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Interpolative Rus-Reich-Ćirić Type Contractions via Simulation Functions

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Abstract

In this paper, we introduce the notion of interpolative Rus-Reich-Ćirić type $\mathcal{Z}$- contractions in the setting of complete metric space. We also consider some immediate consequences of our main results.

1 Introduction and Preliminaries

Let A and B be two Banach spaces. If A and B are algebraically and topologically imbedded in a separated topological linear space, then the pair of A and B is called a Banach couple and is denoted by (A, B). If there is a Banach space E for the Banach couple (A, B) such that the imbedding $A \cap B \subset E \subset A + B$ holds, then E is called and intermediate space of (A, B).

Let (A, B) and (C, D) be two Banach couples. A linear mapping T acting from the space $A + B$ to $C + D$ is called a *bounded operator* $from$ (A, B) to (C, D) if the restrictions of T to the spaces A and B are bounded operators from A to C and B to D, respectively.

We denote by $L(AB, CD)$ the linear space of all bounded operators from the couple (A, B) to the couple (C, D). This is a Banach space in the norm

$$\|T\|_{L(AB,CD)} = \max\{\|T\|_{A\to B}, \|T\|_{C\to D}\}.$$

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Definition 1.1 ([38])**.** *Let (A, B) and (C, D) be two Banach couples, and E (respectively F) be intermediate for the spaces of the Banach couple (A, B) (respectively (C, D)). The triple (A, B, E) is called an interpolation triple, relative to (C, D, F), if every bounded operator from (A, B) to (C, D) maps E to F.*

A triple (A, B, E) is said to be an interpolation triple of type γ $(0 \leq \alpha \leq 1)$ relative to (C, D, F) if it is an interpolation triple and the following inequality holds:

$$\|T\|_{E\to F} \leq c\|T\|^{\gamma}_{A\to B} \cdot \|T\|^{1-\gamma}_{C\to D},$$

for some constant c.

Very recently, inspired from the interpolation theory, an attractive fixed point result via interpolation was reported in [28]. More precisely, in [28], the notion of interpolative Kannan contraction was introduced as follows: For a metric space (X, d), a mapping $T : X \to X$ is called an interpolative Kannan contraction if

$$d(Tx, Ty) \leq \lambda \left[d(x, Tx)\right]^{\gamma} \cdot \left[d(y, Ty)\right]^{1-\gamma}, \tag{1}$$

for all $x, y \in X$ with $x, y \in X \backslash Fix(T)$, where $Fix(T)$ is the set of all fixed point of T, $\lambda \in [0, 1)$ and $\gamma \in (0, 1)$. The main result in [28] is the following.

Theorem 1.1 ([28])**.** *Let (X, d) be a complete metric space and T be an interpolative Kannan type contraction. Then T has a fixed point in X.*

In [28], an example was given to show that the interpolative Kannan type contraction is more effective than the classical Kannan contraction. This initial result was followed by further extensions see e.g. [29, 30].

On the other hand, in 2015, Khojasteh *et al.* [37] introduced the notion of *simulation function.*

Definition 1.2. *(See [37]) A* simulation function *is a mapping $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ satisfying the following conditions:*

(ζ_1) $\zeta(0, 0) = 0$;

(ζ_2) $\zeta(t, s) < s - t$ *for all* $t, s > 0$;

(ζ_3) *if $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then*

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0. \tag{2}$$

In the same year, 2015, this notion was refined by Argoubi et al. *[1] by removing the first axiom* (ζ_1)*. Indeed, it is derived form* (ζ_2)*. From now on, we consider the simulation functions in the sense of Argoubi* et al. *[1], that is,* ζ *satisfies only* (ζ_2) *and* (ζ_3)*. In the sequel, the the letter* $\mathcal{Z}$ *will denote the family of all simulation functions* $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ *that satisfy* (ζ_2) *and* (ζ_3)*. Notice also that the axiom* (ζ_2) *yields that*

$$\zeta(t,t) < 0 \text{ for all } t > 0. \tag{3}$$

Example 1.1. *(See e.g.[37, 40, 3]) Let* $\phi_i : [0,\infty) \to [0,\infty)$ *be continuous functions with* $\phi_i(t) = 0$ *if, and only if,* $t = 0$*. For* $i = 1,2,3,4,5,6$*, we define the mappings* $\zeta_i : [0,\infty) \times [0,\infty) \to \mathbb{R}$*, as follows*

