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The Solvability of First Type Boundary Value Problem for a Schrödinger Equation

Nigar Yildirim Aksoy*

Department of Mathematics, Faculty of Sciences and Arts, Kafkas University, 36100, Kars, Turkey

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Abstract

The paper presents an first type boundary value problem for a Schrödinger equation. The aim of paper is to give the existence and uniqueness theorems of the boundary value problem using Galerkin's method. Also, a priori estimate for its solution is given.

Keywords: First Type Boundary value problem, Schrödinger equation, Galerkin's method.
AMS 2010 codes: 35Q41, 35K20, 35A01.

1 Introduction

The fundamental equation of quantum mechanics, Schrödinger eqution, is the basic non-relativistic wave equation which describes the behaviour of a single particle (on systems of particles) in a field of force. Its solution is called a wave function which give us information about the particle's behavior in time and space and the square of the wave function states the probability of finding the location of an particle in a given area. Schrödinger equation is roughly similar to Newton's equation. Schrödinger equation does for a quantum mechanical particle what Newton's Second Law does for a classical particle. When we solve Newton's equation we can find the position of a particle as depend on time. But, when we solve Schrödinger's equation we get a wave function stated the probability of finding the particle in some region in space varies as a function of time.

As known, Schrödinger equation is a partial differential equation, that is, its solution is a function, not a number. This means that some partial differential equations can be solved, some can't, and some can only be estimated. The exact and numerical solutions of the various partial differential equations have been investigated by using the different methods in works [1-5], [7-11], [13-14], [16-25]. In this study, we analyze a Schrödinger equation whose solution is an estimate of particle. Many researchers have been studied the approximate and exact solutions of Schrödinger equation by using different methodologies such as Adomian Decomposition Method (ADM) [10, 16], Homotopy Perturbation Method (HPM) [3, 5, 14], Homotopy Analysis Method (HAM)

*Corresponding author.
Email address: nyaksoy55@hotmail.com

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[2, 4], Variational Iteration Method (VIM) [7, 18], Galerkin's Method [1, 8-9, 11, 13, 17, 19-21]. Besides, there is various solution techniques for Schrödinger equation.

In this paper, we regard a first type boundary value problem for linear Schrödinger equation in the form:

$$i\frac{\partial\psi}{\partial t}+a_0\frac{\partial^2\psi}{\partial x^2}+ia_1(x,t)\frac{\partial\psi}{\partial x}-a_2(x)\psi+v(t)\psi=f(x,t) \tag{1.1}$$

$$\psi(x,0)=\varphi(x),\ x\in(0,l) \tag{1.2}$$

$$\psi(0,t)=\psi(l,t)=0,\ t\in(0,T), \tag{1.3}$$

where $\psi=\psi(x,t)$ is a wave function, l and T are positive numbers. Here, we will use the notations: $x\in I=(0,l)$, $t\in[0,T]$, $\Omega_t=(0,l)\times(0,t)$, $\Omega=\Omega_T$, $i=\sqrt{-1}$. $a_0>0$ is a given real number, the functions $a_1(x,t)$, $a_2(x)$, $v(t)\in L_2(0,T)$ are the measurable real-valued which satisfy the conditions, respectively,

$$|a_1(x,t)|\le\mu_1,\left|\frac{\partial a_1(x,t)}{\partial x}\right|\le\mu_2,\left|\frac{\partial^2 a_1(x,t)}{\partial x^2}\right|\le\mu_3 \text{ for } almost\ all\ (a.a)\ (x,t)\in\Omega, \tag{1.4}$$
$$\mu_1,\mu_2,\mu_3=const.>0,$$

$$0<a_2(x)\le\mu_4 \text{ for } a.a\ x\in I,\ \mu_4=const.>0, \tag{1.5}$$

$$\|v(t)\|_{L_2(0,T)}\le b_0,\ b_0=const.>0, \tag{1.6}$$

$\varphi(x)$ and $f(x,t)$ are given complex-valued functions such that

$$\varphi\in\mathring{W}_2^2(I),\ f\in\mathring{W}_2^{2,0}(\Omega). \tag{1.7}$$

Here, the spaces $\mathring{W}_2^2(I)$, $\mathring{W}_2^{2,0}(\Omega)$ are Sobolev space and are defined in [12] widely.

