weak Semantics of QC $\mathcal{SHIQ}$

Syntax	Weak Semantics
A	$A^{\mathcal{I}} = \langle +A, -A \rangle$, where $+A, -A \subseteq \Delta^{\mathcal{I}}$
R	$R^{\mathcal{I}} = \langle +R, -R \rangle$, where $+R, -R \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$
o	$o^{\mathcal{I}} \in \Delta^{\mathcal{I}}$
$\top$	$\langle \Delta^{\mathcal{I}}, \emptyset \rangle$
$\bot$	$\langle \emptyset, \Delta^{\mathcal{I}} \rangle$
$C_1 \sqcap C_2$	$\langle +C_1 \cap +C_2, -C_1 \cup -C_2 \rangle$
$C_1 \sqcup C_2$	$\langle +C_1 \cup +C_2, -C_1 \cap -C_2 \rangle$
$\neg C$	$(\neg C)^{\mathcal{I}} = \langle -C, +C \rangle$
$\overline{C}$	$\overline{C}^{\mathcal{I}} = \langle \Delta^{\mathcal{I}} \setminus +C, \Delta^{\mathcal{I}} \setminus -C \rangle$
$\exists R.C$	$\langle \{x \mid \exists y, (x, y) \in +R$ and $y \in +C\}, \{x \mid \forall y, (x, y) \in +R$ implies $y \in -C\} \rangle$,
$\forall R.C$	$\langle \{x \mid \forall y, (x, y) \in +R$ implies $y \in +C\}, \{x \mid \exists y, (x, y) \in +R$ and $y \in -C\} \rangle$
$\geq nR.C$	$\langle \{x \mid \sharp(\{y.(x, y) \in +R\}$ and $y \in +C) \geq n\}, \{x \mid \sharp(\{y.(x, y) \in +R\}$ and $y \notin -C) < n\} \rangle$
$\leq nR.C$	$\langle \{x \mid \sharp(\{y.(x, y) \in +R\}$ and $y \notin -C) \leq n\}, \{x \mid \sharp(\{y.(x, y) \in +R\}$ and $y \in +C) > n\} \rangle$

In QC $\mathcal{SHIQ}$, the concept inclusion under weak satisfaction is defined by the internal inclusion in four-valued DLs because other inclusion, namely, material inclusion and strong inclusion can be transformed into internal inclusion (see [6]).

The following property shows that there exists a close relationship between weak models and 4-models in four-valued DLs defined in [6].

Proposition 1 *Let C, D be concepts, a an individual, R a role and $\mathcal{I}$ an interpretation in QC $\mathcal{SHIQ}$. We have*
(1) $\mathcal{I} \models_w C(a)$ *iff* $\mathcal{I} \models_4 C(a)$*;*
(2) $\mathcal{I} \models_w C \sqsubseteq D$ *iff* $\mathcal{I} \models_4 C \sqsubset D$.

The following proposition shows that intuitive equivalence w.r.t. the complement of a concept is satisfied under QC weak semantics.

Proposition 2 *Let $\mathcal{I}$ be an interpretation and let C, D be concepts, we have*

$$\mathcal{I} \models_w C \sqsubseteq D \textit{ iff } \mathcal{I} \models_w \overline{C} \sqcup D(a) \textit{ for any individual } a \in N_I.$$

The QC weak satisfaction has the drawback that it does not satisfy some basic inference rules, such as MP, MT and DS. Therefore, we define a QC strong interpretation that redefines interpretation of disjunction of concepts and conjunction of concepts.

Definition 3 *Let $\mathcal{I}$ be a pair $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ with $\Delta^{\mathcal{I}}$ as the domain, where $\cdot^{\mathcal{I}}$ is a function assigning elements of $\Delta^{\mathcal{I}}$ to individuals, subsets of $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ to concepts and subsets of $(\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}})^2$ to roles. $\mathcal{I}$ is a* strong interpretation *in QC $\mathcal{SHIQ}$ if the conditions in Table 2, except conditions for conjunction of concepts and disjunction of concepts, are satisfied and the following conditions hold, let $C_i^{\mathcal{I}} = \langle +C_i, -C_i \rangle$ for $i = 1, 2$:*
conjunction of concepts:

$(C_1 \sqcap C_2)^{\mathcal{I}} = \langle +C_1 \cap +C_2, (-C_1 \cup -C_2) \cap (-C_1 \cup \overline{+C_2}) \cap (\overline{+C_1} \cup -C_2)\rangle$
disjunction of concepts:
$(C_1 \sqcup C_2)^{\mathcal{I}} = \langle (+C_1 \cup +C_2) \cap (\overline{-C_1} \cup +C_2) \cap (+C_1 \cup \overline{-C_2}), -C_1 \cap -C_2\rangle$

Compared with the weak interpretation, the strong interpretation of disjunction of concepts tightens the condition that an individual is known to belong to a concept; and the strong interpretation of conjunction of concepts is defined by relaxing the condition that an individual known to be not contained in the extension of a concept. The strong interpretation for the disjunction and conjunction characterizes the relationship between concepts and individuals with holding resolution rule.