(i) $\zeta_1(t,s) = \phi_1(s) - \phi_2(t)$ *for all* $t,s \in [0,\infty)$*, where* $\phi_1(t) < t \leq \phi_2(t)$ *for all* $t > 0$*.*

(ii) $\zeta_2(t,s) = s - \dfrac{f(t,s)}{g(t,s)}t$ *for all* $t,s \in [0,\infty)$*, where* $f,g : [0,\infty)^2 \to (0,\infty)$ *are two continuous functions with respect to each variable such that* $f(t,s) > g(t,s)$ *for all* $t,s > 0$*.*

(iii) $\zeta_3(t,s) = s - \phi_3(s) - t$ *for all* $t,s \in [0,\infty)$*.*

(iv) If $\varphi : [0,\infty) \to [0,1)$ *is a function such that* $\limsup_{t\to r^+} \varphi(t) < 1$ *for all* $r > 0$*, and define*

$$\zeta_4(t,s) = s\,\varphi(s) - t \qquad \text{for all } s,t \in [0,\infty).$$

(v) If $\eta : [0,\infty) \to [0,\infty)$ *is an upper semi-continuous mapping such that* $\eta(t) < t$ *for all* $t > 0$ *and* $\eta(0) = 0$*, and define*

$$\zeta_5(t,s) = \eta(s) - t \qquad \text{for all } s,t \in [0,\infty).$$

(vi) If $\phi : [0,\infty) \to [0,\infty)$ *is a function such that* $\int_0^\varepsilon \phi(u)du$ *exists and* $\int_0^\varepsilon \phi(u)du > \varepsilon$*, for each* $\varepsilon > 0$*, and define*

$$\zeta_6(t,s) = s - \int_0^t \phi(u)du \qquad \text{for all } s,t \in [0,\infty).$$

It is clear that each function ζ_i $(i = 1,2,3,4,5,6)$ *forms a simulation function.*

For further examples and more details on simulation functions see e.g. [37, 40, 3, 4, 5, 14, 15, 25, 26, 27].

Suppose (X, d) is a metric space, T is a self-mapping on X and $\zeta \in \mathcal{Z}$. We say that T is a $\mathcal{Z}$-*contraction* with respect to ζ [37], if

$$\zeta(d(Tx, Ty), d(x, y)) \geq 0 \qquad \text{for all } x, y \in X. \tag{4}$$

Again from (ζ_2), we have the following inequality

$$d(Tx, Ty) \neq d(x, y) \text{ for all distinct } x, y \in X. \tag{5}$$

Thus, we conclude that T cannot be an isometry whenever T is a $\mathcal{Z}$-contraction. In other words, if a $\mathcal{Z}$-contraction T in a metric space has a fixed point, then it is necessarily unique.

Theorem 1.2. *Every $\mathcal{Z}$-contraction on a complete metric space has a unique fixed point.*

The concept of comparison function is introduced by Rus [42] and it has been extensively studied by several of authors to expand more general form of contraction type mappings.

Definition 1.3. *[42] An increasing function $\phi : [0, \infty) \to [0, \infty)$ is said to be a comparison if $\phi^n(t) \to 0$ as $n \to \infty$ for every $t \in [0, \infty)$, where ϕ^n is the n-th iterate of ϕ.*

The collection of all comparison functions will be denoted by Φ.

Let Ψ be the family of nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following condition :

(Ψ_2) $\displaystyle\sum_{n=1}^{+\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the n^{th} iterate of ψ.