We investigate the solutions of the boundary value problem (1.1)-(1.3) (BVP) under conditions (1.4)-(1.7). For this purpose, we shall apply a well known Galerkin's method to BVP. Also, we obtain an estimate for solutions of equation (1.1).

According to Galerkin's method, some fundamental system of linearly independent functions in the studied spaces is choosen and the approximate solutions by means of linearly independent functions are constituted. The solution of BVP is obtained as limits of approximate solutions calculated by this method [11].

2 The Existence and Uniqueness of Solutions

In this section, we define the solution of BVP and prove a theorem which states the existence and uniqueness of the solution of BVP.

Definition 2.1. *A function $\psi(x,t)\in\mathring{W}_2^{2,1}(\Omega)$ is said to be a solution of BVP,if it holds the conditions (1.1)-(1.3) for a.a $(x,t)\in\Omega$, where $W_2^{2,1}(\Omega)$ is a Sobolev space of all elements in $L_2(\Omega)$ having generalized derivatives up to order 2 and 1 with respect to variables x and t, respectiely, inclusive in $L_2(\Omega)$. Also, $\mathring{W}_2^{2,1}(\Omega)$ is a subspace of $W_2^{2,1}(\Omega)$ and $\mathring{W}_2^{2,1}(\Omega)=W_2^{2,1}(\Omega)\cap\mathring{W}_2^{1,0}(\Omega)$.*

Theorem 2.1. *Let the conditions (1.4)-(1.7) be satisfied. Then, BVP has a unique solution $\psi(x,t)\in\mathring{W}_2^{2,1}(\Omega)$ which holds the estimate*

$$||\psi||^2_{\mathring{W}_2^{2,1}(\Omega)}\le c_0\left(||\varphi||^2_{\mathring{W}_2^2(I)}+||f||^2_{\mathring{W}_2^{2,0}(\Omega)}\right), \tag{2.1}$$

where c_0 is a positive constant independent from φ and f.

Proof. We will prove the theorem (2.1) by the Galerkin's method. By this method, the approximate solutions are searched in the form:

$$\psi^N(x,t) = \sum_{k=1}^{N} C_k^N(t) u_k(x), \tag{2.2}$$

where the functions $u_k = u_k(x)$ for $k = 1,2,..$ generate a fundamental system in the space $\mathring{W}_2^2(I)$ and are eigenfunctions corresponding to the eigenvalues λ_k of the problem:

$$\begin{aligned} Lu_k(x) &= -a_0 \frac{d^2 u_k(x)}{dx^2} + a_2(x) u_k(x) = \lambda_k u_k(x),\ x \in I \\ u_k(0) &= u_k(l) = 0 \text{ for } k = 1,2,... \end{aligned}$$

This is a Sturm-Liouville problem. So, its eigenvalues are real and nonnegative and the eigenfunctions $u_k = u_k(x)$ corresponding to the eigenvalues λ_k are real and ortogonal in the spaces $L_2(I)$, $\mathring{W}_2^1(I)$, $\mathring{W}_2^2(I)$. Assume

$$||u_k||_{\mathring{W}_2^2(I)} \leq d_k \text{ for } k = 1,2,.., \tag{2.3}$$

where d_k are positive constants and let $u_k = u_k(x)$ for $k = 1,2,..$ be an orthonormal basis in the space $L_2(I)$. Also, in (2.2), the coefficients $C_k^N(t) = \left(\psi^N(.,t), u_k\right)_{L_2(I)} = \left(\psi^N, u_k\right)$ hold the system