Definition 4 *Let $\models_s$ be a satisfiability relation between a set of weak interpretation and a set of axioms, called $strong\ satisfaction$. For a strong interpretation $\mathcal{I}$, we define $\models_s$ as follows, where C, C_1, C_2 are concepts, R, R_1, R_2 are roles and a is an individual:*
(1) $\mathcal{I} \models_s R_1 \sqsubseteq R_2$ *iff* $+R_1 \subseteq +R_2$*, if* $R_i^{\mathcal{I}} = \langle +R_i, -R_i\rangle$*,* $i = 1, 2$*;*
(2) $\mathcal{I} \models_s$ *Trans*(R) *iff* $+R = (+R)^+$*, if* $R^{\mathcal{I}} = \langle +R, -R\rangle$ *and* $(R^+)^{\mathcal{I}} = \langle +R^+, -R^+\rangle$*;*
(3) $\mathcal{I} \models_s C(a)$ *iff* $a^{\mathcal{I}} \in +C$ *where* $C^{\mathcal{I}} = \langle +C, -C\rangle$*;*
(4) $\mathcal{I} \models_s R(a, b)$ *iff* $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in +R$ *where* $R^{\mathcal{I}} = \langle +R, -R\rangle$*;*
(5) $\mathcal{I} \models_s C_1 \sqsubseteq C_2$ *iff* $\overline{-C_1} \subseteq +C_2$*,* $+C_1 \subseteq +C_2$ *and* $-C_2 \subseteq -C_1$*,*
for $i = 1, 2$*,* $C_i^{\mathcal{I}} = \langle +C_i, -C_i\rangle$*;*
(6) $\mathcal{I} \models_s a \neq b$ *iff* $a^{\mathcal{I}} \neq b^{\mathcal{I}}$*.*

In Definition 4, when defining the strong satisfaction of a concept inclusion under a QC strong interpretation, we use conditions for interpreting a concept inclusion by internal inclusion, material inclusion and strong inclusion in four-valued DLs. By doing so, our strong satisfaction satisfies intuitive equivalence that is falsified by four-valued satisfaction relations.

Proposition 3 *Let $\mathcal{I}$ be an interpretation and let C, D be concepts. We have*

$$\mathcal{I} \models_s C \sqsubseteq D \text{ iff } \mathcal{I} \models_s \neg C \sqcup D(a) \text{ for any individual } a \in N_I.$$

Proposition 2 and Proposition 3 provide the theoretical base of transforming the problem of reasoning with ABoxes and terminologies into the problem of reasoning with ABoxes.

The following proposition provides a slightly different view on the QC semantics of disjunction of concepts.

Proposition 4 *Let C_1 and C_2 be two concepts and a be an individual. We have*

$$\mathcal{I} \models_s C_1 \sqcup C_2(a) \text{ iff } \begin{array}{l} \text{for some } C_i, a^{\mathcal{I}} \in +C_i \text{ and } a^{\mathcal{I}} \notin -C_i; \text{or} \\ \text{for all } C_i, a^{\mathcal{I}} \in +C_i \text{ and } a^{\mathcal{I}} \in -C_i; \end{array}$$

where $C_i^{\mathcal{I}} = \langle +C_i, -C_i\rangle$ *and* $i = 1, 2$*.*

Proposition 5 *Let $\mathcal{I}$ be an interpretation and ϕ be an axiom in QC $\mathcal{SHIQ}$ respectively.*

$$\text{If } \mathcal{I} \models_s \phi \text{ then } \mathcal{I} \models_w \phi.$$

Proposition 5 shows that a strong model is a weak model. However, the converse does not hold. For instance, given an ABox $\mathcal{A} = \{C(a), \neg C(a)\}$. We can construct an interpretation $\mathcal{I}$ such that $a^{\mathcal{I}} \in +C$, $a^{\mathcal{I}} \in -C$ and $a^{\mathcal{I}} \notin +D$ where $C^{\mathcal{I}} = \langle +C, -C \rangle$ and $D^{\mathcal{I}} = \langle +D, -D \rangle$. Then $\mathcal{I} \not\models_s C \sqcup D(a)$ because $\mathcal{I} \not\models_s D(a)$. Clearly, $\mathcal{I} \models_w C \sqcup D(a)$.