The functions in the class of Ψ are called (c)-comparison functions and hence $\Psi \subset \Phi$. Fundamental properties of (c)-comparison functions are collected below:

Lemma 1.1. *(See e.g. [42]) If $\psi \in \Psi$, then the following hold:*

- *(i)* $(\psi^n(t))_{n \in \mathbb{N}}$ *converges to* 0 *as* $n \to \infty$ *for all* $t \in \mathbb{R}^+$*;*
- *(ii)* $\psi(t) < t$, *for any* $t \in \mathbb{R}^+$;
- *(iii)* ψ *is continuous at* 0*;*
- *(iv)* *the series* $\sum_{k=1}^{\infty} \psi^k(t)$ *converges for any* $t \in \mathbb{R}^+$.

The notion of α-admissible mappings [41] and the concept of triangular α-admissible mappings [36] were reconsidered and refined by Popescu [39] in the following way:

Definition 1.4. *[39] Let $\alpha : X \times X \to [0, \infty)$ be a mapping and $X \neq \emptyset$. A self-mapping $T : X \to X$ is said to be an $\alpha-$orbital admissible if for all $s \in X$, we have*

$$\alpha(s, Ts) \geq 1 \Rightarrow \alpha(Ts, T^2 s) \geq 1. \tag{6}$$

Furthermore, an $\alpha-$orbital admissible mapping T is called triangular α-orbital admissible if it holds the following condition:

(TO) $\alpha(s, t) \geq 1$ and $\alpha(s, Tt) \geq 1$ implies that $\alpha(s, Tt) \geq 1$, for all $s, t \in X$.

It is obvious that each $\alpha-$admissible mapping is an α-orbital admissible mapping but not the converse see e.g. [39]. For further attractive results, more examples with details see e.g. [2, 4]-[9]-[10]-[13],[17],[22],[18],[19], [36], [23], [24] and the references therein.

In this paper, we introduce a new interpolative contraction by using the simulation function together with the admissible mappings in the context of complete metric spaces. More precisely, we shall revisit one of the the renowned results in the fixed point theory that was proved independently by Rus, Reich and Ćirić see e.g. [43, 44, 45, 46, 47]. For the sake of the completeness of the paper, we recollect here:

Theorem 1.3. *Let (X, d) be a complete metric spaces and $T : X \to X$ be a Rus-Reich-Ćirić contraction mapping, i.e.,*

$$d(Tx, Ty) \leq \lambda \left[d(x, y) + d(x, Tx) + d(y, Ty) \right], \tag{7}$$

for all $x, y \in X$, where $\lambda \in \left[0, \frac{1}{3}\right)$. Then T has a unique fixed point.

Note that the theorem above was proved independently by Rus [46, 47] and Reich [43, 44, 45] and Ćirić. Notice that several variation of Rus-Reich-Ćirić contraction (7) can be stated also as

$$d(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + cd(y, Ty),$$

where a, b, c are nonnegative real numbers such that $0 \leq a + b + c < 1$.

2 Main results

We start with the following definition.

Definition 2.1. *Let T be a self-mapping defined on a metric space (X, d). If there exist $\zeta \in \mathcal{Z}$, $\psi \in \Psi$, $\gamma, \beta \in (0,1)$ with $\gamma + \beta < 1$ and $\alpha : X \times X \to [0, \infty)$ such that*

$$\zeta(\alpha(x,y)d(Tx,Ty), \psi(R(x,y))) \geq 0 \qquad \text{for all } x, y \in X, \tag{8}$$

where

$$R(x,y) := [d\,(x,y)]^{\beta} \cdot [d\,(x,Tx)]^{\gamma} \cdot [d\,(y,Ty)]^{1-\gamma-\beta},$$

then we say that T is an interpolative Rus-Reich-Ćirić type $\mathcal{Z}$-contraction *with respect to ζ.*

If $\alpha(x,y) = 1$, then T turns into a $\mathcal{Z}$*-contraction with respect to* ζ.