$$\left(i\frac{\partial \psi^N}{\partial t}, u_k\right) = \left(L\psi^N, u_k\right) - i\left(a_1(x,t)\frac{\partial \psi^N}{\partial x}, u_k\right) - \left(v(t)\psi^N, u_k\right) + \left(f, u_k\right), \tag{2.4}$$

and initial conditions

$$C_k^N(0) = \left(\psi^N(.,0), u_k\right)_{L_2(I)} = (\varphi, u_k) = \varphi_k \tag{2.5}$$

for $k = 1,2,..,N$, where $\varphi_k \to \varphi$ strongly in $\mathring{W}_2^2(I)$. The system (2.4) is a system of first order linear nonhomogeneous ordinary differential equations with constant coefficients with respect to the unknowns $C_k^N(t)$ and system (2.4) with (2.5) is a Cauchy problem. From [15], it is written that the problem (2.4)-(2.5) has locally at least one solution on $[0,T]$.

We assert that the problem (2.4)-(2.5) has global solution on $[0,T]$. To prove it, we give the next lemma: □

Lemma 2.1. *The coefficients $C_k^N(t)$ provide the estimate*

$$\int_0^T \sum_{k=1}^{N} \left|C_k^N(t)\right|^2 + \int_0^T \sum_{k=1}^{N} \left|\frac{dC_k^N(t)}{dt}\right|^2 \leq ||\psi^N||^2_{\mathring{W}_2^{2,1}(\Omega)} \leq c_1 \left(||\varphi||^2_{\mathring{W}_2^2(I)} + ||f||^2_{\mathring{W}_2^{2,0}(\Omega)}\right) \tag{2.6}$$

for $N = 1,2,..,$ where the positive constant c_1 does not depend on N.

The proof of lemma (2.1) is carried out as in [20].

We now turn to proof of the theorem (2.1). It follows from the Lemma (2.1) that all possible solutions of the Cauchy problem (2.4)-(2.5) are uniformly bounded on $[0,T]$, which implies that problem (2.4)-(2.5) has one global solution on $[0,T]$.

Let's define a family of functions $l_{N,k}(t)$ for $k,N = 1,2,...$ such that $l_{N,k}(t) = \left(\psi^N(.,t), u_k\right)_{L_2(I)}$. From the estimate (2.6), it is seen that the functions $l_{N,k}(t)$ for $k,N = 1,2,...$ are uniformly bounded on $[0,T]$. So, the inequality

$$\max_{0\leq t\leq T} |l_{N,k}(t)| \leq c_2,$$

is written, where the positive constant c_2 is independent from N,k.

Now, let's show that the functions $l_{N,k}(t)$ are equicontinuous for fixed k and $N \geq k,\ \ N,k = 1,2,...$ on the interval $[0,T]$. For this purpose, we integrate the $k-th$ equation in (2.4) on $(t,t+\Delta t)$ and so, we get

$$i\left(l_{N,k}(t+\Delta t)-l_{N,k}(t)\right) = a_0\int_t^{t+\Delta t}\int_0^l \frac{\partial\psi^N}{\partial x}\frac{du_k}{dx}dxd\tau - i\int_t^{t+\Delta t}\int_0^l a_1(x,t)\frac{\partial\psi^N}{\partial x}u_k dxd\tau +$$
$$\int_t^{t+\Delta t}\int_0^l a_2(x)\psi^N u_k dxd\tau - \int_t^{t+\Delta t}\int_0^l v(\tau)\psi^N u_k dxd\tau + \int_t^{t+\Delta t}\int_0^l f u_k dxd\tau. \tag{2.7}$$