Proposition 6 *Let C be a concept, R a role and n a natural number. For any QC weak (or QC strong) interpretation $\mathcal{I}$, we have*
(1) $(\neg(\leq nR.C))^{\mathcal{I}} = (> nR.C)^{\mathcal{I}}$;
(2) $(\neg(\geq nR.C))^{\mathcal{I}} = (< nR.C)^{\mathcal{I}}$;
(3) $(\exists R.C)^{\mathcal{I}} = (\geq 1R.C)^{\mathcal{I}}$;
(4) $(\forall R.C)^{\mathcal{I}} = (< 1R.\neg C)^{\mathcal{I}}$.

Proposition 7 *Let A be a concept name, C, D concepts, a, b individuals, R a role and n a natural number in QC $\mathcal{SHIQ}$. If $\mathcal{I}$ is a QC weak or QC strong interpretation, then the follows properties hold.*

(1) $\mathcal{I} \models_x \overline{A}(a)$ *iff* $\mathcal{I} \not\models_x A(a)$
(2) $\mathcal{I} \models_x \overline{\overline{A}}(a)$ *iff* $\mathcal{I} \models_x A(a)$
(3) $\mathcal{I} \models_x \overline{\neg A}(a)$ *iff* $\mathcal{I} \models_x \neg\overline{A}(a)$
(4) $\mathcal{I} \models_x \overline{(\forall R.C)}(a)$ *iff* $\mathcal{I} \models_x \exists R.\overline{C}(a)$
(5) $\mathcal{I} \models_x \overline{(\exists R.C)}(a)$ *iff* $\mathcal{I} \models_x \forall R.\overline{C}(a)$
(6) $\mathcal{I} \models_x \overline{\top}(a)$ *iff* $\mathcal{I} \models_x \bot(a)$
(7) $\mathcal{I} \models_w \overline{C} \sqcup \overline{D}(a)$ *iff* $\mathcal{I} \models_w \overline{C \sqcap D}(a)$
(8) $\mathcal{I} \models_w \overline{C} \sqcap \overline{D}(a)$ *iff* $\mathcal{I} \models_w \overline{C \sqcup D}(a)$
(9) $\mathcal{I} \models_x \overline{(\geq nR.C)}(a)$ *iff* $\mathcal{I} \models_x \leq n\text{-}1R.C(a)$
(10) $\mathcal{I} \models_x \overline{(\leq nR.C)}(a)$ *iff* $\mathcal{I} \models_x \geq n\text{+}1R.C(a)$

where $\models_x$ is a place-holder for both $\models_w$ and $\models_s$.

Definition 5 *Given a QC ontology $\mathcal{O}$ and an axiom ϕ in $\mathcal{SHIQ}$, we say $\mathcal{O}$* quasi-classically entails *(for short* QC entails*) ϕ, denoted by $\mathcal{O} \models_Q \phi$, iff for every interpretation $\mathcal{I}$, for any axiom ψ of $\mathcal{O}$ if $\mathcal{I} \models_s \psi$ then $\mathcal{I} \models_w \phi$. In this case, $\models_Q$ is called a* quasi-classical entailment (relation) *(for short* QC entailment*) between $\mathcal{O}$ and ϕ.*

The following example shows that $\models_Q$ is non-trivializable in the sense that when an ontology $\mathcal{O}$ is classically inconsistent.

Example 1 *Given an ABox $\mathcal{A} = \{B(a), \neg B(a)\}$ and a concept name A in QC $\mathcal{SHIQ}$. It is clear that $\mathcal{A}$ is classically inconsistent. However $\mathcal{A} \models_Q A(a)$ does not hold. This is because there exists an interpretation $\mathcal{I}$ such that $a^{\mathcal{I}} \in +B$ and $a^{\mathcal{I}} \in -B$ where $B^{\mathcal{I}} = \langle +B, -B \rangle$. So $\mathcal{I} \models_s B \sqcap \neg B(a)$, but $\mathcal{I} \not\models_w A(a)$ since $A(a)$ does not occur in $\mathcal{A}$.*

The following proposition shows that $\models_Q$ satisfies the *resolution rule*.

Proposition 8 *Let C, D, E be concepts and a be an individual.*

$$\{C \sqcup D(a), \neg C \sqcup E(a)\} \models_Q D \sqcup E(a).$$

By Proposition 8, $\models_Q$ satisfies three important inferences: *MP*, *MT* and *DS*.