Remark 2.1. *If T is an α-admissible* $\mathcal{Z}$-contraction *with respect to ζ, then*

$$\alpha(x,y)d(Tx,Ty) < \psi(R(x,y))) \quad \text{for all } x, y \in X. \tag{9}$$

To prove the assertion, we assume that $x \neq y$. Then $d(x,y) > 0$. If $Tx = Ty$, then $\alpha(x,y)d(Tx,Ty) = 0 < \psi(R(x,y)))$. Otherwise, $Tx \neq Ty$, then $d(Tx,Ty) > 0$. If $\alpha(x,y) = 0$, then the inequality is satisfied trivially. So assume that $\alpha(x,y) > 0$ and applying (ζ_2) with (8), we derive that

$$0 \leq \zeta(\alpha(x,y)d(Tx,Ty), \psi(R(x,y)))) < \psi(R(x,y))) - \alpha(x,y)d(Tx,Ty),$$

so (9) holds.

We can now state the main result of this paper.

Theorem 2.1. *Let (X, d) be a complete metric space, $\zeta \in \mathcal{Z}$. If a self-mapping $T : X \to X$ forms an interpolative Rus-Reich-Ćirić type* $\mathcal{Z}$-contraction *with respect to ζ and satisfies*

(*i*) *T is triangular $\alpha-$orbital admissible,*

(*ii*) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,*

(*iii*) *T is continuous,*

then there exists $u \in X$ such that $Tu = u$.

Proof. On account of the assumption (*ii*), there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Starting with this initial point $x_0 \in X$ an iterative sequence $\{x_n\}$ is constructed by $x_{n+1} = Tx_n$ for all $n \geq 0$. Throughout the proof, without loss of generality, we assume that

$$d(x_n, x_{n+1}) > 0, \text{ for all } n = 0, 1, \ldots. \tag{10}$$

Indeed, if there exists an k_0 such that $x_{k_0} = x_{k_0+1}$, then $u = x_{k_0}$ becomes a fixed point of T which completes the proof. Accordingly, we suppose that $x_n \neq x_{n+1}$ for all n, that is, (10) holds.

By taking the assumption (ii) into account and by regarding that T is $\alpha-$orbital admissible, we obtain that

$$\alpha(x_n, x_{n+1}) \geq 1, \text{ for all } n = 0, 1, \ldots. \tag{11}$$

From (8) and (11), it follows that for all $n \geq 1$, we have

$$\begin{aligned} 0 &\leq \zeta(\alpha(x_n, x_{n-1})d(Tx_n, Tx_{n-1}), \psi(R(x_n, x_{n-1}))) \\ &= \zeta(\alpha(x_n, x_{n-1})d(x_{n+1}, x_n), \psi(R(x_n, x_{n-1}))) \\ &< \psi(R(x_n, x_{n-1})) - \alpha(x_n, x_{n-1})d(x_{n+1}, x_n). \end{aligned} \tag{12}$$

Consequently, we derive that

$$d(x_n, x_{n+1}) \leq \alpha(x_n, x_{n-1})d(x_n, x_{n+1}) < \psi(R(x_n, x_{n-1})) < R(x_n, x_{n-1}), \tag{13}$$

for all $n = 1, 2, \ldots$, where,

$$\begin{aligned} R(x_n, x_{n-1}) &= [d(x_n, x_{n-1})]^{\beta} \cdot [d(x_n, Tx_n)]^{\gamma} \cdot [d(x_{n-1}, Tx_{n-1})]^{1-\gamma-\beta}. \\ &= [d(x_n, x_{n-1})]^{\beta} \cdot [d(x_n, x_{n+1})]^{\gamma} \cdot [d(x_{n-1}, x_n)]^{1-\gamma-\beta}. \\ &= \cdot [d(x_n, x_{n+1})]^{\gamma} \cdot [d(x_{n-1}, x_n)]^{1-\gamma}. \end{aligned}$$

By a simple elimination, the inequality (15) implies that

$$[d(x_n, x_{n+1})]^{1-\gamma} \leq \lambda [d(x_{n-1}, x_n)]^{1-\gamma}. \tag{14}$$

Hence, we conclude that the sequence $\{d(x_n, x_{n-1})\}$ is non-decreasing and bounded from below by zero. Moreover, we deduce, from the monotonicity of $\{d(x_n, x_{n-1})\}$, that $R(x_n, x_{n-1}) \leq d(x_n, x_{n-1})$ and consequently, the inequality (13) turns into