After taking the absolute value of (2.7), applying the Cauchy-Schwarz inequality, we obtain

$$|l_{N,k}(t+\Delta t)-l_{N,k}(t)| \leq a_0\int_t^{t+\Delta t}\left\|\frac{\partial\psi^N(.,\tau)}{\partial x}\right\|_{L_2(I)}\left\|\frac{du_k}{dx}\right\|_{L_2(I)}d\tau +$$
$$\mu_1\int_t^{t+\Delta t}\left\|\frac{\partial\psi^N(.,\tau)}{\partial x}\right\|_{L_2(I)}\|u_k\|_{L_2(I)}d\tau + \mu_4\int_t^{t+\Delta t}\left\|\psi^N(.,\tau)\right\|_{L_2(I)}\|u_k\|_{L_2(I)}d\tau +$$
$$\int_t^{t+\Delta t}|v(\tau)|\left\|\psi^N(.,\tau)\right\|_{L_2(I)}\|u_k\|_{L_2(I)}d\tau + \int_t^{t+\Delta t}\|f(.,\tau)\|_{L_2(I)}\|u_k\|_{L_2(I)}d\tau \tag{2.8}$$

by means of the conditions (1.4), (1.5). Using (2.3) in (2.8), if we take account the inequality

$$\int_t^{t+\Delta t}|v(\tau)|\left\|\psi^N(.,\tau)\right\|_{L_2(I)}\|u_k\|_{L_2(I)}d\tau \leq d_k\int_t^{t+\Delta t}|v(\tau)|\left\|\psi^N(.,\tau)\right\|_{L_2(I)}d\tau$$
$$\leq d_k\left(\max_{0\leq\tau\leq T}\left\|\psi^N(.,\tau)\right\|_{L_2(I)}\right)\int_t^{t+\Delta t}|v(\tau)|d\tau$$
$$\leq d_k\sqrt{\Delta t}\left(\max_{0\leq\tau\leq T}\left\|\psi^N(.,\tau)\right\|_{L_2(I)}\right)\left(\int_0^T|v(\tau)|^2d\tau\right)^{\frac{1}{2}}$$
$$\leq b_0 d_k\sqrt{\Delta t}\left(\max_{0\leq\tau\leq T}\left\|\psi^N(.,\tau)\right\|_{L_2(I)}\right)$$

and apply Cauchy-Schwarz inequality with respect to t to all term at right-hand side of (2.8), we obtain

$$|l_{N,k}(t+\Delta t)-l_{N,k}(t)| \leq d_k\sqrt{\Delta t}a_0\left(\int_0^T\left\|\frac{\partial\psi^N}{\partial x}\right\|^2_{L_2(I)}d\tau\right)^{\frac{1}{2}} + d_k\sqrt{\Delta t}\mu_1\left(\int_0^T\left\|\frac{\partial\psi^N}{\partial x}\right\|^2_{L_2(I)}d\tau\right)^{\frac{1}{2}}$$
$$+d_k\sqrt{\Delta t}\mu_4\left(\int_0^T\left\|\psi^N\right\|^2_{L_2(I)}d\tau\right)^{\frac{1}{2}} + b_0 d_k\sqrt{\Delta t}\left(\max_{0\leq\tau\leq T}\left\|\psi^N(.,\tau)\right\|_{L_2(I)}\right)$$
$$+d_k\sqrt{\Delta t}\left(\int_0^T\|f\|^2_{L_2(I)}d\tau\right)^{\frac{1}{2}}.$$

In above inequality, using the estimate (2.6), we get

$$|l_{N,k}(t+\Delta t)-l_{N,k}(t)| \leq c_3 d_k (\Delta t)^{\frac{1}{2}} \tag{2.9}$$

for $k,N = 1,2,...,$where the positive constant c_3 does not depend on $N,k,\Delta t$. It follows from (2.9), that the functions $l_{N,k}(t)$ are equicontinuous on $[0,T]$.