Example 2 *Suppose $\mathcal{O}$ is an empty QC ontology.*
(1) *Consider the concept $A \sqcup \neg A$. We have $\mathcal{O} \not\models_Q A \sqcup \neg A(a)$, since $\mathcal{O}$ strongly satisfies every formula in $\mathcal{O}$, but $\mathcal{O}$ does not weakly satisfy $A \sqcup \neg A(a)$.*
(2) *Consider the concept $A \sqcup \overline{A}$. We have $\mathcal{O} \models_Q A \sqcup \overline{A}(a)$, since $\mathcal{O}$ strongly satisfies every axiom in $\mathcal{O}$ and $\mathcal{O} \models_w A(a)$ or $\mathcal{O} \models_w \overline{A}(a)$, i.e., $\mathcal{O} \models_w A \sqcup \overline{A}(a)$.*

Example 2 shows that, for some concept name A in QC $\mathcal{SHIQ}$, the concept $A \sqcup \overline{A}$ can be considered as the top concept $\top$ under QC entailment. However, this is not the case for concept $A \sqcup \neg A$.

In the following, we show that some properties of QC entailment satisfied in QC propositional logic (see [9]) still hold in QC $\mathcal{SHIQ}$.

Proposition 9 *Given a QC ontology $\mathcal{O}$, axioms ϕ, ψ, two concepts C, D and an individual a, the following properties hold.*
(1) (Reflexivity) $\mathcal{O} \cup \{\phi\} \models_Q \phi$.
(2) (Monotonicity) *if* $\mathcal{O} \models_Q \phi$ *then* $\mathcal{O} \cup \{\psi\} \models_Q \phi$.
(3) (And-introduction) *if* $\mathcal{O} \models_Q C(a)$ *and* $\mathcal{O} \models_Q D(a)$ *then* $\mathcal{O} \models_Q C \sqcap D(a)$.
(4) (Or-elimination) *if* $\mathcal{O} \cup \{C(a)\} \models_Q \phi$ *and* $\mathcal{O} \cup \{D(a)\} \models_Q \phi$ *then* $\mathcal{O} \cup \{C \sqcup D(a)\} \models_Q \phi$.

The following proposition shows that QC entailment is weaker than classical entailment in $\mathcal{SHIQ}$.

Proposition 10 *Let $\mathcal{O}$ be an ontology and ϕ be an axiom in $\mathcal{SHIQ}$.*

$$\text{If } \mathcal{O} \models_Q \phi \text{ then } \mathcal{O} \models \phi.$$

The converse of Proposition 10 does not hold. In Example 1, $\mathcal{A} \models \phi$ for any axiom because $\mathcal{A}$ is classically inconsistent while $\mathcal{A} \not\models_Q A(a)$.

Proposition 11 *Let $\mathcal{O}$ be an ontology and ϕ be an axiom in $\mathcal{SHIQ}$.*

$$\text{If } \mathcal{O} \models_4 \phi \text{ then } \mathcal{O} \models_Q \phi.$$

In QC $\mathcal{SHIQ}$, a strong interpretation $\mathcal{I}$ *satisfies* a terminology $\mathcal{T}$ (role hierarchy $\mathcal{R}$) iff $\mathcal{I} \models_s C \sqsubseteq D$ ($R \sqsubseteq S$) for each $C \sqsubseteq D$ in $\mathcal{T}$ ($R \sqsubseteq S$ in $\mathcal{R}$). In this case, $\mathcal{I}$ is called a *QC model* of $\mathcal{T}$, written $\mathcal{I} \models_s \mathcal{T}$ (a *QC model* of $\mathcal{R}$, written $\mathcal{I} \models_s \mathcal{R}$). A strong interpretation $\mathcal{I}$ *satisfies* an ABox $\mathcal{A}$ iff $\mathcal{I} \models_s \phi$ for each assertion ϕ in $\mathcal{A}$, where ϕ is one of the following forms: $C(a)$, $R(a, b)$, $a \neq b$. In this case, $\mathcal{I}$ is called a *QC model* of $\mathcal{A}$, written $\mathcal{I} \models_s \mathcal{A}$.

The following proposition shows the close relationship between the QC entailment and the QC model.