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \alpha(x_n, x_{n-1})d(x_n, x_{n+1}) < \psi(R(x_n, x_{n-1})) \\ &< R(x_n, x_{n-1}) \leq d(x_n, x_{n-1}). \end{aligned} \tag{15}$$

Accordingly, there exists $L \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n-1}) = L \geq 0$. We shall prove that

$$\lim_{n\to\infty} d(x_n, x_{n-1}) = 0. \tag{16}$$

Suppose, on the contrary that $L > 0$. Note that from the inequality (15), we derive that

$$\lim_{n\to\infty} \alpha(x_n, x_{n-1})d(x_n, x_{n+1}) = L, \tag{17}$$

and

$$\lim_{n\to\infty} R(x_n, x_{n+1}) = L. \tag{18}$$

Letting $s_n = \alpha(x_n, x_{n-1})d(x_n, x_{n+1})$ and $t_n = R(x_n, x_{n-1})$ and taking (ζ_3) into account, we get that

$$0 \leq \limsup_{n\to\infty} \zeta(\alpha(x_n, x_{n-1})d(x_{n+1}, x_n), R(x_n, x_{n-1})) < 0 \tag{19}$$

which is a contradiction. Thus, we have $L = 0$.

Now, we shall prove that the iterative sequence $\{x_n\}$ is Cauchy. Again we use the method of *Reductio ad absurdum.* Suppose, on the contrary that, $\{x_n\}$ is not a Cauchy sequence. Thus, there exists $\varepsilon > 0$, for all $N \in \mathbb{N}$, there exist $n, m \in \mathbb{N}$ with $n > m > N$ and $d(x_m, x_n) > \varepsilon$. On the other hand, from (16), there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) < \varepsilon \text{ for all } n > n_0. \tag{20}$$

Consider two partial subsequences x_{n_k} and x_{m_k} of x_n such that

$$n_0 \leq n_k < m_k < m_{k+1} \text{ and } d(x_{m_k}, x_{n_k}) > \varepsilon \text{ for all } k. \tag{21}$$

Notice that

$$d(x_{m_{k-1}}, x_{n_k}) \leq \varepsilon \text{ for all } k, \tag{22}$$

where m_k is chosen as a least number $m \in \{n_k, n_{k+1}, n_{k+2}, \ldots\}$ such that (21) is satisfied. We also mention that $n_k + 1 \leq m_k$ for all k.. In fact, the case $n_k + 1 \leq m_k$ is impossible due to (20),(21). Thus, $n_k + 2 \leq m_k$ for all k. It yields that

$$n_k + 1 < m_k < m_k + 1 \text{ for all } k.$$

On account of (21),(22) and the triangle inequality, we derive that

$$\begin{aligned} \varepsilon &< d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{m_k-1}) + \varepsilon \text{ for all } k. \end{aligned} \tag{23}$$

Due to (16), we deduce that

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon. \tag{24}$$

Again by the triangle inequality, we derive that

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k}) \text{ for all } k.$$

Analogously, we have

$$d(x_{m_k+1}, x_{n_k+1}) \leq d(x_{m_k+1}, x_{m_k}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_k+1}) \text{ for all } k.$$

Combining the two inequalities above together with (16) and (23), we find that

$$\lim_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) = \varepsilon. \tag{25}$$

Particularly, there exists $n_1 \in \mathbb{N}$ such that for all $k \geq n_1$ we have

$$d(x_{m_k}, x_{n_k}) > \frac{\varepsilon}{2} > 0 \text{ and } d(x_{m_k+1}, x_{n_k+1}) > \frac{\varepsilon}{2} > 0. \tag{26}$$