Thus, from Ascoli-Arzela's theorem [6], we can extract the subsequence $\{l_{N_m,k}(t)\}$ from sequences $\{l_{N,k}(t)\}$ for fixed k and $m = 1,2,..$ such that

$$l_{N_m,k}(t) \ \underrightarrow{uniformly} \ l_k(t) \ \text{ on } \ [0,T] \tag{2.10}$$

Let

$$\psi(x,t) = \sum_{k=1}^{\infty} l_k(t) u_k(x) \tag{2.11}$$

and we claim that the subsequence $\left\{\psi^{N_m}\right\}$ weakly converges to $\psi(x,t)$ in $L_2(I)$, which this convergence are uniformly with respect to the variable t. That is, there is a positive number ε such that $\left|\left(\psi^{N_m}(x,t)-\psi(x,t),g\right)_{L_2(I)}\right| < \varepsilon$ for all $t \in [0,T]$, $\forall g \in L_2(I)$. Since the space $L_2(I)$ is a separable Hilbert space, we can write any element g of $L_2(I)$ in the form $g = \sum\limits_{k=1}^{\infty} (g,u_k)_{L_2(I)} u_k$. Thus, it is written that

$$\begin{aligned}
&\left(\psi^{N_m}(x,t)-\psi(x,t),g\right)_{L_2(I)} = \left(\psi^{N_m}-\psi, \sum_{k=1}^{\infty}(g,u_k)_{L_2(I)} u_k\right)_{L_2(I)} \\
&= \left(\psi^{N_m}-\psi, \sum_{k=1}^{s}(g,u_k)_{L_2(I)} u_k\right)_{L_2(I)} + \left(\psi^{N_m}-\psi, \sum_{k=s+1}^{\infty}(g,u_k)_{L_2(I)} u_k\right)_{L_2(I)} \\
&= \sum_{k=1}^{s}(g,u_k)_{L_2(I)} \left(\psi^{N_m}(x,t)-\psi(x,t),u_k\right)_{L_2(I)} + \left(\psi^{N_m}-\psi, \sum_{k=s+1}^{\infty}(g,u_k)_{L_2(I)} u_k\right)_{L_2(I)} \\
&\leq \left(\sum_{k=1}^{s}\left|(g,u_k)_{L_2(I)}\right|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{s}|l_{N_m,k}(t)-l_k(t)|^2\right)^{\frac{1}{2}} \\
&\quad + \left\|\psi^{N_m}-\psi\right\|_{L_2(I)} \left\|\sum_{k=s+1}^{\infty}(g,u_k)_{L_2(I)} u_k\right\|_{L_2(I)}.
\end{aligned} \tag{2.12}$$

Since $g \in L_2(I)$, $\left(\sum\limits_{k=1}^{s}\left|(g,u_k)_{L_2(I)}\right|^2\right)^{\frac{1}{2}} = \|g\|_{L_2(I)} < +\infty$ for big enough values of s. Also, since $l_{N_m,k}(t) \to l_k(t)$ as $N_m \to \infty$ it is written that for any $\varepsilon > 0$

$$\left(\sum_{k=1}^{s}\left|(g,u_k)_{L_2(I)}\right|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{s}|l_{N_m,k}(t)-l_k(t)|^2\right)^{\frac{1}{2}} \leq c_3 \frac{\varepsilon}{2} \tag{2.13}$$

for big enough values of s, where $c_3 > 0$ is independent from N_m. Similarly, it is clear that

$$\left\|\psi^{N_m}-\psi\right\|_{L_2(I)} \left\|\sum_{k=s+1}^{\infty}(g,u_k)_{L_2(I)} u_k\right\|_{L_2(I)} \leq c_4 \left(\sum_{k=s+1}^{\infty}\left|(g,u_k)_{L_2(I)}\right|^2\right)^{\frac{1}{2}}, \tag{2.14}$$