Proposition 12 *Let $\mathcal{O}$ be a QC ontology, C, D two concepts and a an individual.*
(1) $\mathcal{O} \models_Q C(a)$ *iff there is no any QC model of* $\mathcal{O} \cup \{\overline{C}(a)\}$.
(2) $\mathcal{O} \models_Q C \sqsubseteq D$ *iff there is no any QC model of* $\mathcal{O} \cup \{C \sqcap \overline{D}(t)\}$ *for some new individual t not occurring in $\mathcal{O}$.*

4 Reasoning in Quasi-Classical Description Logic $\mathcal{SHIQ}$

4.1 The QC Consistency Problem

In QC $\mathcal{SHIQ}$, a concept C is *QC satisfiable* w.r.t. a QC ABox $\mathcal{A}$ if there exists a QC model $\mathcal{I}$ of $\mathcal{A}$ such that $\mathcal{I} \models_s C(a)$ for some individual a, and *QC unsatisfiable* w.r.t. $\mathcal{A}$ otherwise. A concept C is *QC satisfiable* w.r.t. a QC role hierarchy $\mathcal{R}$ and a QC terminology $\mathcal{T}$ if there exists a QC model $\mathcal{I}$ of $\mathcal{R}$ and $\mathcal{T}$ such that $\mathcal{I} \models_s C(a)$ for some individual a, and *QC unsatisfiable* w.r.t. $\mathcal{R}$ and $\mathcal{T}$ otherwise. A QC ABox $\mathcal{A}$ is *QC consistent* if there exists a QC model $\mathcal{I}$ of $\mathcal{A}$, and *QC inconsistent* otherwise. A QC ABox $\mathcal{A}$ is *QC consistent* w.r.t. a QC role hierarchy $\mathcal{R}$ and a QC terminology $\mathcal{T}$ if there exists a QC model $\mathcal{I}$ of $\mathcal{R}$ and $\mathcal{T}$ such that $\mathcal{I}$ is a QC model of $\mathcal{A}$, and *QC inconsistent* w.r.t. $\mathcal{R}$ and $\mathcal{T}$ otherwise.

We are able to show that the complexity of QC consistency checking in QC $\mathcal{SHIQ}$ is in the same level as that of satisfiability check in $\mathcal{SHIQ}$.

Proposition 13 *Checking QC consistency of a QC $\mathcal{SHIQ}$ ABox w.r.t. a QC role hierarchy $\mathcal{R}$ and a QC terminology $\mathcal{T}$ is EXPTIME-Complete.*

4.2 The Inference Problems

In QC $\mathcal{SHIQ}$, there are two basic inference problems given as follows.

- *instance checking:* an individual a is called a *QC instance* of a concept C w.r.t. a QC ABox $\mathcal{A}$ iff for any QC model $\mathcal{I}$ of $\mathcal{A}$, $\mathcal{I}$ is a QC model of $C(a)$.
- *subsumption:* a concept C *QC subsumes* a concept D w.r.t. a QC role hierarchy $\mathcal{R}$ and a QC terminology $\mathcal{T}$ iff for any QC model $\mathcal{I}$ of $\mathcal{R}$ and $\mathcal{T}$, $\mathcal{I}$ is a QC model of $C \sqsubseteq D$.

Proposition 14 *Given a QC $\mathcal{SHIQ}$ ontology $\mathcal{O}$, two concepts C, D and an individual a, we have*
(1) $\mathcal{O} \models_Q C(a)$ *iff* $\mathcal{O} \cup \{\overline{C}(a)\}$ *is QC inconsistent.*
(2) $\mathcal{O} \models_Q C \sqsubseteq D$ *iff* $\mathcal{O} \cup \{C \sqcap \overline{D}(t)\}$ *is QC inconsistent for some new individual t not occurring in $\mathcal{O}$.*

Proposition 14 shows that two basic inference problems can be reduced to the problem of QC consistency checking.

5 Conclusions

In this paper, we presented QC $\mathcal{SHIQ}$ to handle inconsistency in $\mathcal{SHIQ}$. The syntax of $\mathcal{SHIQ}$ was extended by introducing the notion of the complement of a concept. QC semantics was defined by two kinds of semantics: weak semantics and strong semantics. We showed that both of them satisfy some important inference rules. A QC entailment relation based on strong interpretations and weak interpretations was introduced and we showed that this entailment relation satisfies some desirable properties.

We introduced the notion of QC consistency of a QC ontology and two basic QC entailment problems and showed that these two entailment problems can be reduced to the problem of QC consistency checking. Furthermore, we showed that QC consistency problem of ABoxes is EXPTIME-Complete. As a future work, following [8], we will develop an algorithm based on the classical tableau for querying. In the other direction, we will consider employing a classical DL reasoner, such as KAON2 or Pellet, to implement paraconsistent reasoning based on QC semantics. Further, we will also consider introducing QC semantics into more expressive DLs such as $\mathcal{SHOIN}(D)$.

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