Moreover, since T is triangular α-orbital admissible, we have

$$\alpha(x_{m_k}, x_{n_k}) \geq 1. \tag{27}$$

Regarding the fact T is an α-admissible $\mathcal{Z}$-*contraction* with respect to ζ, together with (26) and (27) we get that

$$\begin{aligned} 0 &\leq \zeta(\alpha(x_{m_k}, x_{n_k})d(Tx_{m_k}, Tx_{n_k}), \psi(R(x_{m_k}, x_{n_k}))) \\ &= \zeta(\alpha(x_{m_k}, x_{n_k})d(x_{m_k+1}, x_{n_k+1}), \psi(R(x_{m_k}, x_{n_k}))) \\ &< \psi(R(x_{m_k}, x_{n_k})) - \alpha(x_{m_k}, x_{n_k})d(x_{m_k+1}, x_{n_k+1}), \end{aligned} \tag{28}$$

for all $k \geq n_1$, where

$$\begin{aligned} R(x_{m_k}, x_{n_k}) &= [d(x_{m_k}, x_{n_k})]^{\beta} \cdot [d(x_{m_k}, Tx_{m_k})]^{\gamma} \cdot [d(x_{n_k}, Tx_{n_k})]^{1-\gamma-\beta}. \\ &= [d(x_{m_k}, x_{n_k})]^{\beta} \cdot [d(x_{m_k}, x_{m_k+1})]^{\gamma} \cdot [d(x_{n_k}, x_{n_k+1})]^{1-\gamma-\beta}. \end{aligned} \tag{29}$$

Consequently, we have

$$0 < d(x_{m_k+1}, x_{n_k+1}) < \alpha(x_{m_k}, x_{n_k})d(x_{m_k+1}, x_{n_k+1}) < \psi(R(x_{m_k}, x_{n_k})) < R(x_{m_k}, x_{n_k}),$$

for all $k \geq n_1$. Letting $n, m \to \infty$ in the inequality above, and keeping in mind the observations in (16), (30), (25), (28) and (29), we find that

$$\lim_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) = 0, \tag{30}$$

which is a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence. Owing to the fact that (X, d) is a complete metric space, there exists $u \in X$ such that

$$\lim_{n\to\infty} d(x_n, u) = 0. \tag{31}$$

Since T is continuous, we derive from (31) that

$$\lim_{n\to\infty} d(x_{n+1}, Tu) = \lim_{n\to\infty} d(Tx_n, Tu) = 0. \tag{32}$$

From (31), (32) and the uniqueness of the limit, we conclude that u is a fixed point of T, that is, $Tu = u$. $\square$

Theorem 2.2. *Let (X, d) be a complete metric space and let $T : X \to X$ be an α-admissible* $\mathcal{Z}$-contraction *with respect to ζ. Suppose that*

(*i*) *T is triangular α-orbital admissible;*

(*ii*) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(*iii*) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.*

Then there exists $u \in X$ such that $Tu = u$.

Proof. Following the proof of Theorem 2.1, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \geq 0$, converges for some $u \in X$. From (11) and condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \geq 1$ for all k. Applying (8), for all k, we get that

$$\begin{aligned} 0 &\leq \zeta(\alpha(x_{n(k)}, u)d(Tx_{n(k)}, Tu), \psi(R(x_{n(k)}, u))) \\ &= \zeta(\alpha(x_{n(k)}, u)d(x_{n(k)+1}, Tu), \psi(R(x_{n(k)}, u))) \\ &< \psi(R(x_{n(k)}, u)) - \alpha(x_{n(k)}, u)d(x_{n(k)+1}, Tu), \end{aligned} \tag{33}$$

which is equivalent to

$$d(x_{n(k)+1}, Tu) = d(Tx_{n(k)}, Tu) \leq \alpha(x_{n(k)}, u)d(Tx_{n(k)}, Tu) \leq \psi(R(x_{n(k)}, u)). \tag{34}$$

Letting $k \to \infty$ in the above equality, we have $d(u, Tu) = 0$, that is, $u = Tu$. □

3 Consequences

In this section, we shall illustrate that several existing fixed point results in the literature can be derived from our main results by regarding Example 1.1.

If $\psi \in \Psi$ and we define

$$\zeta_E(t, s) = \psi(s) - t \qquad \text{for all } s, t \in [0, \infty),$$

then ζ_E is a simulation function (cf. Example 1.1 (v)).