where $c_4 > 0$ is independent from N_m. Since the series $\sum_{k=s+1}^{\infty} \left|(g,u_k)_{L_2(I)}\right|^2$ is the rest of Fourier series of the function $g \in L_2(I)$, if we regard the converging of the series $\sum_{k=1}^{\infty} \left|(g,u_k)_{L_2(I)}\right|^2$, we can write $\sum_{k=s+1}^{\infty} \left|(g,u_k)_{L_2(I)}\right|^2 \leq \frac{\varepsilon^2}{4}$ for any $\varepsilon > 0$. Thus, consideringly this inequality if we use (2.13) and (2.14) in (2.12), we achieve

$$\left|\left(\psi^{N_m}(x,t) - \psi(x,t), g\right)_{L_2(I)}\right| < \varepsilon$$

as $N_m \to \infty$ for $\forall g \in L_2(I)$, $\forall t \in [0,T]$ and $\forall \varepsilon > 0$, which follows that the sequence $\left\{\psi^{N_m}(x,t)\right\}$ weakly converges to $\psi(x,t)$ in $L_2(I)$ as uniformly with respect to t.

For $N = N_m$, since the subsequence $\left\{\psi^{N_m}\right\}$ is uniformly bounded from (2.6), we can extract a subsequence which weakly converges in $W_2^{2,1}(\Omega)$ to $\psi(x,t)$ defined by formula (2.11). For simplicity, let's denote this subsequence as $\left\{\psi^{N_m}(x,t)\right\}$. That is, limit relations

$$\left\{\psi^{N_m}\right\} \xrightarrow{weakly} \psi(x,t) \text{ in } L_2(\Omega) \tag{2.15}$$

$$\left\{\frac{\partial \psi^{N_m}}{\partial x}\right\} \xrightarrow{weakly} \frac{\partial \psi(x,t)}{\partial x} \text{ in } L_2(\Omega) \tag{2.16}$$

$$\left\{\frac{\partial^2 \psi^{N_m}}{\partial x^2}\right\} \xrightarrow{weakly} \frac{\partial^2 \psi(x,t)}{\partial x^2} \text{ in } L_2(\Omega) \tag{2.17}$$

$$\left\{\frac{\partial \psi^{N_m}}{\partial t}\right\} \xrightarrow{weakly} \frac{\partial \psi(x,t)}{\partial t} \text{ in } L_2(\Omega). \tag{2.18}$$

are written.Thus, by using the limit relations (2.15)-(2.18) and the weakly lower semicontinuity of the norm on $L_2(\Omega)$,if we take the lower limit of estimate (2.6) for $N = N_m$ and as $m \to \infty$ we have the inequalities

$$\begin{aligned}
||\psi||^2_{L_2(\Omega)} &\leq \underline{\lim}_{m\to\infty} \left(||\psi^{N_m}||^2_{L_2(\Omega)}\right) \leq \underline{\lim}_{m\to\infty} \left(c_1 \left(||\varphi||^2_{\mathring{W}_2^2(I)} + ||f||^2_{\mathring{W}_2^{2,0}(\Omega)}\right)\right) \\
\left\|\frac{\partial \psi}{\partial x}\right\|^2_{L_2(\Omega)} &\leq \underline{\lim}_{m\to\infty} \left(\left\|\frac{\partial \psi^{N_m}}{\partial x}\right\|^2_{L_2(\Omega)}\right) \leq \underline{\lim}_{m\to\infty} \left(c_1 \left(||\varphi||^2_{\mathring{W}_2^2(I)} + ||f||^2_{\mathring{W}_2^{2,0}(\Omega)}\right)\right) \\
\left\|\frac{\partial^2 \psi}{\partial x^2}\right\|^2_{L_2(\Omega)} &\leq \underline{\lim}_{m\to\infty} \left(\left\|\frac{\partial^2 \psi^{N_m}}{\partial x^2}\right\|^2_{L_2(\Omega)}\right) \leq \underline{\lim}_{m\to\infty} \left(c_1 \left(||\varphi||^2_{\mathring{W}_2^2(I)} + ||f||^2_{\mathring{W}_2^{2,0}(\Omega)}\right)\right) \\
\left\|\frac{\partial \psi}{\partial t}\right\|^2_{L_2(\Omega)} &\leq \underline{\lim}_{m\to\infty} \left(\left\|\frac{\partial \psi^{N_m}}{\partial t}\right\|^2_{L_2(\Omega)}\right) \leq \underline{\lim}_{m\to\infty} \left(c_1 \left(||\varphi||^2_{\mathring{W}_2^2(I)} + ||f||^2_{\mathring{W}_2^{2,0}(\Omega)}\right)\right)
\end{aligned}$$

which is equivalent to

$$||\psi||^2_{W_2^{2,1}(\Omega)} \leq 4c_1 \left(||\varphi||^2_{\mathring{W}_2^2(I)} + ||f||^2_{\mathring{W}_2^{2,0}(\Omega)}\right),$$

it follows that the limit function $\psi(x,t)$ provides the estimate (2.1) and $\psi \in W_2^{2,1}(\Omega)$.

Now, let's show that the function $\psi(x,t)$ provides the equation (1.1) for $a.a$ $(x,t) \in \Omega$. After multiplying the k-*th* equation in (2.4) for $N = N_m$ with any continuous function $\overline{\eta}_k(t)$ and let's sum the obtained equalities on k from 1 to $N' \leq N_m$ and finally integrate over $[0,T]$.Ultimately,we achieve the identity

$$\int_{\Omega} \left[i\frac{\partial \psi^{N_m}}{\partial t} + a_0 \frac{\partial^2 \psi^{N_m}}{\partial x^2} + ia_1(x,t)\frac{\partial \psi^{N_m}}{\partial x} - a(x)\psi^{N_m} + v(t)\psi^{N_m} - f\right] \overline{\eta}^{N'}(x,t)dx = 0, \tag{2.19}$$

where $\overline{\eta}^{N'}(x,t)=\sum_{k=1}^{N'}\overline{\eta}_k(t)u_k(x)$, $N'\le N_m$.Thus, taking the limit of (2.19) for $N=N_m$ as $m\to\infty$ and then using the limit relations (2.15)-(2.18), we obtain the identity

$$\int_{\Omega}\left[i\frac{\partial\psi}{\partial t}+a_0\frac{\partial^2\psi}{\partial x^2}+ia_1(x,t)\frac{\partial\psi}{\partial x}-a(x)\psi+v(t)\psi-f\right]\overline{\eta}^{N'}(x,t)dx=0,$$

where $\overline{\eta}^{N'}(x,t)=\sum_{k=1}^{N'}\overline{\eta}_k(t)u_k(x)$, $N'\le N_m$. Since the set of functions $\overline{\eta}^{N'}(x,t)$ are dense in $L_2(\Omega)$, if we take the limit of above integral identity for $N'\to\infty$ we get the following identity for any $\eta(x,t)\in L_2(\Omega)$

$$\int_{\Omega}\left[i\frac{\partial\psi}{\partial t}+a_0\frac{\partial^2\psi}{\partial x^2}+ia_1(x,t)\frac{\partial\psi}{\partial x}-a(x)\psi+v(t)\psi-f\right]\overline{\eta}(x,t)dx=0. \tag{2.20}$$

From (2.20), we can easily say that the limit function $\psi(x,t)$ holds (1.1) for $a.a\ (x,t)\in\Omega$.

Similarly to the paper [20], we prove that the conditions (1.2) and (1.3) is fulfilled by limit function $\psi(x,t)$. Thus, we arrive $\psi\in\mathring{W}_2^{2,1}(\Omega)$.