Corollary 3.1. *Let (X, d) be a complete metric space, $\zeta \in \mathcal{Z}$. Let a self-mapping $T : X \to X$ satisfy*

$$\alpha(x, y)d(Tx, Ty) \leq \psi(R(x, y)), \text{ for all } x, y \in X \setminus Fix(T).$$

Suppose also that

(i) T is triangular $\alpha-$orbital admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) T is continuous.

Then there exists $u \in X$ such that $Tu = u$.

Proof. Taking $\zeta_E(t,s) = \psi(s) - t$ for all $s, t \in [0, \infty)$ in Theorem 2.1, we get that

$$\alpha(x,y)d(Tx,Ty) \leq \psi(R(x,y)), \text{ for all}$$

We skip the details. □

Corollary 3.2. *Let (X,d) be a complete metric space, $\zeta \in \mathcal{Z}$. If a self-mapping $T: X \to X$ satisfies*

$$d(Tx,Ty) \leq \psi(R(x,y)), \text{ for all } x, y \in X \setminus Fix(T),$$

then there exists $u \in X$ such that $Tu = u$.

Proof. Take $\alpha(x,y) = 1$ for all $x, y \in X$ in Corollary 3.1. □

Definition 3.1. *Let (X,d) be a metric space. We say that the self-mapping $T : X \to X$ is an* interpolative Rus-Reich-Ćirić type contraction, *if there exist a constant $\lambda \in [0,1)$ and $\gamma, \beta \in (0,1)$ such that*

$$d\left(Tx,Ty\right) \leq \lambda R(x,y) \tag{35}$$

for all distinct $x, y \in X \setminus F_T(X)$, where

$$R(x,y) := \left[d\left(x,y\right)\right]^{\beta} \cdot \left[d\left(x,Tx\right)\right]^{\gamma} \cdot \left[d\left(y,Ty\right)\right]^{1-\gamma-\beta}$$

Corollary 3.3. *Let (X,d) be a complete metric space and T be an interpolative Rus-Reich-Ćirić type contraction. Then T has a fixed point in X.*

Proof. For $\lambda \in (0,1)$, take $\psi(t) = \lambda$ for all $x, y \in X$ in Corollary 3.2. □

Example 3.1. *Let $X = \{1,3,4,7\}$ be a set endowed with a standard metric $d(x,y) = |x-y|$.*

d(x,y)	*1*	*3*	*4*	*7*
1	*0*	*2*	*3*	*6*
3	*2*	*0*	*1*	*4*
4	*3*	*1*	*0*	*3*
7	*6*	*4*	*3*	*0*

We define a self-mapping T on X by $T : \begin{pmatrix} 1 & 3 & 4 & 7 \\ 4 & 7 & 4 & 3 \end{pmatrix}$. *It is clear that T is not Rus-Reich-Ćirić contraction. Indeed, there is no $\lambda \in [0, \frac{1}{3})$ such that the following inequality is fulfilled:*

$$\begin{aligned} d(T1, T3) = d(4,7) = 3 \quad &\leq \lambda(d(1,3) + d(T1,1) + d(3,T3)) \\ &= \lambda(d(1,3) + d(4,1) + d(3,7)) \\ &= \lambda(2+3+4) = 9\lambda. \end{aligned}$$

On the other hand, for $\gamma = \beta = \frac{1}{16}$ and $\lambda = \frac{4}{5}$, the self-mapping T forms an interpolative Rus-Reich-Ćirić type contraction and 4 *is the desired unique fixed point of T. Note that in the setting of interpolative Rus-Reich-Ćirić type contraction, the constant lies between 0 and 1 although in the classical version it is restricted with 1/3. Notice also that this constructive example can be imbedded in several known examples.*

Conclusion and Discussion

It is clear that we can list several consequences of our main results by defining the mapping ζ in a proper way like in the Example 1.1. In particular, inspired from the results in [24] , we are able to get several existing fixed point theorems in the various settings (in the context of *partially ordered set endowed with a metric*, in the setting of *cyclic contraction* etc.) regarding Theorem 2.1 (and hence Theorem 2.2). We omit the details since they can be easily obtained by verbatim of [24].

Competing interests

The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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