Finally, let's prove the uniquenes of solution of BVP in $\mathring{W}_2^{2,1}(\Omega)$. To that end, we consider two different solutions ψ and ζ in $\mathring{W}_2^{2,1}(\Omega)$. Let's denote $\rho(x,t)=\psi(x,t)-\zeta(x,t)$. Then, the function $\rho(x,t)$ satisfies the following boundary value problem:

$$i\frac{\partial\rho}{\partial t}+a_0\frac{\partial^2\rho}{\partial x^2}+ia_1(x,t)\frac{\partial\rho}{\partial x}-a(x)\rho+v(t)\rho=0 \tag{2.21}$$

$$\rho(x,0)=0,\ x\in I \tag{2.22}$$

$$\rho(0,t)=\rho(l,t)=0,\ t\in(0,T). \tag{2.23}$$

To obtain the uniquenes of the solution of BVP in $\mathring{W}_2^{2,1}(\Omega)$, if we multiply the equation (2.21) by $\overline{\rho}(x,t)$ and later integrate over Ω_t, we have

$$\int_{\Omega_t}\left[i\frac{\partial\rho}{\partial t}\overline{\rho}+a_0\frac{\partial^2\rho}{\partial x^2}\overline{\rho}+ia_1(x,t)\frac{\partial\rho}{\partial x}\overline{\rho}-a(x)|\rho|^2+v(\tau)|\rho|^2\right]dxd\tau=0.$$

In above equality, if we apply the formula of partial integration we obtain

$$\int_{\Omega_t}\left[i\frac{\partial\rho}{\partial t}\overline{\rho}-a_0\left|\frac{\partial\rho}{\partial x}\right|^2+ia_1(x,t)\frac{\partial\rho}{\partial x}\overline{\rho}-a(x)|\rho|^2+v(\tau)|\rho|^2\right]dxd\tau=0. \tag{2.24}$$

Then, subtracting the complex conjugate of (2.24) from itself, we get

$$\int_{\Omega_t}\left[\frac{\partial\left(|\rho|^2\right)}{\partial t}+a_1(x,t)\left(\frac{\partial\rho}{\partial x}\overline{\rho}+\frac{\partial\overline{\rho}}{\partial x}\rho\right)\right]dxd\tau=0,$$

which is equivalent to

$$\|\rho(.,t)\|_{L_2(I)}^2+\int_{\Omega_t}\frac{\partial}{\partial x}\left(a_1(x,t)|\rho|^2\right)dxd\tau=\int_{\Omega_t}\frac{\partial a_1(x,t)}{\partial x}|\rho|^2dxd\tau. \tag{2.25}$$

If we use (2.22), (2.23) in (2.25), we get

$$\|\rho(.,t)\|^2_{L_2(I)} \leq \int\limits_{\Omega_t} \left|\frac{\partial a_1(x,t)}{\partial x}\right| |\rho|^2 dxd\tau \leq \mu_3 \int\limits_0^t \|\rho(.,\tau)\|^2_{L_2(I)} d\tau.$$

Thus, if we apply Gronwall's lemma to the above inequality, we have

$$0 \leq \|\rho(.,t)\|^2_{L_2(I)} \leq 0. \tag{2.26}$$

The inequality (2.26) implies that $\|\rho(.,t)\|^2_{L_2(I)} = 0$ for *any* $t \in [0,T]$. So, $\psi(x,t) = \zeta(x,t)$ for *any* $t \in [0,T]$ and $a.a\ x \in I$. i.e., BVP has a unique solution in $\overset{\circ}{W}{}^{2,1}_2(\Omega)$. We arrive at the conslusion of the theorem (2.1).

3 Conclusion

In this paper, Galerkin's method have been succesfully applied to the linear Schrödinger equation with special gradient term. It was shown that the solution of BVP exists and it is unique. Also, an estimate satisfied by the solution function is obtained. Studied problem consists of a special gradient term and the coefficients of equation are more general than the former works. Especially, the coefficient a_1 depends on both the variables x and t. Because of the distinctness of considered equation with conditions, our problem differs from previous works in the literature.

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